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553. SOME NUMBERS RELATED TO THE STIRLING NUMBERS
OF THE FIRST AND SECOND KIND*

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1. The STIRLING numbers of the first and second kind are defined by

$$(1.1) \quad x(x+1) \cdots (x+n-1) = \sum_{k=0}^n S_1(n, k) x^k$$

and

$$(1.2) \quad x^n = \sum_{k=0}^n S(n, k) x(x-1) \cdots (x-k+1),$$

respectively. They satisfy the recurrences

$$(1.3) \quad S_1(n+1, k) = S_1(n, k-1) + nS_1(n, k),$$

$$(1.4) \quad S(n+1, k) = S(n, k-1) + kS(n, k).$$

It is not difficult to show that $S_1(n, n-k)$ and $S(n, n-k)$ are polynomials in n of degree $2k$. Moreover

$$(1.5) \quad S_1(n, n-k) = \sum_{j=0}^{k-1} S_1'(k, j) \binom{n}{2k-j} \quad (k > 0)$$

and

$$(1.6) \quad S(n, n-k) = \sum S'(k, j) \binom{n}{2k-j} \quad (k > 0).$$

The coefficients $S_1'(k, j)$, $S'(k, j)$ were introduced by JORDAN [10, Ch. 4] and WARD [12]. We are using the notation of [6]. See also [13].

The Eulerian numbers $A_{n,k}$ are usually defined by ([3], [11, Ch. 8])

$$(1.7) \quad \frac{1-\lambda}{1-\lambda e^{x(1-\lambda)}} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k=1}^n A_{n,k} \lambda^k.$$

They satisfy the recurrence

$$(1.8) \quad A_{n+1, k} = (n-k+2) A_{n, k-1} + k A_{n, k}$$

as well as

$$(1.9) \quad x^n = \sum_{k=1}^n A_{n, k} \binom{x+k-1}{n}.$$

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The representation (1.9) suggests that it may be of interest to consider representations of this kind for $S_1(n, n-k)$ and $S(n, n-k)$. Some general results on such representations are discussed in some detail in [8] but will not be needed. We shall show that

$$(1.10) \quad S_1(n, n-k) = \sum_{j=1}^k B_1(k, j) \binom{n+j-1}{2k}$$

and

$$(1.11) \quad S(n, n-k) = \sum_{j=1}^k B(k, j) \binom{n+j-1}{2k}.$$

Moreover we find that

$$(1.12) \quad B_1(k, j) = B(k, k-j+1) \quad (1 \leq j \leq k).$$

We shall also show that

$$(1.13) \quad B_1(k, j) = a_{k,j},$$

where the $a_{k,j}$ are numbers that first occurred in an entirely different connection [4].

Combinatorial interpretations of the $S_1(n, k)$, $S(n, k)$, $S'_1(n, k)$, $S'(n, k)$ are known [6], [11, Ch. 4]. However no combinatorial interpretation of $B_1(n, k)$ or $B(n, k)$ is known.

2. Since $S_1(n, n-k)$ is a polynomial in n of degree $2k$, we may put

$$(2.1) \quad S_1(n, n-k) = \sum_{j=0}^{2k} B_1(k, j) \binom{n+j-1}{2k}.$$

However it follows from (1.5) that $B_1(k, 0) = 0$, $B_1(k, j) = 0$ ($k < j \leq 2k$). Thus (2.1) reduces to

$$(2.2) \quad S_1(n, n-k) = \sum_{j=1}^k B_1(k, j) \binom{n+j-1}{2k}.$$

By (1.3),

$$(2.3) \quad S_1(n+1, n-k+1) = S_1(n, n-k) + nS_1(n, n-k+1).$$

Thus

$$\begin{aligned} \sum_{j=1}^k B_1(k, j) \binom{n+j}{2k} - \sum_{j=1}^k B_1(k, j) \binom{n+j-1}{2k} &= n \sum_{j=1}^{k-1} B_1(k-1, j) \binom{n+j-1}{2k-2} \\ &= \sum_{j=1}^{k-1} B_1(k-1, j) \binom{n+j-1}{2k-2} ((n-2k+j+1) + (2k-j-1)) \\ &= \sum_{j=1}^{k-1} B_1(k-1, j) \left\{ (2k-1) \binom{n+j-1}{2k-1} \right. \\ &\quad \left. + (2k-j-1) \left(\binom{n+j}{2k-1} - \binom{n+j-1}{2k-1} \right) \right\}, \end{aligned}$$

so that

$$\sum_{j=1}^k B_1(k, j) \binom{n+j-1}{2k-1} = (2k-1) \sum_{j=1}^{k-1} B_1(k-1, j) \binom{n+j-1}{2k-1} + \sum_{j=2}^k (2k-j) B_1(k-1, j-1) \binom{n+j-1}{2k+1} - \sum_{j=1}^{k-1} (2k-j-1) B_1(k-1, j) \binom{n+j-1}{2k-1}.$$

