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COMPLETE PROPOSITIONAL CONNECTIVES

by ROGER F. WHEELER in Leicester, England

By a well-known theorem due to POST [1], a propositional connective in 2-valued logic, with truth function $f(p_1, p_2, \ldots, p_n)$, will generate by iteration all the 2^{2^n} truth functions of n variables if and only if

(1) $$f(p, p, \ldots, p) = \neg p$$

(2) $$f(\neg p_1, \neg p_2, \ldots, \neg p_n) \neq \neg f(p_1, p_2, \ldots, p_n).$$

For the case $n = 2$, it is also well-known that there are just 2 functions (joint denial and alternative denial) which have this property, and, for $n = 3$, R. A. CUNINGHAME-GREEN [2] has found that there are 16 essentially distinct connectives which are capable of generating all the ternary functions.

The object of the present paper is to study such complete propositional connectives in 2-valued logic systematically and to describe methods of enumerating them when $n = 4$ and $n = 5$.

0	0	0	0	—
1	0	0	0	*a*
0	1	0	0	*b*
0	0	1	0	*c*
0	0	0	1	*d*
1	1	0	0	*ab*
1	0	1	0	*ac*
1	0	0	1	*ad*
0	1	1	0	*bc*
0	1	0	1	*bd*
0	0	1	1	*cd*
1	1	1	0	*abc*
1	1	0	1	*abd*
1	0	1	1	*acd*
0	1	1	1	*bcd*
1	1	1	1	*abcd*

Table 1

Suppose the 2^n possible sets of truth values for $p_1, p_2, \ldots, p_n$ are arranged in the following order, which is illustrated in table 1 for the case $n = 4$. The first number written down is the number consisting only of zeros, and this is followed by all the numbers with only one unit digit, then all the numbers with just 2 units and so on, finishing with 11 . . . 1. Within each group the numbers are arranged lexically, giving the digits, for example, precedence from left to right. Table 1 also shows an equivalent alphabetic reference system for the argument-sets and for many purposes this scheme will be found to be more illuminating than the sets of truth values.

When the sets of truth values are ordered according to this plan, it is possible to describe POST's criteria in a new and graphic way. If the valuation of f is given for each of these sets in turn, then f is the truth function for a complete connective if and only if the resulting series of digits $q_1, q_2, \ldots, q_r$ ($r = 2^n$) has the following features.

(1) $q_1 = 1$ and $q_r = 0$.

(2) The sequence $q_1, q_2, \ldots, q_r$ is *altered* when 0 and 1 are interchanged and the order of the digits is simultaneously reversed.

[Note, in passing, that an alternative ordering scheme, which at first sight might seem more natural, is to list the 2^n sets in ascending binary order, and, in fact, both methods allow POST's criteria to be restated in the above form. It will, however. be found that the ordering chosen is more convenient for concentrating one's attention on those features of the tabulation which are significant for ensuring completeness and avoiding duplication.]

The 16 essentially distinct connectives on 3 propositions p_1, p_2, p_3 which have the necessary property are tabulated systematically in table 2. The 8 different sets of truth values for (p_1, p_2, p_3) appear as headings for the columns, and 1 representative of each of the 16 functions is recorded in each row.

Table 2 displays, first, the function in which $q_2, \ldots, q_7$ contains no unit digit, then all those in which this sequence contains just one unit digit and so on. It will be noticed that with the systematic arrangement used here, the bottom half of the table can be obtained at once as a mirror-image, so to speak, of the top half, the sequence $q_2, \ldots, q_7$ being derived from its 'reflexion' by interchange of 0 and 1. The rows of Table 2 correspond, in order, to the columns in Mr. CUNINGHAME-GREEN's table [2] numbered

1 3 2 5 4 6 9 7 10 8 12 11 14 13 15 16.

The column headed N is the key to successful enumeration. It records the number of times each function will occur when every permutation of p_1, p_2, p_3 is counted separately, and all possible truth functions satisfying POST's criteria are recorded. The total number of these is, of course, $8 \times 7 = 56$, since q_1 and q_8 are fixed, q_2, q_3, q_4 can be chosen in 2^3 ways, and the other 3 places then filled in $(2^3 - 1)$ ways, (the 'missing' one being the one which, by its symmetry, would violate POST's second condition). For example, as shown in table 3, the function in line 5 will appear 6 times, and so on. The reason for preferring, as headings for the columns, the alphabetic symbols a, b, c, d mentioned earlier will now be clear. Experience has shown that it is much easier to decide what number N to assign by using these

