of the following sketch. Assume that $\dot{F}_{N-1} = O(\epsilon^N)$. Now we have

$$\dot{F}_{N-1} = \frac{\partial F_{N-1}}{\partial x}\dot{x} + \ldots + \frac{\partial F_{N-1}}{\partial w}\dot{w},$$

and if we substitute for $\dot{x}, \ldots, \dot{w}$ from the differential equations we have

$$\dot{F}_{N-1} = \sum_0^r \epsilon^n \phi_n(x, y, z, w).$$

The ϕ_n with $n < N$ must be identically 0, and we have

$$\dot{F}_{N-1} = \epsilon^N \phi_N(H, K, L, M) + O(\epsilon^{N+1}),$$

$$\phi_N = \sum c_{pqrs} H^p \ldots M^s.$$

Now, since λ_2/λ_1 is irrational, ϕ_N is of the form

$$\phi_N = P(r, \rho) - \dot{f}_N(H, \ldots, M).$$

If we can prove $P = 0$ we arrive at $\dot{F}_N = O(\epsilon^{N+1})$ and have made the step from N to $N+1$.

The proof that $P = 0$ calls for the argument of § 13, which remains a key proof. But § 8 (about the x_m, etc.) would disappear, and only § 13 of §§ 9–15 is ultimately needed.

From § 16 on we are aiming at the long time T of Theorem 4. The proofs here are inevitably rather complicated whatever the approach, but the new one would achieve at least some simplification.

I have now carried the results farther, to the point of finding explicit asymptotic equations to the orbits (as opposed to a mere pair of 'integrals' connecting the original $x, y, \dot{x}, \dot{y}$), valid over the time T. The proofs use the new approach, and would be accessible in due course to a reader who wishes to avoid being immersed too much in the detail of the present paper.]

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CONSTRUCTION AND APPLICATION OF A CLASS OF MODULAR FUNCTIONS (II)†

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Introduction

In this paper we continue our discussion of the class G_n of entire modular functions on $\Gamma_0(n)$ introduced in (**3**). To avoid certain inessential complications arising from the transformation formula for the Dedekind η function, only those n relatively prime to 6 were considered there. Here we remove this restriction and give a different version of Theorem 1 of (**3**) valid for arbitrary n. With suitable modifications of a trivial character all the results of (**3**) remain valid.

As in (**3**), let F_n denote the totality of entire modular functions on $\Gamma_0(n)$ and E_n the subclass of F_n consisting of those functions of F_n with non-negative valence at all the parabolic points of the fundamental region [illegible] of $\Gamma_0(n)$, other than $\tau = i\infty$. H_n is the subclass of G_n [illegible] defined. Thus $H_n \subset E_n$. The valence of a function of E_n will always mean its valence at $\tau = i\infty$. A polynomial basis consisting of functions whose Fourier expansions at $\tau = i\infty$ in the uniformizing variable $x = \exp 2\pi i\tau$ possess integral coefficients will be called an integral polynomial basis. We will also construct here integral polynomial bases for [illegible] 10, 14, 15, 21, 22, 26, and give modular equations of [illegible]. In addition we study in detail the functions of H_n for [illegible].

We assume familiarity with the contents of (**3**), although much of this paper is self-contained. In many cases details of proofs or calculations which closely parallel corresponding discussions in (**3**) are omitted. The meaning of terms or symbols undefined here will be found in (**3**). Formulae of (**3**) will be referred to as (3. i).

The modular functions g

We must first prove suitable analogues of Lemmas 1 and 2 of (**3**). Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \qquad (c > 0).$$

Define $$\alpha(A) = s(a, c) - (a+d)/12c, \qquad \bar{A} = \begin{pmatrix} c & d \\ a & b \end{pmatrix},$$

so that $\bar{A}$ is also an element of Γ.

† The preparation of this paper was supported in part by the Office of Naval Research.

[illegible] *that* [illegible] *Then*

[illegible] $= -1/4$.

[illegible] $12c$ [illegible] $12a$.

[illegible] reciprocity formula for the Dedekind sum

[illegible] $1/4$ [illegible] $12c$ [illegible] $12a$ [illegible] $12ac$.

[illegible] we deduce

[illegible] *that* $a > 0$, $c > 0$. *Then if* $(c, 6) = 1$

[illegible] (mod 2) (2)

[illegible] (mod 2) (3)

[illegible] proof.

[illegible] Γ and [illegible] a non-zero integer then Δ_m will denote the [illegible] $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of Δ such that $(a, m) = 1$.