It follows that

$$(2.4) \quad B_1(k, j) = (2k-j) B_1(k-1, j-1) + j B_1(k-1, j) \quad (1 \leq j \leq k).$$

Since $S_1(n, n-1) = \binom{n}{2}$, it is evident that

$$(2.5) \quad B_1(1, 1) = 1, \quad B_1(1, j) = 0 \quad (j > 1).$$

The numbers $B_1(k, j)$ are completely determined by (2.4) and (2.5).

The following table is easily computed.

$B_1(k, j)$:

$k \backslash j$	1	2	3	4	5
1	1				
2	1	2			
3	1	8	6		
4	1	22	58	24	
5	1	52	328	444	120

Turning next to $S(n, k)$, we have

$$(2.6) \quad S(n, n-k) = \sum_{j=1}^k B(k, j) \binom{n+j-1}{2k}.$$

By (1.4),

$$(2.7) \quad S(n+1, n-k+1) = S(n, n-k) + (n-k-1) S(n, n-k+1),$$

so that

$$\begin{aligned} \sum_{j=1}^k B(k, j) \binom{n+j}{2k} - \sum_{j=1}^k B(k, j) \binom{n+j-1}{2k} &= \sum_{j=1}^{k-1} B(k-1, j) \binom{n+j-1}{2k-2} ((n-2k+j+1) + (k-j)) \\ &= (2k-1) \sum_{j=1}^{k-1} B(k-1, j) \binom{n+j-1}{2k-1} \\ &\quad + \sum_{j=1}^{k-1} (k-j) B(k-1, j) \left(\binom{n+j}{2k-1} - \binom{n+j-1}{2k-1} \right). \end{aligned}$$

It follows that

$$(2.8) \quad B(k, j) = (k-j+1) B(k-1, j-1) + (k+j-1) B(k-1, j) \quad (1 \leq j \leq k).$$

Since $S(n, n-1) = \binom{n}{2}$ we have

$$(2.9) \quad B(1, 1) = 1, \quad B(1, j) = 0 \quad (j > 1).$$

The $B(k, j)$ are completely determined by (2.8) and (2.9).

3. In (2.8) replace j by $k-j+1$ and we get

$$(3.1) \quad B(k, k-j+1) = jB(k-1, k-j) + (2k-j)B(k-1, k-j+1) \quad (1 \leq j \leq k).$$

If we put $\bar{B}(k, j) = B(k, k-j+1)$, (3.1) becomes

$$(3.2) \quad \bar{B}(k, j) = j\bar{B}(k-1, j) + (2k-j)B(k-1, j-1) \quad (1 \leq j \leq k).$$

Also, by (2.9),

$$(3.3) \quad \bar{B}(1, 1) = 1, \quad \bar{B}(1, j) = 0 \quad (j > 1).$$

Therefore, comparison with (2.4) and (2.5) yields

$$(3.4) \quad B_1(k, j) = B(k, k-j+1) \quad (1 \leq j \leq k).$$

An explicit formula for $B(k, j)$ can be found in the following way. Multiply both sides of (2.6) by x^n and sum over n . Since

$$\begin{aligned} \sum_{n=k}^{+\infty} x^{n-k} \sum_{j=1}^k B(k, j) \binom{n+j-1}{2k} &= \sum_{j=1}^k B(k, j) x^{k-j+1} \sum_{n=k}^{+\infty} \binom{n+j-1}{2k} x^{n-2k+j-1} \\ &= \sum_{j=1}^k B(k, j) x^{k-j+1} \sum_{n=0}^{+\infty} \binom{n+2k}{2k} x^n = (1-x)^{-2k-1} \sum_{j=1}^k B(k, j) x^{k-j+1} \\ &= (1-x)^{-2k-1} \sum_{j=1}^k B(k, k-j+1) x^j \end{aligned}$$

we get

$$(3.5) \quad \sum_{j=1}^k B(k, k-j+1) x^j = (1-x)^{2k+1} \sum_{t=0}^{+\infty} S(t+k, t) x^t \quad (k > 0).$$

Comparison of coefficients gives

$$(3.6) \quad B(k, k-j+1) = \sum_{t=0}^j (-1)^{j-t} \binom{2k+1}{j-t} S(t+k, t)$$

and therefore

$$(3.7) \quad B(k, k-j+1) = \sum_{t=0}^j (-1)^{j-t} \binom{2k+1}{j-t} \frac{1}{t!} \sum_{s=0}^t (-1)^{t-s} \binom{t}{s} S_1^{t+k}.$$