		a	b	c	ab	ac	bc			
p_1	0	1	0	0	1	1	0	1	N	
p_2	0	0	1	0	1	0	1	1		
p_3	0	0	0	1	0	1	1	1		
$f(p_1, p_2, p_3)$	1	0	0	0	0	0	0	0	1	$\binom{6}{0}$
	1	1	0	0	0	0	0	0	3	$\binom{6}{1}$
	1	0	0	0	1	0	0	0	3	
	1	1	1	0	0	0	0	0	3	$\binom{6}{2}$
	1	1	0	0	1	0	0	0	6	
	1	1	0	0	0	0	1	0	3	
	1	0	0	0	1	1	0	0	3	
	1	1	1	0	0	1	0	0	6	$\binom{6}{3}-2^3$
	1	0	0	1	1	0	1	0	6	
	1	1	1	1	0	0	1	0	3	$\binom{6}{4}$
	1	0	1	1	1	1	0	0	3	
	1	0	1	1	0	1	1	0	6	
	1	0	0	1	1	1	1	0	3	
	1	1	1	1	0	1	1	0	3	$\binom{6}{5}$
	1	0	1	1	1	1	1	0	3	
	1	1	1	1	1	1	1	0	1	$\binom{6}{6}$
	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8		
									56	2^6-2^3

Table 2

letters, rather than the sets of truth values. It has also been found extremely helpful to think geometrically in many situations, by regarding a, b, c, d as the vertices of a tetrahedron whose edges and faces are the double and triple letter symbols. This sort of geometric language will be used freely in the sequel, whenever it seems appropriate.

a	b	c	ab	ac	bc
1	0	0	1	0	0
1	0	0	0	1	0
0	1	0	1	0	0
0	1	0	0	0	1
0	0	1	0	1	0
0	0	1	0	0	1
q_2	q_3	q_4	q_5	q_6	q_7

Table 3

The problem of listing all the functions for a given number of variables, therefore, is only a question of doing this trivial enumeration systematically, in order to ensure that the list is exhaustive and excludes duplicates. For this purpose, the last column at the right of table 2 provides a crucial check. It records the total number of functions in which the group $q_2, \ldots, q_7$ contains $0, 1, \ldots, 6$ unit digits. The number of functions containing 3 units must, of course, be adjusted to remove the 2^3 'forbidden' members. In the general case, if the functions for which $q_2, \ldots, q_{r-1}$ contain u unit digits are complete, their total, allowing for all permutations, will be $\binom{r-2}{u}$, except when $u = \frac{1}{2}(r-2)$, in which case it will be $\binom{r-2}{u} - 2^u$.

Hence, by the exercise of a little patience, the above method will generate all the permissible truth functions in a purely routine way. Unless, however, one is particularly anxious to possess a complete list for reference, it is more interesting to enumerate them some method more economical than merely compiling a table and counting the number of entries. Two methods for doing this will be discussed, and a description of each method will be given by applying it to the enumeration of the functions of 4 arguments. The first method is an immediate outcome of the above process of systematic tabulation, but the second method, which in turn developed from the first, is rather more efficient. In both cases, we start by ignoring POST's second condition and show that there are 996 essentially distinct functions with 4 arguments which satisfy the first condition, and then show that just 16 of these are symmetric and so fail to satisfy the second criterion, so that the number of complete propositional connectives on 4 variables is 980.

The same total, 980, has been obtained independently by Mr. CUNINGHAME-GREEN, using a Ferranti 'Pegasus' digital computer.

Method 1

We concentrate on the number of unit digits (from 0 to 14) contained in the sequence $q_2, \ldots, q_{15}$. Suppose the single letter columns a, b, c, d contain u_1 units $(0 \leq u_1 \leq 4)$, the double letter columns $ab, \ldots, cd$ contain u_2 units $(0 \leq u_2 \leq 6)$ and the triple letter columns contain u_3 units. Such a function will be given the reference symbol $[u_1, u_2, u_3]$ and the total number of such functions is clearly

$$\binom{4}{u_1} \cdot \binom{6}{u_2} \cdot \binom{4}{u_3} = T[u_1, u_2, u_3], \text{ say.}$$

Thus a group of functions of type (e.g.) [2, 2, 1], [1, 2, 2], [2, 2, 3], [3, 2, 2], [2, 4, 1], [1, 4, 2], [2, 4, 3], [3, 4, 2] will all contain the same number of members. But furthermore, and this is the important point, the number of essentially distinct functions in each group will be the same. Hence, to reduce the amount of computation, any one member of the group can be taken as representative, and the solutions which arise can be counted with the appropriate multiplicity, shown in column M in table 5. As a check, note that the sum of the numbers in this column M is

$$(4+1)(6+1)(4+1) = 175.$$

Obviously, the most convenient choice is $0 \leq u_2 \leq 3$, $0 \leq u_1, u_3 \leq 2$, with, say, $u_3 \leq u_1$, and this is the choice adopted below.