[illegible] *that* [illegible] *in* Δ, *where* $(t, m) = 1$. *Then* Δ [illegible] *of* Δ_m. [illegible] *function invariant* [illegible] *respect* [illegible] *the* [illegible] *a function on* Δ.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta$. Then

$$M = \begin{pmatrix} 1 & tx \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + xtc & * \\ * & * \end{pmatrix}$$

[illegible] 1, x can be determined so that $(a - xtc, m) = 1$ [illegible] this is a consequence of Dirichlet's theorem [illegible]. For [illegible] in Δ_m. Thus $M = S^{tx} M_0$ is a product of elements of [illegible]

[illegible] for every n we conclude that $\Gamma_0(n)$ may be generated by [illegible] $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $(a, 6) = 1$. Thus it is only necessary to show [illegible] invariance of a function with respect to these substitutions to show its [illegible] for $\Gamma_0(n)$. Also, it suffices to consider only those substitutions [illegible] both a and nc are positive.

[illegible] now prove our generalization of Theorem 1 of (**3**). It turns out [illegible] general an additional requirement is necessary which becomes [illegible] when $(n, 6) = 1$. The conditions imposed on the sequence $\{r_\delta\}$

below are just those which imply that the functions in question have integral valence at all the parabolic points of Q_n. As in (**3**), we put

$$\phi_\delta = \phi_\delta(\tau) = \eta(\delta\tau)/\eta(\tau).$$

If δ is a divisor of n, then δ' will denote the conjugate divisor; i.e. $\delta\delta' = n$.

THEOREM 1. *Suppose that $n > 1$ and that $\{r_\delta\}$ is a sequence of integers indexed by the positive divisors δ of n such that $r_1 = 0$,*

$$v(i\infty) = \tfrac{1}{24} \sum_{\delta | n} (\delta - 1) r_\delta \text{ is an integer,} \tag{4}$$

$$v(0) = \tfrac{1}{24} \sum_{\delta | n} (\delta' - n) r_\delta \text{ is an integer,} \tag{5}$$

$$\prod_{\delta | n} \delta^{r_\delta} \text{ is a rational square.} \tag{6}$$

Then the function

$$g = g(\tau) = \prod_{\delta | n} \phi_\delta^{r_\delta}$$

is a function on $\Gamma_0(n)$.

Proof. We need only show the invariance of g for those substitutions $A = \begin{pmatrix} a & b \\ nc & d \end{pmatrix}$ of $\Gamma_0(n)$ for which $(a, 6) = 1$ and a and nc are both positive, in view of the remarks following Lemma 3. As in (**3**), we find that

$$g(A\tau) = \exp(-i\pi\lambda)\, g(\tau),$$

where

$$\lambda = \sum_{\delta | n} \{\chi(A_\delta) - \chi(A)\} r_\delta$$

and

$$A_\delta = \begin{pmatrix} a & \delta b \\ \delta' c & d \end{pmatrix}.$$

Formula (3) of Lemma 2 now implies that

$$\lambda \equiv 2ab\, v(i\infty) - 2ac\, v(0) + \tfrac{1}{2} \sum_{\delta | n} \left\{1 - \left(\frac{\delta}{a}\right)\right\} r_\delta$$

$$\equiv \tfrac{1}{2} \sum_{\delta | n} \left\{1 - \left(\frac{\delta}{a}\right)\right\} r_\delta \pmod{2},$$

by (4) and (5). Thus

$$\exp(-i\pi\lambda) = \prod_{\delta | n} \left(\frac{\delta}{a}\right)^{r_\delta}$$

which is 1, by (6). The theorem is thus proved.

Let $\nu(n)$ denote the genus of $\Gamma_0(n)$. It is known (see (**2**)) that $\nu(n)$ is given by

$$\nu(n) = 1 + \tfrac{1}{12}\psi(n) - \tfrac{1}{4}\lambda(n) - \tfrac{1}{3}\mu(n) - \tfrac{1}{2}\xi(n), \tag{7}$$

where

$$\psi(n) = n \prod_{p | n} \left(1 + \frac{1}{p}\right), \qquad \xi(n) = \sum_{d | n} \phi((d, d')),$$

the number of incongruent solutions modulo n of $x^2+1 \equiv 0$ and ρ [illegible] is the number of incongruent solutions modulo n of

$$x^2-x+1 \equiv 0 \pmod{n}.$$

From (7) it is quite easy to obtain the following expressions for $\nu(2q)$, $\nu(3q)$, where q is a prime:

$$\nu(2q) = \left[\frac{q-3}{4}\right] \qquad (q \neq 2),$$

$$\nu(3q) = \left[\frac{q-2}{3}\right].$$

We wish to study in detail the functions of G_n for $n = pq$, where p is 2 or 3 and q is a prime [illegible] p. Put $Q = \frac{1}{2}(q-1)$. Formulae (3.21) and (3.22) apply and we have that

