Mousetrap

Richard K. Guy, Richard J. Nowakowski.

February 10, 1993

It's all so unimportant. That's what makes it so very interesting.

Agatha Christie: *The Murder of Roger Ackroyd.*

Cayley [8, 9] introduced a permutation problem he called **Mousetrap** which is loosely based on the card game Treize. Suppose that the numbers $1, 2, \ldots, n$ are written on cards, one to a card. After shuffling (permuting) the cards, start counting the deck from the top card down. If the number on the card does not equal the count, transfer the card to the bottom of the deck and continue counting. If the two are equal then set the card aside and start counting again from 1. The game is **won** if all the cards have been set aside.

Table 1 gives all the permutations on 4 cards and the order in which they are set aside. The game is won on just 6 occasions out of the 24.

Table 1. Six winning games in four-card Mousetrap.

permutation	1234	1243	1324	1342	1423	1432	2134	2143	2314	2341	2413	2431
cards set aside	1	1342	12	1	1234	1423	3214		4			3142
permutation	3124	3142	3214	3241	3412	3421	4123	4132	4213	4231	4312	4321
cards set aside	4		21	23				3	2134	2		

Cayley proposed two questions.

1. For each n find all the winning permutations of $1, 2, \ldots, n$.

2. For each n find the number of permutations that eliminate precisely i cards for each i, $1 \leq i \leq n$.

An answer for question 2 would give an answer for question 1, but very little is known about either. In this paper, we give some results for question 2 and raise some other questions.

In [9], Cayley lists all the possible outcomes for $n = 4$. Steen [46] notes that Cayley made some errors. He goes on to calculate, for any n, the number of permutations that have i, $1 \leq i \leq n$, as the first card set aside. This number he denotes by $a_{n,i}$ and he used $b_{n,i}$ ($c_{n,i}$) to denote the number of permutations that have 1 (respectively 2) as the first hit and i as the second. He obtained the recurrence relations

$$a_{n,i} = a_{n,i-1} - a_{n-1,i-1}, \qquad b_{n,i} = a_{n-1,i-1}$$

$$c_{n,i} = c_{n,1} - (i-1)c_{n-1,1} + \sum_{k=2}^{i-2}(-1)^k \frac{i(i-1-k)}{2} c_{n-k,1} \quad \text{for } n > i+1$$

and used them to show that for $0 \leq i \leq n$,

$$a_{n,0} = na_{n-1,0} + (-1)^n, \quad a_{0,0} = 1$$

$$a_{n,i} = \sum_{k=0}^{i}(-1)^k \binom{i}{k}(n-1-k)!$$

$$b_{n,i} = a_{n-1,i-1} = a_{n-2,i-2} - a_{n-3,i-2}$$

$$c_{n,i} = \sum_{k=1}^{i-3}(-1)^{k+i-1}\frac{k(k+3)}{2}(n-i+k-1)! - (i-1)(n-3)! + (n-2)!$$

Steen denoted the sums of the $a_{n,i}$ $b_{n,i}$ $c_{n,i}$ taken over $0 \leq i \leq n$ (but omitting $i = 0$, $i = 1$, $i = 2$ respectively) by a_n b_n c_n and further showed that

$$a_n = na_{n-1} + (-1)^{n+1} \qquad b_n = a_{n-1}$$

and deduced a complicated expression for c_n from his formula for $c_{n,i}$ which unfortunately holds for neither $i = n$ nor $i = n-1$. We will give formulas for $c_{n,n}$ and $c_{n,n-1}$ and will see that

$$c_n = (n-2)(n-2)! - [\![\frac{1}{e}((n-1)! - (n-2)! - 2(n-3)!)]\!],$$

where $[\![x]\!]$ is the nearest integer to x.

Table 2 lists the numbers of permutations of n, $1 \leq n \leq 9$ which set aside exactly i cards, $0 \leq i \leq n$.