For simplicity, Greek letters have been used as arbitrary reference symbols for various partition types and the significance of these will now be explained. Any given selection of single and triple letters corresponds to a partition of a, b, c, d into one of the 5 possible types shown in the left part of table 4.

α	$\{4\}$	$abcd$	1
β	$\{3.1\}$	$abc\|d$	4
γ	$\{2^2\}$	$ab\|cd$	6
δ	$\{2.1^2\}$	$ab\|c\|d$	12
ε	$\{1^4\}$	$a\|b\|c\|d$	24

$\bar{\alpha}$	3
$\bar{\gamma}$	12

Table 4

Thus, the selection of a, b, c, abd, bcd corresponds to a partition $ac|b|d$ of type δ in which a and c are interchangeable but b and d are distinguishable from a and c and from each other. The selection of a, d, abd, acd corresponds to a partition $ad|bc$ of type γ which has 2 distinguishable pairs, and so on. The number of ways of obtaining a partition of each given type will be called its *frequency* and this is shown at the right of table 4, (e.g. for δ, it is $4!/[2!\,1!\,1!]$).

As soon as the double letter symbols are introduced, however, extra patterns arise. Two new distributions become significant, and these extend the above list. (1) The selection of 2 adjacent 'edges', such as ab and bd partitions the vertices into type δ (in this case, $ad|b|c$). But the selection of a pair of opposite edges such as ab and cd leaves all the vertices on exactly the same footing, while dividing them into 2 *in*distinguishable pairs. The frequency of this type of distribution is 3 and we shall call it $\bar{\alpha}$. [Note, for future reference, that it is equivalent to a linking of the vertices by a closed quadrilateral, such as ad, db, bc, ca since this automatically omits a pair of opposite edges, and vice versa.]

(2) The selection of 3 edges such as ab, ac, bc gives a partition of type β, and so does a group like ad, bd, cd. But the selection of 3 edges forming an open chain, such as ab, bc, cd introduces a new feature. It sorts the vertices into a γ pattern ($ad|bc$), but from every such partition of the vertices can be derived 2 different chains of edges, namely ab, bc, cd and ac, cb, bd, which both divide the vertices into the same 2 distinguishable pairs ($ad|bc$). This type of distribution will be referred to as $\bar{\gamma}$; its frequency is 12.

It will readily be found that these 7 cases are the only ones which can occur, no matter what selection of single, double and triple letter symbols is made.

[When two groups of double letter symbols are chosen independently, there is just one further case which can arise. This is not required for enumerating the functions of 4 arguments, but a similar feature will arise later with five-variable functions, and this is the best place to introduce the idea. It occurs only when 2 distributions of type $\bar{\alpha}$ are applied separately; e.g. if one pair of opposite edges of the tetrahedron is to be coloured red and another pair is to be notched. When the pairs chosen for these operations do not coincide, we obtain a quadrilateral with its sides alternately coloured red and notched. The vertices of the tetrahedron are still indistinguishable, but the distribution now has frequency 6 and will be called $\bar{\alpha}$.]

For each specimen type $[u_1, u_2, u_3]$ we now work out all the possible arrangements, classifying each according to its distribution type. Thus, for [1, 1, 1] we have the 6 essentially distinct possibilities shown below; (this total is given in column D of table 5).

$$\begin{array}{ccc|c} a & ab & abc & \varepsilon \\ a & ab & acd & \delta \\ a & ab & bcd & \delta \end{array} \qquad \begin{array}{ccc|c} a & bc & abc & \delta \\ a & bc & abd & \varepsilon \\ a & bc & bcd & \delta \end{array}$$

Now comes an invaluable check. The number of occurrences of 4δ and 2ε is equal to

$$4 \cdot 12 + 2 \cdot 24 = 96,$$

which is the correct value for $T\,[1, 1, 1]$. It is this possibility of continual checking which gives one confidence that no cases have been overlooked or duplicated.