Analogous to (3.6), we have also

$$(3.8) \quad B_1(k, k-j+1) = \sum_{t=0}^j (-1)^{j-t} \binom{2k+1}{j-t} S_1(t+k, t).$$

In view of (3.4) we may state

$$(3.9) \quad B(k, j) = \sum_{t=0}^j (-1)^{j-t} \binom{2k+1}{j+t} S_1(t+k, t).$$

By (1.11) and (3.9),

$$\begin{aligned} S(n, n-k) &= \sum_{j=1}^k \binom{n+j-1}{2k} \sum_{t=0}^j (-1)^{j-t} \binom{2k+1}{j-t} S_1(t+k, t) \\ &= \sum_{t=0}^k S_1(t+k, t) \sum_{j=t}^k (-1)^{j-t} \binom{2k+1}{j-t} \binom{n+j-1}{2k}. \end{aligned}$$

The inner sum on the extreme right can be summed by SAALCHÜTZ'S theorem [1, p. 9] and we find, after some manipulation, that

$$(3.10) \quad \sum_{j=1}^k (-1)^{j-t} \binom{2k+1}{j-t} \binom{n+j-1}{2k} = \binom{k+n}{k-t} \binom{k-n}{k+t}.$$

Therefore we have

$$(3.11) \quad S(n, n-k) = \sum_{t=0}^k \binom{k+n}{k-t} \binom{k-n}{k+t} S_1(t+k, t).$$

To express $S_1(n, k)$ in terms of $S(n, k)$, we take

$$\begin{aligned} S_1(n, n-k) &= \sum_{j=1}^k B_1(k, j) \binom{n+j-1}{2k} = \sum_{j=1}^k B(k, k-j+1) \binom{n+j-1}{2k} \\ &= \sum_{j=1}^k \binom{n+j-1}{2k} \sum_{t=0}^j (-1)^{j-t} \binom{2k+1}{j-t} S(t+k, t) \\ &= \sum_{t=0}^k S(t+k, t) \sum_{j=t}^k (-1)^{j-t} \binom{2k+1}{j-t} \binom{n+j-1}{2k}. \end{aligned}$$

Hence, by (3.10),

$$(3.12) \quad S_1(n, n-k) = \sum_{t=0}^k \binom{k+n}{k-t} \binom{k-n}{k+t} S(t+k, t).$$

Note that the coefficients on the right of (3.11) and (3.12) are the same. The formulas (3.11) and (3.12) are not new; for references see [7] and [9].

4. While we have obtained an explicit formula for $B(k, j)$ in (3.7), we have been unable to find a simple generating function. However the following results are of some interest in this connection.

Put

$$(4.1) \quad e^{nz} = \sum_{r=0}^n \frac{(nz)^r}{r!} + \frac{(nz)^n}{n!} S_n(z),$$

where n is a positive integer and z is an arbitrary complex number. BUCKHOLTZ [2] proved that, for $k \geq 1$,

$$S_n(z) = \sum_{r=0}^{k-1} n^{-r} U_r(z) + O(n^{-k}),$$

uniformly in a certain region of the z -plane. The coefficients $U_r(z)$ are determined by

$$(4.2) \quad U_r(z) = (-1)^r \left(\frac{z}{1-z} \frac{d}{dz} \right)^r \frac{z}{1-z}.$$

It follows from (4.2) that

$$(4.3) \quad U_r(z) = (-1)^r (1-z)^{-2r-1} Q_r(z),$$

where, for $r \geq 1$, $Q_r(z)$ is a polynomial of degree r with positive integral coefficients.

The writer [4] showed that, if we put

$$(4.4) \quad Q_k(z) = \sum_{j=1}^k a_{kj} z^j \quad (k \geq 1),$$

then the a_{kj} satisfy the recurrence

$$(4.5) \quad a_{kj} = ja_{k-1,j} + (2k-j)a_{k-1,j-1} \quad (1 \leq j \leq k).$$

Moreover

$$(4.6) \quad a_{11} = 1, \quad a_{1j} = 0 \quad (j > 1).$$

Therefore, comparing (4.5) with (3.2), we have at once

$$(4.7) \quad a_{kj} = B_1(k, j) = B(k, k-j+1).$$

Put $f_j(x) = \sum_{k=j}^{\infty} a_{kj} x^k$ ($j = 1, 2, \dots$). Then, by (4.5),

$$f_j(x) = \sum_{k=j}^{\infty} (ja_{k-1,j} + (2k-j)a_{k-1,j-1}) x^k$$

which gives

$$(4.8) \quad (1-jx)f_j(x) = x \left(2x \frac{d}{dx} - j + 2 \right) f_{j-1}(x) \quad (j > 1).$$