$$\left.\begin{aligned} r(\infty) &= a\alpha-b\beta-c\gamma \\ r(0) &= a\alpha-b\beta-c\gamma \\ r(1/p) &= a\alpha-b\beta-c\gamma \\ r(1/q) &= a\alpha-b\beta-c\gamma \end{aligned}\right\} \qquad (8)$$

$$\left.\begin{aligned} &= A\alpha-B\beta-C\gamma \\ &= A\alpha-B\beta-C\gamma \\ &= -A\alpha-B\beta-C\gamma \end{aligned}\right\}. \qquad (9)$$

For $p = 2$:

$$a = \tfrac{1}{12}AQ, \qquad b = \tfrac{1}{4}BQ, \qquad c = \tfrac{1}{12}C(Q-1),$$

where ABC is 6 or 12 and A, B, C are given in the following table:

Q (mod 12)	A	B	C		
0	1	1	12		
2	3	1	2	$\alpha+\beta+\gamma$ even	
3	2	2	3	$\alpha+\beta$ even	
5	6	2	1	$\alpha+\beta+\gamma$ even	(10)
6	1	1	6	$\alpha+\beta+\gamma$ even	
8	3	1	4		
9	2	2	3	$\alpha+\beta+\gamma$ even	
11	6	2	1	$\alpha+\beta$ even	

Notice that $q = 3$ is not included in this table, but will be given separately.

For $p = 3$, a, b, c are given by

$$a = \tfrac{1}{6}AQ, \qquad b = \tfrac{1}{3}BQ, \qquad c = \tfrac{1}{6}C(Q+1).$$

where ABC is 6 or 18 and A, B, C are as follows:

Q (mod 6)	A	B	C	
0	1	1	6	
2	3	3	2	(11)
3	2	1	3	
5	6	3	1	

Finally for $n = 6$, we have that $\alpha+\beta+\gamma$ is even and that

$$a = b = c = \tfrac{1}{2}, \qquad A = 6, \qquad B = 2, \qquad C = 3.$$

We are primarily interested in the subclass H_n of G_n. The functions of H_n are characterized by $r(0) \geqslant 0$, $r(1/p) \geqslant 0$, $r(1/q) \geqslant 0$; or what is the same thing, if we put $r = -r(i\infty)$, by

$$0 \leqslant \alpha \leqslant \frac{r}{2a}, \qquad 0 \leqslant \beta \leqslant \frac{r}{2b}, \qquad 0 \leqslant \gamma \leqslant \frac{r}{2c}.$$

The functions of H_n for which $-r(i\infty)$ is least positive will be called minimal functions. It is of interest to determine these functions. Let $[x, y]$ denote the least common multiple of x and y.

We have the lemma

Lemma 4. *Let a, b, c be positive integers. Let m be the least positive integer such that*

$$a\alpha+b\beta+c\gamma = m, \qquad 0 \leqslant \alpha \leqslant \frac{m}{2a}, \qquad 0 \leqslant \beta \leqslant \frac{m}{2b}, \qquad 0 \leqslant \gamma \leqslant \frac{m}{2c}. \qquad (12)$$

Put $m_0 = 2\min\{[a,b], [a,c], [b,c]\}$. Then $m \leqslant m_0$, and if $m \neq m_0$,

$$m \geqslant a+b+c.$$

If a, b, c form the sides of a triangle then $m = \min\{m_0, a+b+c\}$.

Proof. We show first that m_0 is the least positive solution of (12) for which $\alpha\beta\gamma = 0$. Suppose $\gamma = 0$. Then we seek the least positive integer m_1 such that $a\alpha+b\beta = m_1$, $0 \leqslant a\alpha \leqslant \frac{1}{2}m_1$, $0 \leqslant b\beta \leqslant \frac{1}{2}m_1$. This implies that $a\alpha = b\beta = \frac{1}{2}m_1$ and so the least permissible m_1 is $2[a,b]$ with $\alpha = [a,b]/a$, $\beta = [a,b]/b$. A similar conclusion holds if $\beta = 0$ or $\alpha = 0$, and so m_0 has the stated property. Any other solution m involves $\alpha\beta\gamma \neq 0$, implying $m \geqslant a+b+c$. The first part of the lemma is therefore proved. Further if a, b, c form the sides of a triangle then $m = a+b+c$ is a solution of (12) with $\alpha = \beta = \gamma = 1$. From this the second part of the lemma follows directly.