				1	4	6	12	24	3	12		
M	$u_1\ u_2\ u_3$	T	MT	α	β	γ	δ	ε	$\bar{\alpha}$	$\bar{\gamma}$	D	MD
8	0 0 0	1	8	1							1	8
16	1 0 0	4	64		1						1	16
8	1 0 1	16	128		1		1				2	16
8	2 0 0	6	48			1					1	8
8	2 0 1	24	112				2				2	8
2	2 0 2	36	72			2		1			3	6
8	0 1 0	6	48			1					1	8
16	1 1 0	24	384				2				2	32
8	1 1 1	96	768				4	2			6	48
8	2 1 0	36	288			2		1			3	24
8	2 1 1	144	1152				4	4			8	64
2	2 1 2	216	512			4		8			12	24
8	0 2 0	15	120				1		1		2	16
16	1 2 0	60	960				3	1			4	64
8	1 2 1	240	1920				6	7			13	104
8	2 2 0	90	720			1	2	2		1	6	48
8	2 2 1	360	2880				6	12			18	144
2	2 2 2	540	1080			2	4	19		2	27	54
4	0 3 0	20	80		2					1	3	12
8	1 3 0	80	640		2		2	2			6	48
4	1 3 1	320	1280		2		6	10			18	72
4	2 3 0	120	480				4	2		2	8	32
4	2 3 1	480	1920				8	16			24	96
1	2 3 2	720	720				8	24		4	36	36
175				8	56	56	392	448	8	28		996
	Check $2^{14} = 16384$			8	224	336	4704	10752	24	336		

Table 5

Table 5 has been obtained systematically in this way. The number in the last column is the product of the numbers in the columns M and D, and the total of these products MD is the required number of functions satisfying POST's first condition, namely 996.

Various other cross-checks are possible, and a final check can be done as follows. Work out the total number of occurrences of each distribution type and put it at the foot of the column. Thus, for type γ, it is

$$8 \cdot 1 + 2 \cdot 2 + 8 \cdot 1 + 8 \cdot 2 + 2 \cdot 4 + 8 \cdot 1 + 2 \cdot 2 = 56.$$

The sum of these column totals, of course, is again 996. But if each column total is multiplied by the frequency of its distribution, and the results added, the total must now be 2^{14} or 16384, and this is also the sum of the products MT.

It remains to enumerate those functions, which, by their symmetry, fail to satisfy the second criterion. These 16 functions are typified by the ones in table 6, which shows just half of them, the other half being obtained by the reflexion process noted earlier when discussing ternary functions. (With these symmetric functions, of course, the right half is also a mirror image of the left, with 0 and 1 interchanged.)

a	*b*	*c*	*d*	*ab*	*ac*	*ad*	*bc*	*bd*	*cd*	*abc*	*abd*	*acd*	*bcd*
0	0	0	0	1	1	1	0	0	0	1	1	1	1
0	0	0	0	0	0	0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	0	0	0	1	1	1	0
1	0	0	0	1	1	0	1	0	0	1	1	1	0
1	0	0	0	1	0	0	1	1	0	1	1	1	0
1	0	0	0	0	0	0	1	1	1	1	1	1	0
1	1	0	0	1	1	1	0	0	0	1	1	0	0
1	1	0	0	0	0	0	1	1	1	1	1	0	0

Table 6

If the function is to be symmetric, the aggregate of the vertices, edges and faces omitted from the selection must be the dual of the figure formed by those chosen, and it will also be seen that the 3 edges which occur must either all pass through the same vertex or all lie in the same face, i.e. they must belong to one of the two β types and must not be of type $\bar{\gamma}$. Table 7 is a concise summary table for these functions, analogous to table 5 above, and with similar opportunities for checking.

				4	12		
M	$[u_1, u_2, u_3]$	T	MT	β	δ	D	MD
2	[0, 3, 4] or [4, 3, 0]	8	16	[illegible]		2	4
2	[1, 3, 3] or [3, 3, 1]	32	64	[illegible]	2	4	8
1	[2, 3, 2]	48	48		4	4	4
5				8	8		16
		Check $2^7 = 128$		32	96		

Table 7

Method 2

This method developed from the above procedure and will be easier to understand now that this first process has been explained.

A selection of any number of double letter symbols from 0 to 6 produces a distribution of one of the 7 types $\alpha, \ldots, \varepsilon, \bar{\alpha}, \bar{\gamma}$. Similarly, a selection of any number of the single letter symbols from 0 to 4 produces yet another such division (actually, only 3 types are possible here but that is not important at present) and, finally, a selection of any number of triple letter symbols produces a third distribution, and all these 3 choices can be made independently.

