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KNOTS AND CLASSES OF MÉNAGE PERMUTATIONS

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Introduction

The problème des ménages is to count the ways of seating n husbands and their n wives around a circular table in so that each husband is seated between two wives, neither of whom is his own. Let the wives be seated first and numbered 1, 2, ..., n clockwise around the table. Then each seating arrangement defines a permutation which carries i into j if wife i is seated to the right of the husband of wife j. One is thus led to the problem of counting the permutations of 1, 2, ..., n which, for i = 1, 2, ..., or n, carry i into neither i nor i + 1 modulo n. Such permutations will be called ménage permutations. The literature on the ménage problem dates back to Cayley; its history and a bibliography may be found in I. Kaplansky and J Riordan [1] together with an enumeration of the permutations which violate exactly k of the n restrictions.

Let P be one of the ménage permutations and let C be any one of the n cyclic permutations. Then the permutation $P' = C^{-1}PC$ is also a ménage permutation. If two ménage permutations P and Q satisfy $Q = C^{-1}PC$ for a suitable cyclic permutation C, then P and Q will be called equivalent permutations. In this way the ménage permutations may be gathered into equivalence classes. The number T(n) of equivalence classes will be found here (Eq. (11) below).

In the problème des ménages equivalent permutations describe seating arrangements which are similar in the following sense. Suppose, in the seating plan of the permutation P, that each person's number is decreased by a modulo n. To bring wives #1, 2, ..., n back into the standard places we now move each person 2a places counterclockwise around the table. The result is a new seating plan with permutation $C^{-1}PC$, where C is the cyclic permutation $i \to i + a \bmod n$. Hence if both permutations of family names and rotations around the table are regarded as not producing essentially distinct seating plans, the number of distinct seating plans is our number T(n).

Knots

The problem of classifying the ménage permutations was posed in 1877 by P. G. Tait [3, Vol. I, p. 287] in his study of knots. To explain the connection between our problem and knots we may imagine a knot as a tangled loop of string lying on a table. We suppose the knot to be an alternating one, *i.e.*, as one follows the string it crosses itself alternately over and under as illustrated in Figure 1. Tait

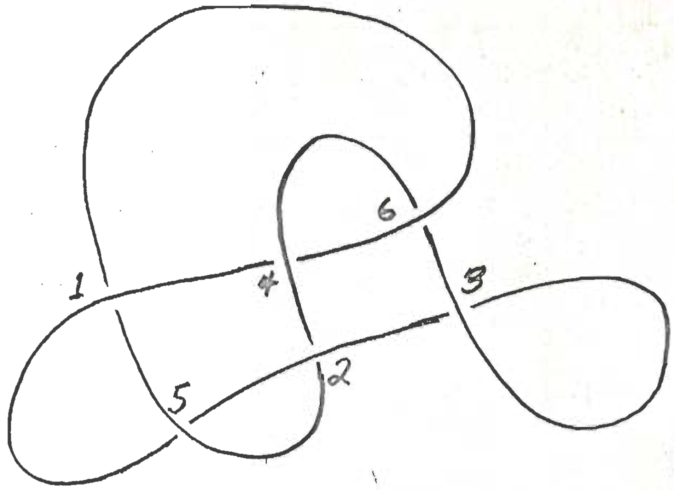

Figure 1.

devised the following compact notation for alternating knots. Pick a crossing and number it 1. Next follow the string which lies above at this crossing and follow it to the next crossing at which it lies above. Number this crossing 2. Continuing one numbers alternate crossings 1, 2, ..., n (n = total number of crossings) as in Figure 1. The knot so numbered defines a permutation in which i is carried into the number of the under-crossing which follows the over-crossing i. In Figure 1 the permutation is $1 \rightarrow 5$, $2 \rightarrow 3$, $3 \rightarrow 6$, $4 \rightarrow 2$, $5 \rightarrow 1$, $6 \rightarrow 4$. This permutation completely characterizes the knot.

A crossing (such as 3 in Figure 1), which appears twice in succession, can be removed by a simple twist. If no such crossing is present the knot is characterized by a ménage permutation. If the crossing #1 were chosen at a different place the crossing numbers would be subjected to a cyclic permutation C and the knot permutation P would change to $C^{-1}PC$; equivalent permutations describe the same knot. Tait hoped to tabulate knots by examining one ménage permutation from each equivalence class.

<u>Pólya's Theorem</u>

To count equivalence classes of ménage permutations we employ a form of a theorem of G. Pólya [2]. This theorem concerns a group G of transformations (symmetries) operating on a set of objects I. In our case G is the multiplicative group of cyclic permutations, I is the set of ménage permutations and the result of applying the transformation C to the object P is the new object $C^{-1}PC$. In Pólya's theorem two objects are called equivalent if one object may be transformed into the other by applying one of the transformations in G to it. Then the number T of different equivalence classes of objects is found to be

$$(1) \qquad T = \frac{1}{g} \sum_{\varphi} N(\varphi)$$

where g is the order of the group G, $N(\varphi)$ is the number of objects left invariant by the transformation φ and the sum is over all members of G. In our case $g = n$ and it remains to find $N(\varphi)$.