The precise behaviour of m is rather recondite, but can certainly be determined from Lemma 6 of (3) or directly from (12). This has been done by K. Goldberg. We content ourselves with the lemma in its present form however, which is sufficient to determine the minimal functions of H_n [illegible] ere p is 2 or 3 and q is a prime $> p$. The results are un- and are summarized in the following theorem.

ppose $n = pq$ where p is 2 or 3 and q is a prime $> p$. 1 (mod $2p$), and p when $q \equiv -1$ (mod $2p$) and put and 1 when $p = 3$. If $n = 15$ then

$$(\tau)\eta(15\tau)^5$$

nction of H_{15}. If $n \neq 15$ then

$$\eta(\tau)^{e''}\eta(pq\tau)^{epq}$$

onal function of H_n.

e give the proof for just one case only. $3q$, where $q = 12r + 11$ is prime. Then from table (11) $a = 6r+5$, $b = 6r+5$, $c = r+1$. Thus sides of a triangle and the second part of Lemma 4 applies. 0 with $\alpha = 1, \beta = 1, \gamma = 0$.

the genus developed earlier gives $v(n) = 4r+3$, and $3r+1$. Also, the function of valence $-12r-10$ becomes

$$(3\tau)^9\eta(q\tau)^3\eta(\tau)^3\eta(3q\tau)^9.$$

b, c which occur do not always form the sides of a triangle, d discussions (of no difficulty) are necessary.

ate value of the valence of a non-constant function of E_n $\Gamma_0(n)$, the results for $n = 15$ and for $q \equiv 1$ (mod $2p$) 3 lends force to the author's conjecture that a poly- the functions of E_n may always be determined by choosing combinations of functions of H_n. In (**3**) integral E and E_{35} were determined; and here we determine E_n for those values of $n \leq 26$ which are products of s. This leaves untreated the cases $n = 33, 34$. Although difficulties are involved, the computations are rather we do not give them here. As in (**3**), the notation for chosen so that subscripts correspond to $-v(i\infty)$.

ases and modular equations

st few functions of low negative valence are given in the following table

	$v(0)$	$v(\frac{1}{2})$	$v(\frac{1}{3})$	$v(i\infty)$	r_2	r_3	r_6
1	1	0	0	-1	-1	1	-5
1	0	0	1	-1	3	9	-9
0	0	1	0	-1	8	4	-8
1	0	1	1	-2	11	13	-17
1	1	1	0	2	7	5	-13
2	1	0	1	2	2	10	-14

Since $\Gamma_0(6)$ is of genus 0, any one of the functions A_1, B_1, C_1 is a basis for E_6. These functions must be linearly related in pairs while any two of A_2, B_2, C_2 and one of A_1, B_1, C_1 must be linearly related, by a simple application of Liouville's theorem. In this way we find the modular equations for $\eta(\tau)$ of level 6

$$a_1 = A_1-5 = B_1-3 = C_1-4 = B_2-C_2-5 = \tfrac{1}{3}(A_2-C_2)-3 = x^{-1}+6x+\ldots. \tag{13}$$

For the functions themselves,

$$A_1 = \eta(\tau)^5\eta(3\tau)/\eta(2\tau)\eta(6\tau)^5$$
$$B_1 = \eta(2\tau)^3\eta(3\tau)^9/\eta(\tau)^3\eta(6\tau)^9$$
$$C_1 = \eta(2\tau)^8\eta(3\tau)^4/\eta(\tau)^4\eta(6\tau)^8$$
$$A_2 = \eta(2\tau)^{11}\eta(3\tau)^{13}/\eta(\tau)^7\eta(6\tau)^{17}$$
$$B_2 = \eta(\tau)\eta(2\tau)^7\eta(3\tau)^5/\eta(6\tau)^{13}$$
$$C_2 = \eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^{10}/\eta(6\tau)^{14}.$$

If we notice in addition that

$$a_3 = A_1B_1C_1 = \eta(2\tau)^{10}\eta(3\tau)^{14}/\eta(\tau)^2\eta(6\tau)^{22}$$

is a function with positive valence at the parabolic points $0, \frac{1}{2}, \frac{1}{3}$ (in fact with valence 1 at each of these points) we can say

THEOREM 3. *$\{a_1\}$, $\{a_1, a_3^{-1}\}$ are integral polynomial bases for E_6, F_6 respectively.*

Of course, a_3 is expressible as a polynomial in a_1 of degree 3. Explicitly,

$$a_3 = a_1^3+2a_1^2-23a_1-60 = (a_1+3)(a_1+4)(a_1-5).$$

This is derivable from relations (13).