We define the *product* of 2 distributions as the aggregate of all the possible distributions which can result when the 4 letters are divided according to both distributions simultaneously but independently. (This concept is the crux of the present method.)

For example, to evaluate $\gamma\delta$ ($\equiv \delta\gamma$), the letters must be divided independently both as γ and as δ. Take for δ a typical partition such as $ab|c|d$ (in which a and b are indistinguishable). The γ partition involves making an additional selection of 2 letters, which then become distinguished from the other pair but not from each other.

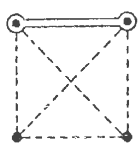
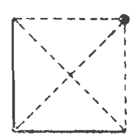
If we select $\left\{\begin{matrix} a, b \\ c, d \\ a, c \text{ or } b, c \\ a, d \text{ or } b, d \end{matrix}\right\}$ the resulting composite partition is of type $\left\{\begin{matrix} \delta \\ \delta \\ \varepsilon \\ \varepsilon \end{matrix}\right\}$.

The frequencies of the γ and δ partitions are 6 and 12 respectively, and so that of the product $\gamma\delta$ must be $6 \cdot 12 = 72$. But the sum of the frequencies of 2δ and 2ε is $2 \cdot 12 + 2 \cdot 24 = 72$, and this provides a valuable check on the accuracy of our classification. Obviously, this check will always work and, in fact, we can regard the symbolic formula

$$\gamma\delta = 2\delta + 2\varepsilon$$

not only as a rule of combination but also as an equation from which a true result can be obtained by substituting for each Greek letter the frequency of its distribution type: $\gamma = 6$, $\delta = 12$, $\varepsilon = 24$.

The labour of building up the multiplication table (table 8) can be lightened in many cases (at least it was for the writer!) by thinking geometrically; e.g. for the $\gamma\delta$ problem above, we can take γ and δ as, say,

γ  and  δ

and fit these tetrahedral patterns together in all possible ways.

The complete multiplication table for $\alpha, \ldots, \bar{\gamma}$ is given in table 8, though actually only the first 3 columns are needed for our present purpose. The first row and first column are also the headings, as multiplication by α leaves every partition unaltered.

1 α	4 β	6 γ	12 δ	24 ε	3 $\bar{\alpha}$	12 $\bar{\gamma}$
4 β	$\beta + \delta$	2δ	$2\delta + \varepsilon$	4ε	δ	2ε
6 γ	2δ	$2\gamma + \varepsilon$	$2\delta + 2\varepsilon$	6ε	$\gamma + \bar{\gamma}$	$2\varepsilon + 2\bar{\gamma}$
12 δ	$2\delta + \varepsilon$	$2\delta + 2\varepsilon$	$2\delta + 5\varepsilon$	12ε	$\delta + \varepsilon$	6ε
24 ε	4ε	6ε	12ε	24ε	3ε	12ε
3 $\bar{\alpha}$	δ	$\gamma + \bar{\gamma}$	$\delta + \varepsilon$	3ε	$\bar{\alpha} + \bar{\bar{\alpha}}$	$3\bar{\gamma}$
12 $\bar{\gamma}$	2ε	$2\varepsilon + 2\bar{\gamma}$	6ε	12ε	$3\bar{\gamma}$	$4\varepsilon + 4\bar{\gamma}$
			← Not needed for the present problem →			

Table 8

The checks already described go a long way, but do not completely eliminate the possibility of error, as it is still possible to have obtained δ where one should have had $\bar{\gamma}$, or to have got an ε instead of 2δ, and the checks so far applied would not have revealed these faults. When we realize, however, that this multiplication of distributions must by its nature be *associative*, we have discovered a final and very convincing method of checking, by working out from the table products like $\beta\gamma\delta$ in 3 different ways, and products like $\beta^2\gamma$ in 2 ways.

We now apply this information to our problem. We tabulate all the possible partition types which can occur when we make a selection of u_1 single letter symbols $(0 \leq u_1 \leq 4)$, or, dually, of u_1 triple letter symbols, and also the possible distributions which can arise when we take u_2 double letter symbols $(0 \leq u_2 \leq 6)$. The sum of the frequencies must, of course, be equal to $\binom{4}{u_1}$ or $\binom{6}{u_2}$ respectively.