Table 2. Numbers of permutations eliminating just i cards.

n \ $i =$	0	1	2	3	4	5	6	7	8	9
1	0	1								
2	1	0	1							
3	2	2	0	2						
4	9	6	3	0	6					
5	44	31	19	11	0	15				
6	265	180	105	54	32	0	84			
7	1854	1255	771	411	281	138	0	330		
8	14833	9949	6052	3583	2057	1366	668	0	1812	
9	133496	89162	55340	32135	19026	12685	6753	4305	0	9978

If $i = 0$, so that no card is set aside then the permutation is a **derangement** ([44], sequence 766), that is, card j is in place j for no value of j. Sloane [44] also contains some related sequences. For example, sequences 1166, 706, 1189, 1450 and 1637 are given by the recurrence $a_n = na_{n-1} + (n-k)a_{n-2}$ with $k = 1$, 2, 3, 4 and 5 respectively, and are concerned with the not unrelated problem of counting permutations containing given numbers of consecutive members([2, 3, 17, 19, 50]). Sequences 1423 and 1186 are Steen's a_n and c_n respectively. The first edition of [44] cited Steen's sometimes erroneous values of c_n for $3 \le n \le 10$. They have been corrected in the forthcoming second edition. They can be calculated from the formula above, and for $1 \le n \le 20$ are

0, 0, 1, 3, 13, 65, 397, 2819, 22831, 207605, 2094121, 23205383, 280224451, 3662810249, 51523391965, 776082247979, 12463259986087, 212573743211549, 3837628837381201, 73108996989052175.

There is a considerable literature on permutations with restricted position [41, 12], which include derangements or the *problème des rencontres* [18, 21, 38], Lucas's *problème des ménages* [28, 4, 5, 7, 14, 24, 27, 29, 40, 42] or Tait's earlier version of it [10, 11, 33, 34, 35, 47], and the more general enumeration of latin rectangles [6, 15, 20, 25, 26, 29, 32, 36, 37, 39, 52, 53, 54, 55, 56]. The first explicit solution of the *problème des ménages* was given by Touchard[48] (see also [49]), the simplest by Kaplansky [22], while Wyman & Moser [51] gave an interesting solution using an exponential generating function. Tait [47] and Gilbert [16] were interested in the possible connexion with knot theory.

But here we restrict ourselves to Mousetrap. The top diagonal, $i = n$, of Table 2 gives answers to Cayley's first question. No sequence beginning in this way appears in [44], although we may be in time to appear in the second edition. Since it is impossible to leave just one card, it follows that the second diagonal, $i = n - 1$, consists of zeros. The row sums are $n!$

Notice that the number of permutations which eliminate at least one card is the nearest integer to $n!(1 - \frac{1}{e})$. This is the special case $j = n - 1$ of a more general problem [43]: find the minimum number of permutations of $1, 2, \ldots, n$ which contain all permutations of a given j-element subset.

We also considered the case where just one card is set aside. In Table 3 the kth column gives the number of permutations in which just the card k is set aside so that the row sums correspond to the column $i = 1$ of Table 2.

Table 3. Numbers of permutations eliminating just card k.

n	$k=$ 1	2	3	4	5	6	7	8	9	10
1	1									
2	0	0								
3	1	0	1							
4	2	1	1	2						
5	9	5	5	3	9					
6	44	31	25	20	16	44				
7	265	203	167	142	117	96	265			
8	1854	1501	1267	1075	932	791	675	1854		
9	14833	12449	10745	9311	8241	7132	6205	5413	14833	
10	133496	114955	101005	88993	78607	70340	62141	55004	48800	133496

Derangement numbers appear in the column $k = 1$ and on the diagonal $k = n$. It is easy to verify that there is a good reason for this. The inclusion-exclusion method used in [22] can help to find summation formulas for the other entries of the table. Let $E^k = (n-k)!/n!$

Theorem 1 *The probability that only the card k is set aside is given by*

$$E(1-E)^{n-2k+1}(1-3E)(1-3E+E^2)^{k-2}$$

if $1 \le k \le n/2$*; by*

$$E[1 - \binom{2p+1}{1}E + \binom{2p}{2}E^2 + \ldots (-1)^i \binom{2p-i+1}{i} E^i + \ldots (-1)^p \binom{p}{p} E^p]^k$$

if $n/2 \le k < n$ *and* $p = (k-1)/(n-k)$ *is an integer; and otherwise by*

$$[1 - \binom{2p+4}{1}E + \ldots (-1)^i \binom{2p-i+5}{i} E^i + \ldots (-1)^{p+2} \binom{p+3}{p+2} E^{p+2}] \times$$

$$[1 - \binom{2p+3}{1}E + \ldots (-1)^i \binom{2p-i+4}{i} E^i + \ldots (-1)^{p+2} \binom{p+2}{p+2} E^{p+2}]^{j-2} \times$$

$$E[1 - \binom{2p}{1}E + \ldots (-1)^i \binom{2p-i+1}{i} E^i + \ldots (-1)^p \binom{p+1}{p} E^p] \times$$

$$[1 - \binom{2p+1}{1}E + \ldots (-1)^i \binom{2p-i+2}{i} E^i + \ldots (-1)^{p+1} \binom{p+1}{p+1} E^{p+1}]^{k-j}$$

where p *is now* $\lfloor (k-1)/(n-k) \rfloor$, *and* $j = k - p(n-k) > 1$, *except that for* $k = n$ *it is asymptotically* $1/nc$.