The reasoning involved in choosing a function with positive valence at all parabolic points other than $\tau = i\infty$ is explained in (**3**) 343.

$n = 10$. Here we find the table

	α	β	γ	$v(0)$	$v(\frac{1}{2})$	$v(\frac{1}{5})$	$v(i\infty)$	r_2	r_5	r_{10}
A_1	0	1	1	1	0	0	-1	-1	1	-3
B_1	1	0	1	0	0	1	-1	1	5	-5
C_1	1	1	0	0	1	0	-1	4	2	4

with

$$A_1 = \eta(\tau)^3\eta(5\tau)/\eta(2\tau)\eta(10\tau)^3$$
$$B_1 = \eta(2\tau)\eta(5\tau)^5/\eta(\tau)\eta(10\tau)^5$$
$$C_1 = \eta(2\tau)^4\eta(5\tau)^2/\eta(\tau)^2\eta(10\tau)^4.$$

Once again the genus is 0, so any one of the functions A_1, B_1, C_1 is a basis for E_{10}. We find as before the modular equations for $\eta(\tau)$ of level 10

$$a_1 = A_1+3 = B_1-1 = C_1-2 = x^{-1}+x+\ldots. \tag{14}$$

$$r_1 = A_1 B_1 C_1 = \eta(2\tau)^4\eta(5\tau)^8/\eta(10\tau)^{12}$$

...e 1 at $\tau = 0, \frac{1}{2}, \frac{1}{5}$ and so will serve as the additional function ...ary to go over to a basis for F_{10}. Thus

...HEOREM 4. $\{e_{10}\}$, $\{a_1, a_3^{-1}\}$ *are integral polynomial bases for* E_{10}, F_{10} ...*ctively.*

...he polynomial identity for a_3 in terms of a_1 is

$$a_3 = a_1^3 - 7a_1 - 6 = (a_1+1)(a_1+2)(a_1-3),$$

...h is derivable from equations (14).

...14. Since $\Gamma_0(14)$ is of genus 1, the situation is somewhat more com-...x. Here we find

β	γ	$v(0)$	$v(\frac{1}{2})$	$v(\frac{1}{7})$	$v(i\infty)$	r_2	r_7	r_{14}
0	1	0	0	2	−2	1	7	−7
1	2	3	0	1	−4	−2	6	−10
1	1	1	2	1	−4	5	7	−11

$$A_2 = \eta(2\tau)\eta(7\tau)^7/\eta(\tau)\eta(14\tau)^7$$

$$A_4 = \eta(\tau)^6\eta(7\tau)^6/\eta(2\tau)^2\eta(14\tau)^{10}$$

$$B_4 = \eta(2\tau)^5\eta(7\tau)^7/\eta(\tau)\eta(14\tau)^{11}.$$

...we lack ...function of valence −3 which we can obtain by choosing ...riate linear combinations of A_2, A_4, B_4. We find

$$a_2 = A_2 - 1 = x^{-2} + x^{-1} + 2x + \ldots,$$

$$a_3 = \tfrac{1}{2}(B_4 - A_4) + 2A_2 - 3 = x^{-3} + 2x^{-1} + 5x + \ldots .$$

...integrality of the coefficients of a_3 may be checked by means of ...7 of (3) and so we obtain

...OREM 5. $\{a_2, a_3\}$, $\{a_2, a_3, B_4^{-1}\}$ *are integral polynomial bases for* E_{14}, ...*ectively.*

...can obtain a modular equation of level 14 for $\eta(\tau)$ by expressing B_4 ...terms of the basis elements a_2, a_3. We find

$$B_4 = a_2^2 - 4a_2 - a_3 - 8.$$

...o, we must have for appropriate constants r_i

$$a_3^2 = r_0 a_2^3 + r_1 a_2 a_3 + r_2 a_2^2 + r_3 a_3 + r_4 a_2 + r_5.$$

...mputation yields

$$a_3^2 = a_2^3 - 3a_2 a_3 + 4a_2^2 - 9a_3 + 4a_2 - 8.$$

$n = 15$. The first few functions are

$$A_2 = \eta(3\tau)\eta(5\tau)^5/\eta(\tau)\eta(15\tau)^5$$

$$A_4 = \eta(\tau)^7\eta(5\tau)/\eta(3\tau)\eta(15\tau)^7$$

$$B_4 = \eta(\tau)^2\eta(3\tau)^4\eta(5\tau)^2/\eta(15\tau)^8$$

$$C_4 = \eta(3\tau)^9\eta(5\tau)^3/\eta(\tau)^3\eta(15\tau)^9$$

with

	α	β	γ	$v(0)$	$v(\frac{1}{3})$	$v(\frac{1}{5})$	$v(i\infty)$	r_3	r_5	r_{15}
A_2	1	0	1	0	0	2	−2	1	5	−5
A_4	0	1	2	4	0	0	−4	−1	1	−7
B_4	1	1	1	2	2	0	−4	4	2	−8
C_4	2	1	0	0	4	0	−4	9	3	−9