Vertices or faces

u_1	0,4	1,3	2
	α	β	γ
$\binom{4}{u_1}$	1	4	6
$2\alpha + 2\beta + \gamma$			

Edges

u_2	0,6	1,5	2,4	3
	α	γ	$\delta + \bar{\alpha}$	$2\beta + \bar{\gamma}$
$\binom{6}{u}$	1	6	15	20
$2\alpha + 2\beta + 2\gamma + 2\delta + 2\bar{\alpha} + \bar{\gamma}$				

Table 9

The aggregate of all the possibilities is shown underneath each part of table 9. (The check sums here are 2^4 and 2^6 respectively.)

The grand total of possible arrangements, therefore, is the result of combining the 3 independent selections for vertices, edges and faces in all possible ways. Hence, it is obtained by expanding the following product, and this is done by using the multiplication table (table 8).

$$\begin{aligned}&(2\alpha + 2\beta + \gamma)^2 (2\alpha + 2\beta + 2\gamma + 2\delta + 2\bar{\alpha} + \bar{\gamma})\\ &= (2\alpha + 2\beta + \gamma)(4\alpha + 12\beta + 12\gamma + 36\delta + 16\varepsilon + 4\bar{\alpha} + 6\bar{\gamma})\\ &= \underline{8\alpha + 56\beta + 56\gamma + 392\delta + 448\varepsilon + 8\bar{\alpha} + 28\bar{\gamma}}.\end{aligned}$$

The sum of the coefficients in this expansion is the required total number of functions, namely 996, and, if yet one more check is needed, the value of the expression when the frequencies are substituted must be 2^{14}, as we observed in method 1.

The symmetric functions can also be enumerated by a similar process. As already noted, the 3 edges must have one of the two β patterns, and once the partition of the vertices has been chosen, that of the faces is automatically determined. Hence the symmetric functions are obtained from the expansion of

$$2\beta(2\alpha + 2\beta + \gamma) = \underline{8\beta + 8\delta},$$

and so there are 16 of them.

Connectives of 5 propositions

The same methods can be applied to the enumeration of the corresponding functions with 5 arguments. Only the second method will be used in this case, as it seems the more efficient.

There are now 5 single-letter symbols a, b, c, d, e, 10 two-letter symbols $ab, \ldots, de$, 10 three-letter and 5 four-letter symbols. Geometrically, the figure which presents itself is a pentatope in 4 dimensions with its vertices, edges, faces and cells, (though since we shall be mainly concerned with the vertices and edges, our attention can be restricted, if desired, to its plane projection, a pentagon with all 10 vertex-connectors inserted).

There are 7 possible partitions of the 5 vertices which can be produced by selections of one- and four-letter symbols, and these are allotted the reference symbols $\alpha, \ldots, \eta$, as shown in table 10.

α	{5}	$abcde$	1
β	{4.1}	$abcd\mid e$	5
γ	{3.2}	$abc\mid de$	10
δ	{3.1²}	$abc\mid d\mid e$	20
ε	{2².1}	$ab\mid cd\mid e$	30
ζ	{2.1³}	$ab\mid c\mid d\mid e$	60
η	{1⁵}	$a\mid b\mid c\mid d\mid e$	120