Enumeration

The typical member of G is a cyclic permutation C_a which carries i into i + a modulo n for i = 1, 2, ---, n. We now construct all $N(C_a)$ ménage permutations P which satisfy $P = C_a^{-1}PC_a$, or equivalently $C_aP = PC_a$. First note that if P sends i into j (which we write $P:i = j$) then, for any k, $C_a^kP:i = C_a^k:j$ whence $PC_a^k:j$ or $PC_a^k:i = C_a^k:j$

$$(2) \qquad P : (i + ka) = j + ka.$$

In (2) the additions are modulo n. The set of numbers of the form ka modulo n contains only w = n/(n,a) different numbers; these are 0, (n,a), 2(n,a), ---, (w-1)(n,a). Using (2) with k = 1, 2, ---, w-1 P will be completely determined from the (n,a) numbers y_r defined by

$$(3) \qquad y_r = P:r \ , \quad r = 1, 2, \cdots, (n,a).$$

In addition to the first (n,a) ménage restrictions

$$(4) \qquad y_r \neq r \ \text{ and } y_r \neq r + 1$$

we must now require, for every pair r, s with $r \neq s$, that

$$(5) \qquad y_r \neq y_s \quad \text{mod } (n,a).$$

To prove that (5) is necessary note first, since (n,a) is itself a multiple of a modulo n, that another form of (2) is

(6) $$P:(i + k(n,a)) = j + k(n,a).$$

If (5) did not hold then for some integer M we would have

$$P:r = P:s + M(n,a) \quad \text{mod } n.$$

Using (6) with $k = -M$ modulo n one concludes

(7) $$P:(r - M(n,a)) = P:s.$$

Since r and s are different integers chosen from 1, ..., (n,a), one concludes $r - M(n,a)$ and s are different modulo n. Then (7) shows that the operation P has no inverse and hence is not a permutation.

Conversely suppose that integers $y_1, y_2, \ldots, y_{(n,a)}$ are given ($y_r = 1, 2, \ldots$, or n) satisfying (4) and (5). Then the permutation P defined by (2) and (3) is a ménage permutation which is left invariant by C_a. For, (5) and (2) again ensure that P is a permutation and (4) and (2) ensure that all the ménage restrictions are satisfied.

To construct sets of integers y_r satisfying (4) and (5) note first that (5) is equivalent to the requirement that the transformation Q

$$r \rightarrow y_r \mod (n,a)$$

shall be a permutation of 1, 2, ..., (n,a). If Q is itself a ménage permutation then there are many different sets of integers y_r which produce the same transformation Q; each y_r may be altered by any multiple of (n,a) without changing Q. The number of ways of picking such sets of integers y_r is

$$w^{(n,a)} \quad \text{where} \quad w = n/(n,a).$$

Q need not be a ménage permutation. Suppose Q violates a ménage restriction, say $Q{:}s = s$ or $Q{:}s = s + 1$. Then if $(n,a) > 1$ the number of ways of picking that particular y_s is reduced to w-1. If Q violates exactly K of the ménage restrictions there are

$$w^{(n,a)-K}\,(w-1)^K \;=\; w^{(n,a)}\left\{1 - \frac{1}{w}\right\}^K$$

choices for the set of y_r. Hence

$$(8) \qquad N(C_a) = w^{(n,a)} \sum_{K=0}^{(n,a)} u_{(n,a),K} \left\{1 - \frac{1}{w}\right\}^K$$

in which $u_{m,K}$ is the number of permutations of $1, \ldots, m$ which violate just K of the ménage restrictions.

The case $(n,a) = 1$ is an exceptional one. Conceptually the numbers $u_{1,K}$ are not yet clearly defined. For example, does the identity permutation $1 \to 1$ violate two ménage restrictions or one? We propose to define $u_{1,K}$ in such a way that (8) becomes true even when $(n,a) = 1$. In this case $N(C_a) = n - 2$ since y_1 can be any of $3, 4, \ldots, n$. We are thus led to the convention

$$u_{1,0} = -1, \quad u_{1,1} = 2, \quad u_{1,2} = u_{1,3} = \cdots = 0.$$

The numbers $u_{m,K}$ are given by Kaplansky and Riordan [1] and by J. Touchard [4].

The generating function $U_m(t) = \sum_K u_{m,K} t^K$ has the expression

$$(9) \qquad U_m(t) = \sum_{k=0}^{m} \frac{2m}{2m-k} \binom{2m-k}{k} (m-k)!(t-1)^k.$$

Note that (9) agrees with our convention $U_1(t) = 2t - 1$. In terms of the generating function, (8) becomes

$$(10) \qquad N(C_a) = \left\{n/(n,a)\right\}^{(n,a)} \quad U_{(n,a)}(1 - (n,a)/n)$$

The number of equivalence classes $T(n)$ can now be computed from (1) using (10). A simplification occurs if the terms $N(C_a)$ of (1) having the same value of (n,a) are grouped together. When $(n,a) = d$, which is a division of n, the number of terms of (1) is $\varphi(n/d)$ where $\varphi(m)$ is Euler's function. Then $T(n)$ is expressed as a sum over divisors of n

$$(11) \qquad T(n) = \frac{1}{n} \sum_{d|n} \varphi(n/d)(n/d)^d \; U_d(1-d/n).$$

When $n = 3, 4, 5, 6, 7, 8, 9, 10$ the numbers $T(n)$ of classes are 1, 2, 5, 20, 87, 616, 4843, and 44128. When $n = 5$ 2484

a set of five non-equivalent ménage permutations (written as rearrangements of 1 2 3 4 5) are 35124, 45123, 51234, 34512, 35214. Of these Tait observes that only the first two have a corresponding knot. The knot for 35124 is the knot of Figure 1 after the crossing #3 has been removed by a twist. The knot for 45123 may be drawn as a five-pointed star (Solomon's pentacle).

References

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