$\Gamma_0(15)$ is also of genus 1, and the discussion is similar to that for $n = 14$. We have

$$a_2 = A_2 - 2 = x^{-2} + x^{-1} + 2x + \ldots,$$

$$a_3 = \tfrac{1}{5}(B_4 - A_4) + 3A_2 - 19 = x^{-3} + x^{-1} + x^2 + \ldots .$$

Also,

$$a_6 = A_2 B_4 = \eta(\tau)\eta(3\tau)^5\eta(5\tau)^7/\eta(15\tau)^{13}$$

has valence 2 at $\tau = 0, \frac{1}{3}, \frac{1}{5}$ and so we can say

THEOREM 6. $\{a_2, a_3\}$, $\{a_2, a_3, a_6^{-1}\}$ *are integral polynomial bases for* E_{15}, F_{15} *respectively.*

Since A_2, A_4, B_4, C_4 must be linearly related, it is simple to find a modular equation of level 15 for $\eta(\tau)$. Such an equation is

$$A_4 - 2B_4 + C_4 - 25A_2 + 100 = 0.$$

Also, for appropriate constants r_i, s_i we have

$$a_6 = r_0 a_2^3 + r_1 a_2 a_3 + r_2 a_2^2 + r_3 a_3 + r_4 a_2 + r_5,$$

$$a_3^2 = s_0 a_2^3 + s_1 a_2 a_3 + s_2 a_2^2 + s_3 a_3 + s_4 a_2 + s_5.$$

Evaluating the constants, we find

$$a_6 = a_2^3 - 4a_2 a_3 - 8a_3 - 7a_2 - 6, \tag{15}$$

$$a_3^2 = a_2^3 - 3a_2 a_3 + 2a_2^2 - 8a_3 - 16a_2 - 33.$$

$n = 21$. This is the last instance of $\nu(n) = 1$ for the n's under consideration. We have the functions

$$A_2 = \eta(3\tau)^3\eta(7\tau)/\eta(\tau)\eta(21\tau)^3$$

$$A_4 = \eta(\tau)^5\eta(7\tau)/\eta(3\tau)\eta(21\tau)^5$$

$$B_4 = \eta(3\tau)\eta(7\tau)^7/\eta(\tau)\eta(21\tau)^7$$

$$C_4 = \eta(\tau)^2\eta(7\tau)^4/\eta(21\tau)^6$$

We still need a function with positive valence at $0, \frac{1}{2}, \frac{1}{11}$ to go over to the basis for F_{22} and $A_5 B_5 C_5$ could be used for this purpose. It is a simple matter to express A_5, B_5, C_5 in terms of a_3, a_4, a_5 by means of the equations given above, which imply that

$$A_5 = 17a_3 - 7a_4 + a_5 - 21$$
$$B_5 = a_3 + a_4 + a_5 + 3$$
$$C_5 = 6a_3 + 4a_4 + a_5 + 12.$$

The function $A_5 B_5 C_5$ is somewhat uneconomical however, since it has valence -15 at $\tau = i\infty$, and a simpler function is

$$A_6 = \eta(2\tau)^6 \eta(11\tau)^6 / \eta(\tau)^2 \eta(22\tau)^{10}$$

having valence 1 at 0 and $\frac{1}{11}$ and valence 4 at $\frac{1}{2}$. We find without difficulty the modular equation of level 22 for $\eta(\tau)$

$$A_6 = a_3^2 + 2a_5 - 3a_4 - 2a_3 - 4$$

and have the theorem:

THEOREM 8. *$\{a_3, a_4, a_5\}$, $\{a_3, a_4, a_5, A_6^{-1}\}$ are integral polynomial bases for E_{22}, F_{22} respectively.*