$\bar{\alpha}$	12
$\bar{\beta}$	15
$\bar{\varepsilon}$	60

$\bar{\bar{\beta}}$	30

Table 10

1 α	5 β	10 γ	20 δ	30 ε	60 ζ	120 η	12 $\bar{\alpha}$	15 $\bar{\beta}$	60 $\bar{\varepsilon}$
5 β	$\beta + \delta$	$\delta + \varepsilon$	$2\delta + \zeta$	$\varepsilon + 2\zeta$	$3\zeta + \eta$	5η	$\bar{\varepsilon}$	$\zeta + \bar{\beta}$	$2\eta + \bar{\varepsilon}$
10 γ	$\delta + \varepsilon$	$\gamma + \varepsilon + \zeta$	$\delta + 3\zeta$	$2\varepsilon + 2\zeta + \eta$	$4\zeta + 3\eta$	10η	$2\bar{\varepsilon}$	$\varepsilon + \zeta + \bar{\varepsilon}$	$4\eta + 2\bar{\varepsilon}$
20 δ	$2\delta + \zeta$	$\delta + 3\zeta$	$2\delta + 4\zeta + \eta$	$6\zeta + 2\eta$	$6\zeta + 7\eta$	20η	2η	$3\zeta + \eta$	10η
30 ε	$\varepsilon + 2\zeta$	$2\varepsilon + 2\zeta + \eta$	$6\zeta + 2\eta$	$2\varepsilon + 4\zeta + 5\eta$	$6\zeta + 12\eta$	30η	$2\eta + 2\bar{\varepsilon}$	$\varepsilon + 2\zeta + 2\eta + \bar{\varepsilon}$	$14\eta + 2\bar{\varepsilon}$
60 ζ	$3\zeta + \eta$	$4\zeta + 3\eta$	$6\zeta + 7\eta$	$6\zeta + 12\eta$	$6\zeta + 27\eta$	60η	6η	$3\zeta + 6\eta$	30η
120 η	5η	10η	20η	30η	60η	120η	12η	15η	60η
12 $\bar{\alpha}$	$\bar{\varepsilon}$	$2\bar{\varepsilon}$	2η	$2\eta + 2\bar{\varepsilon}$	6η	12η	$2\bar{\alpha} + 2\bar{\varepsilon}$	$3\bar{\varepsilon}$	$4\eta + 4\bar{\varepsilon}$
15 $\bar{\beta}$	$\zeta + \bar{\beta}$	$\varepsilon + \zeta + \bar{\varepsilon}$	$3\zeta + \eta$	$\varepsilon + 2\zeta + 2\eta + \bar{\varepsilon}$	$3\zeta + 6\eta$	15η	$3\bar{\varepsilon}$	$\zeta + \eta + \bar{\beta} + \bar{\bar{\beta}}$	$6\eta + 3\bar{\varepsilon}$
60 $\bar{\varepsilon}$	$2\eta + \bar{\varepsilon}$	$4\eta + 2\bar{\varepsilon}$	10η	$14\eta + 2\bar{\varepsilon}$	30η	60η	$4\eta + 4\bar{\varepsilon}$	$6\eta + 3\bar{\varepsilon}$	$28\eta + 4\bar{\varepsilon}$

Table 11

When two- and three-letter symbols are added, there are 3 extra distribution types:

$\bar{\alpha}$ (frequency $\frac{1}{2} \cdot 4!$) occurs when the vertices are joined by a continuous, closed, chain of 5 edges.

$\bar{\beta}$ (frequency 15) arises when there is one odd vertex and the other 4 are joined by a pair of opposite edges, (or, by what is completely equivalent, a continuous, closed, chain of 4 edges).

$\bar{\varepsilon}$ (frequency 60) occurs when the vertices are partitioned into type ε, but the 2 pairs can be linked in 2 ways; it is essentially our previous type $\bar{}$ with the addition of a fifth, distinguishable, vertex.

Examples of each of these types will be given presently; (see table 13). The type $\bar{\bar{\beta}}$ (frequency 30) is merely the type $\bar{\bar{\alpha}}$ for 4 vertices, (see above), with the addition of a fifth, distinguishable vertex; it arises only in one product, $\bar{\beta}^2$, (never in individual factors). So its multiplication table is not needed, (unless a check is desired for $\bar{\beta}^2$, using the associative law).

Vertices or cells

u_1	0,5	1,4	2,3
	α	β	γ
$\binom{5}{u_1}$	1	5	10
$x = 2(\alpha + \beta + \gamma)$			

Edges or faces

u_2	0,10	1,9	2,8	3,7	4,6	5
	α	γ	$\varepsilon + \bar{\beta}$	$\gamma + \delta + \varepsilon + \bar{\varepsilon}$	$\beta + \gamma + 2\zeta + \bar{\beta} + \bar{\varepsilon}$	$2\varepsilon + 2\zeta + \bar{\alpha} + \bar{\varepsilon}$
$\binom{10}{u_2}$	1	10	45	120	210	252
$y = 2\alpha + 2\beta + 6\gamma + 2\delta + 6\varepsilon + 6\zeta + \bar{\alpha} + 4\bar{\beta} + 5\bar{\varepsilon}$						

Table 12

This time, the multiplication table (table 11) must be constructed for all 10 distributions, $\alpha, \ldots, \eta, \bar{\alpha}, \bar{\beta}, \bar{\varepsilon}$, because now that edge and face symbols are both needed. any pairing of Greek letters may arise. The same checks as before can be applied.

The distribution types produced by selecting u_1 vertices (or cells) and u_2 edges (or faces) are given in table 12. The check sums for x and y are 2^5 and 2^{10} respectively. The way in which the results for $u_2 = 4$ and $u_2 = 5$ were obtained is illustrated by the series of sketches in table 13; the other cases will present no difficulty.