For example, since $f_1(x) = x/(1-x)$, $f_2(x) = \frac{2x^2}{(1-x)^2(1-2x)}$, so that

$$(4.9) \quad a_{k2} = 2^{k+1} - 2k - 2 \quad (k = 0, 1, 2, \dots).$$

Generally, it follows from (4.8) that

$$(4.10) \quad f_j(x) = \frac{x^j P_j(x)}{(1-x)^j (1-2x)^{j-1} \dots (1-jx)} \quad (j = 1, 2, \dots),$$

where $P_j(x)$ is a polynomial in x of degree $< \frac{1}{2}j(j-1)$, $j > 1$, that satisfies the recurrence

$$(4.11) \quad P_j(x) = (1-x) \dots (1-(j-1)x) \left\{ 2x \frac{d}{dx} + 2 \sum_{s=1}^{j-1} \frac{s(j-s)x}{1-sx} + j \right\} P_{j-1}(x) \quad (j > 1).$$

For example, since $P_1(x) = 1$, we have

$$P_2(x) = (1-x) \left(2x \frac{d}{dx} + \frac{2x}{1-x} + 2 \right) \cdot 1 = 2,$$

$$P_3(x) = (1-x)(1-2x) \left(2x \frac{d}{dx} + \frac{4x}{1-x} + \frac{4x}{1-2x} + 3 \right) \cdot 2 = 6 - 2x - 6x^2.$$

It follows from (4.10) or otherwise that

$$(4.12) \quad a_{kj} = \sum_{s=0}^j \Phi_{j,s}(k) (j-s)^k,$$

where $\Phi_{j,s}(k)$ is a polynomial in k of degree s . Indeed, by (3.7),

$$\begin{aligned} B(k, k-j+1) &= \sum_{s=0}^j s^k \sum_{t=s}^j (-1)^{j-t} \binom{2k+1}{j-t} \binom{t}{s} \frac{s^t}{t!} \\ &= \sum_{s=0}^j (j-s)^k \sum_{t=j-s}^j (-1)^{j-t} \binom{2k+1}{j-t} \binom{t}{s} \frac{s^t}{t!}, \end{aligned}$$

so that

$$(4.13) \quad \Phi_{j,s}(k) = \sum_{t=0}^s (-1)^t \binom{2k+1}{t} \binom{j-t}{s} \frac{s^t}{(j-t)!}.$$

Using (3.7) we get

$$\begin{aligned} a_{k2} &= B(k, k-1) = 2^{k+1} - 2k - 2, \\ a_{k3} &= B(k, k-2) = \frac{1}{2} 3^{k+2} - (2k+2) 2^{k+1} + \frac{1}{2} + \binom{2k+2}{3}. \end{aligned}$$

REFERENCES

1. W. N. BAILEY: *Generalized Hypergeometric Series*. Cambridge, 1935.
2. J. D. BUCKHOLTZ: *Concerning an approximation of Copson*. Proc. Amer. Math. Soc. 14 (1963), 564-568.
3. L. CARLITZ: *Eulerian numbers and polynomials*. Math. Mag. 33 (1959), 47-60.
4. L. CARLITZ: *The coefficients in an asymptotic expansion*. Proc. Amer. Math. Soc. 16 (1965), 248-252.
5. L. CARLITZ: *The coefficients in an asymptotic expansion and certain related numbers*. Duke Math. J. 35 (1968), 83-90.
6. L. CARLITZ: *Note on the numbers of Jordan and Ward*. Duke Math. J. 38 (1971), 783-790.
7. L. CARLITZ: *Note on Nörlund's polynomial $B_n(z)$* . Proc. Amer. Math. Soc. 11 (1960), 452-455.
8. L. CARLITZ: *Polynomial representations and compositions I*. Houston J. Math. 1 (1975), 23-45.
9. H. W. GOULD: *Stirling number representation problems*. Proc. Amer. Math. Soc. 11 (1960), 447-451.
10. C. JORDAN: *Calculus of Finite Differences*. New York, 1947.
11. J. RIORDAN: *An Introduction to Combinatorial Analysis*. New York, 1958.
12. M. WARD: *The representation of Stirling's numbers and Stirling's polynomials as sums of factorials*. Amer. J. Math. 56 (1934), 87-95.
13. M. ŽIVKOVIĆ: *On a representation of Stirling's numbers of the first kind*. These Publications, № 498 — № 541 (1975), 217-221.