$n = 26$. Again the genus is 2, and we have the functions

$$A_3 = \eta(2\tau)^1 \eta(13\tau)^2 / \eta(\tau)^2 \eta(26\tau)^1$$
$$A_7 = \eta(\tau)^7 \eta(13\tau)^5 / \eta(2\tau)^3 \eta(26\tau)^9$$
$$B_7 = \eta(\tau)^3 \eta(13\tau)^9 / \eta(2\tau) \eta(26\tau)^{11}$$
$$C_7 = \eta(2\tau) \eta(13\tau)^{13} / \eta(\tau) \eta(26\tau)^{13}$$
$$A_8 = \eta(\tau) \eta(2\tau) \eta(13\tau)^{11} / \eta(26\tau)^{13}$$

with the table

	α	β	γ	$v(0)$	$v(\frac{1}{2})$	$v(\frac{1}{13})$	$v(i\infty)$	r_2	r_{13}	r_{26}
A_3	3	1	0	0	3	0	-3	4	2	1
A_7	1	2	1	6	0	1	-7	-3	5	-9
B_7	4	1	1	3	0	4	-7	-1	9	-11
C_7	7	0	1	0	0	7	-7	1	13	-13
A_8	6	1	1	2	1	5	-8	1	11	-13

From the three functions of valence -7 we can determine two of valence -6. Together with A_3^2 this gives two of valence -5, and finally one of valence -4, so that we have enough functions to obtain a basis for E_{26}. In addition A_8 is a function with positive valence at $0, \frac{1}{2}, \frac{1}{13}$.

Again the computations are not too involved, and we find

$$a_3 = A_3 - 2 = x^{-3} + 2x^{-2} + x^{-1} + 2x + \ldots,$$
$$a_4 = \tfrac{1}{10}A_3^2 - \tfrac{4}{5}A_3 - \tfrac{1}{40}A_7 + \tfrac{3}{40}B_7 - \tfrac{1}{20}C_7 + \tfrac{7}{10}$$
$$= x^{-4} - x^{-1} + x + \ldots,$$
$$a_5 = \tfrac{1}{10}A_3^2 - \tfrac{4}{5}A_3 + \tfrac{3}{80}A_7 - \tfrac{1}{20}B_7 + \tfrac{1}{80}C_7 - \tfrac{3}{10}$$
$$= x^{-5} - x^{-2} + 2x^{-1} + x^2 + \ldots.$$

Please enter 2507 - 2513

Coefficient of modular functions Type 2

needs name cards

we have the modular equation of level 26 for $\eta(\tau)$

$$1 \quad a_3 a_5 \quad 3a_3 a_4 \cdot 3a_3^2 - 7a_5 - 23a_4 - 13a_3 - 24.$$

can say

$a_4, a_1, \ldots, a_3, a_4, a_5, A_3^{-1}$ *are integral polynomial bases for* ...*ively*.

...ord or two in conclusion about the cases $n = 33$ and $n = 34$. ...33) and $E_0(34)$ are of genus 3, so the minimal functions of E_{33} ...valence 4. The minimal function of H_{33} has valence

$$-10 = -3.3 - 1,$$

...d by theorem 2 since 11 ... -1 (mod 6); while the minimal fun... ...H_{34} has valence -4 ... $-3-1$ which is also in agreement with ...2 since 17 ... 1 (mod 4). Each case involves working with some ...tions of H_n to obtain bases for E_n and F_n. It seems hardly worth ...reproduce these calculations here.

...alculations described in this paper depended upon a knowledge ...first few coefficients in the Fourier expansion for the functions η at ... These were computed initially by hand from the tables given in ...d then recomputed on the electronic digital computer (a 704) of the ...nal Bureau of Standards in Washington, D.C. We give here the ...ents of all powers of x to the tenth, and refer the reader to (3) where ...thod of computation is explained in detail.

Tables of coefficients

2507 2508

$n = 6$

	a_1	A_2	B_2	C_2	a_3	a_5^{-1}
					1	1
−2		1	1	1	2	−2
−1	1	7	−1	−2	−5	9
0	0	24	−8	−3	−24	−4
1	6	50	2	−4	−23	28
2	4	58	26	22	76	18
3	3	3	27	30	249	118
4	−12	−120	−24	−12	168	80
5	8	−200	−136	−128	−599	504
6	12	−39	−135	−147	−1670	466
7	30	402	162	132	−1026	1631
8	20	728	568	548	3272	2160
9	30	246	486	516	8529	5466
10	72	1200	−624	−552	5232	7498

ignore signs

7298

~~6706~~ 6707 6708 6709

$n = 10$

6710

	a_1	a_3	a_5^{-1}
−3		1	1
−2		0	0
−1	1	−4	4
0	0	0	0
1	1	2	14
2	2	−8	8
3	2	8	40
4	−2	32	32
5	−1	−5	105
6	0	−16	112
7	−4	28	284
8	−2	−64	320
9	5	−138	702
10	2	40	840

2509

$n = 14$

	a_2	A_4	B_4	B_4^{-1}	a_3
−4		1	1	1	
−3		−6	1	−1	1
−2	1	11	−3	4	0
−1	1	−2	−2	−5	2
0	0	−7	0	15	0
1	2	−10	−3	−19	5
2	2	6	6	45	4
3	3	26	−2	−52	2
4	4	18	−10	118	4
5	−2	−72	26	−137	10
6	−1	1	15	281	0
7	1	42	0	−316	−4
8	−4	44	16	625	−12
9	−2	−14	−49	−695	−9
10	−6	−150	−10	1331	8

ignore signs!