On this occasion, we shall consider the symmetric functions first, because any one selection of (say) vertices and edges uniquely fixes one symmetric function

Hence the number of symmetric functions is obtained by expanding xy and the total number of functions (after some very heavy arithmetic!) by expanding $(xy)^2$.

$$\begin{aligned} xy &= 2(\alpha + \beta + \gamma)(2\alpha + 2\beta + 6\gamma + 2\delta + 6\varepsilon + 6\zeta + \bar{\alpha} + 4\bar{\beta} + 5\bar{\varepsilon}) \\ &= \underline{4\alpha + 12\beta + 28\gamma + 36\delta + 84\varepsilon + 188\zeta + 120\eta + 2\bar{\alpha} + 16\bar{\beta} + 54\bar{\varepsilon}} \\ (xy)^2 &= \underline{16\alpha + 240\beta + 1008\gamma + 7440\delta + 31248\varepsilon + 659184\zeta + 8584768\eta} \\ &\qquad \underline{+ 24\bar{\alpha} + 768\bar{\beta} + 48360\bar{\varepsilon} + 256\bar{\bar{\beta}}}. \end{aligned}$$

So there is a total of 9333312 functions, of which 544 are symmetric, leaving 9332768 admissible.

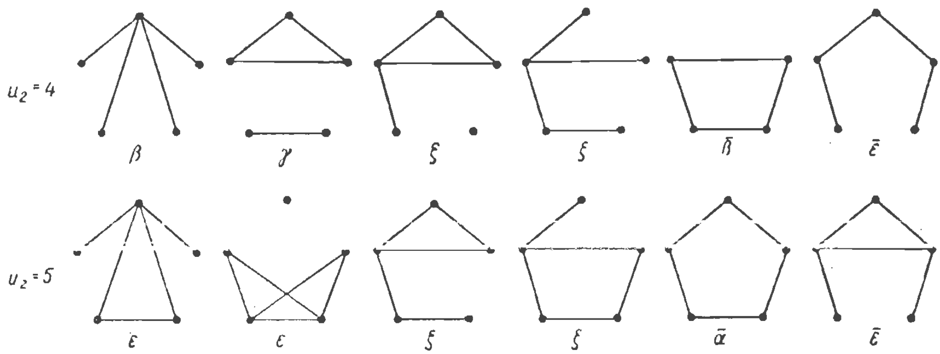


Table 13

Conclusion

This is as far as the writer has taken the problem at present. The functions of 5 variables required a much more complicated enumeration of cases and much more tedious algebra than those of 4, but they were not significantly more difficult. The duality

vertex $\leftrightarrow$ face, edge $\leftrightarrow$ edge,

was replaced by

vertex $\leftrightarrow$ cell, edge $\leftrightarrow$ face.

With 6 or 7 arguments, however, where a third dual relation will appear, it seems that the labour involved, even with the techniques above, will be considerable. It had been hoped that a general pattern for n arguments might have emerged by this time, but this hope does not seem to have been realized. [Nevertheless, the expressions for the total number of functions have one property which is perhaps worthy of a passing mention. With 4 variables, the coefficients of β, γ, δ, ε and $\bar{\gamma}$ are all divisible by 7, and with 5 variables, the coefficients of δ, ε, ζ, η and $\bar{\varepsilon}$ are all divisible by 31.] With this type of problem, however, it is quite likely that any formula (if there is a simple one) would take different forms according to whether n

is even or odd, reflecting the presence or absence of a self-dual element in the geometric picture, and, if this is so, our information so far would be rather scanty to hazzard any useful predictions.

The final table, table 14, gives a summary of the results obtained.

Number of propositions	Number of connectives satisfying Post's first condition	Number of these which fail to satisfy Post's second condition	Number of complete propositional connectives
1	1	1	0
2	3	1	2
3	20	4	16
4	996	16	980
5	9333312	544	9332768

Table 14

In conclusion, I wish to express my thanks to Professor R. L. Goodstein, who has kindly read the MS of this paper, as my interest in this problem was first stimulated by a lecture he gave in which Post's theorem was discussed.

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[1] E. L. Post, The 2-valued Iterative Systems of Mathematical Logic (1941).
[2] R. A. Cuninghame-Green, Single Primitive Ternary Connectives for the 2-valued Propositional Calculus. This Zeitschr. **5** (1959).

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