2510

$n = 15$

	a_2	A_4	B_4	C_4	a_3	a_6	a_5^{-1}
						1	
−6						−1	1
−5						−1	1
−4		1	1	1		−5	2
−3		−7	−2	3	1	5	8
−2	1	14	−1	9	0	−1	10
−1	1	8	−2	13	1	12	24
0	0	−56	9	24	0	3	53
1	2	34	4	24	0	25	74
2	4	51	−4	41	1	−25	153
3	0	−36	−6	24	6	−43	280
4	2	−8	−8	42	6	−54	436
5	−1	−25	0	0	2	−2	793
6	3	−16	−26	39	7	20	1322
7	−4	−23	22	−33	−3	25	2085
8	2	97	32	17	−7	207	3510
9	−3	102	42	−93	−21	101	5648
10	1	−25	−25	0	3		8796

2511

$n = 21$

[illegible]	[illegible]	B_4	C_4	a_3	a_8	a_8^{-1}
					1	1
					−1	1
	1	1	1		−1	2
	5	1	−2	1	−3	6
1	5	2	−1	0	3	8
1	11	2	2	−1	4	13
0	−20	4	1	0	0	29
0	−1	5	2	1	−4	44
2	7	7	−2	1	2	66
1	1	2	−4	1	10	122
2	24	6	6	−2	7	184
−1	−7	2	2	1	−20	269
0	22	8	−7	5	−21	448
0	−7	−1	−4	1	0	668
2	1	1	−8	1	5	972
−2	−7	−5	10	−3	12	1505
1	11	8	5	0	−25	2205

2512

$n = 22$

[illegible]	B_5	C_5	A_6	A_6^{-1}	a_3	a_4	a_5
			1	1			
1	1	1	2	−2			1
−7	1	4	−1	5		1	0
17	1	6	−2	−10	1	0	0
14	2	8	−1	22	0	2	0
2	2	13	−6	−40	1	2	−1
−21	3	12	0	75	0	0	0
36	4	14	4	−130	2	0	2
13	5	24	−5	230	2	3	0
26	6	18	4	−382	0	4	2
24	[illegible]	20	6	636	0	4	4
10	[illegible]	32	−4	−1016	2	4	4
9	1	20	−7	1633	2	3	−4
4	1	15	6	−2540	1	2	1
17	7	16	13	3942	0	3	4
[illegible]	[illegible]	[illegible]	4	−5978	2	−4	2
13	[illegible]	−20	43	9057	−2	−5	12

ignore signs always!

2513

$n = 26$

	a_3	A_3^2	A_7	B_7	C_7	A_8	A_8^{-1}	a_4	a_5
−8						1	1		
−7			1	1	1	−1	1		
−6		1	−7	−3	1	−2	3		
−5		4	17	1	1	1	4		1
−4		6	−14	2	2	0	9	1	0
−3	1	8	2	2	2	2	12	0	0
−2	2	13	−21	−1	3	1	23	0	−1
−1	1	12	36	−4	4	0	31	−1	2
0	0	14	13	1	5	0	54	0	0
1	2	24	−26	−2	6	−2	73	1	0
2	0	18	−24	0	8	1	118	2	1
3	3	20	10	2	10	−2	159	−1	0
4	2	32	12	4	12	−2	246	1	2
5	0	24	−17	−1	15	−11	340	2	2
6	2	31	29	−7	5	13	500	0	3
7	2	36	57	25	9	21	684	2	3
8	2	14	−66	−10	14	−11	984	0	−2
9	1	8	−26	−18	6	2	1341	−1	0
10	0	16	−20	−20	12	−22	1883	0	2

REFERENCES

1. L. E. Dickson, *Modern elementary theory of numbers* (University of Chicago Press, 1943).
2. R. Fricke, *Lehrbuch der Algebra* (F. Vieweg und Sohn, 1926 (Dritter Band)).
3. M. Newman, 'Construction and application of a class of modular functions', *Proc. London Math. Soc.* (3) 7 (1957) 334–50.
4. —— 'A table of the coefficients of the powers of $\eta(\tau)$', *Proc. Kon. Nederl. Akad. Wetensch. Ser* A57 = *Indag. Math.* 18 (1956) 204–16.

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