Proof. Let p_{ij} be the probability that the permutation puts i in the jth place. In what follows $p_{ij}p_{kl}$ is interpreted as meaning the probability of both events happening (so that $p_{ij}p_{ik} = 0 = p_{ji}p_{ki}$ if $j \neq k$).

The probability that just card 1 is set aside is $p_{11}(1-p_{23})(1-p_{34})\ldots(1-p_{n-1,n})(1-p_{n2})$. All the terms are independent so this reduces to $E(1-E)^{n-1}$.

If k is in the first half of the range, $1 < k \leq \lfloor n/2 \rfloor$, then the expression is

$$\Pi_{i=1}^{k-1}(1-p_{i,i}) \times p_{k,k} \times \Pi_{i=1}^{k-1}(1-p_{i,k+i}) \times \Pi_{i=1}^{n-k-1}(1-p_{k+i,2k+i})$$

The terms can be put into k groups: the ith group, $i < k$, contains the terms corresponding to the restrictions on the ith card and the $k+i$th card. Specifically,

$$(1-p_{1,1})(1-p_{n-k+1,1})(1-p_{1,k+1})(1-p_{n,k+1}) \text{ and } (1-p_{i,i})(1-p_{n-k+i,i})(1-p_{i,k+i})$$

for $2 \leq i < k$ and, for $i = k$, $p_{k,k}\Pi_{i=1}^{n-k}(1-p_{k+i,2k+i})$.

For example take $k = 4$. We display the groups in columns.

$(1-p_{11})\times$	$(1-p_{22})\times$	$(1-p_{33})\times$	$p_{44}[(1-p_{59})\ldots(1-p_{n-4,n})]\times$
$(1-p_{n-3,1})\times$	$(1-p_{n-2,2})\times$	$(1-p_{n-1,3})\times$	
$(1-p_{n5})\times$			
$(1-p_{15})\times$	$(1-p_{26})\times$	$(1-p_{37})$	

The terms in each of these groups except the last are not pairwise independent but any two terms from different groups are independent. The first group reduces to $1 - 4E + 3E^2$. Groups 2 through $k-1$ each reduce to $1 - 3E + E^2$ and the last to $E(1-E)^{n-2k}$. In general then, for $k \leq n/2$ the probability that only k is set aside is given by

$$E(1-E)^{n-2k}(1-4E+3E^2)(1-3E+E^2)^{k-2} = E(1-E)^{n-2k+1}(1-3E)(1-3E+E^2)^{k-2}$$

In the second half of the range, $n/2 < k < n-1$, the dependence pattern becomes more complicated. We give two examples.

(a) $n = 14,\ k = 9$:

$$\begin{array}{llllll} p_{9,9}\times & 1-p_{14,10} & 1-p_{1,10} & 1-p_{1,1} & 1-p_{6,1} & 1-p_{6,6} & 1-p_{11,6} \\ & & 1-p_{2,11} & 1-p_{2,2} & 1-p_{7,2} & 1-p_{7,7} & 1-p_{12,7} \\ & & 1-p_{3,12} & 1-p_{3,3} & 1-p_{8,3} & 1-p_{8,8} & 1-p_{13,8} \\ & & 1-p_{4,13} & 1-p_{4,4} & 1-p_{9,4} & & \\ & & 1-p_{5,14} & 1-p_{5,5} & 1-p_{10,5} & & \end{array}$$

(b) $n = 13,\ k = 9$:

$$\begin{array}{lllllll} p_{9,9}\times & 1-p_{13,10} & 1-p_{1,10} & 1-p_{1,1} & 1-p_{5,1} & 1-p_{5,5} & 1-p_{9,5} \\ & & 1-p_{2,11} & 1-p_{2,2} & 1-p_{6,2} & 1-p_{6,6} & 1-p_{10,6} \\ & & 1-p_{3,12} & 1-p_{3,3} & 1-p_{7,3} & 1-p_{7,7} & 1-p_{11,7} \\ & & 1-p_{4,13} & 1-p_{4,4} & 1-p_{8,4} & 1-p_{8,8} & 1-p_{12,8} \end{array}$$

Note that the entries in each line are not pairwise independent, but any two entries on different lines are independent. Specifically, the groups are

$$p_{k,k}(1-p_{n,k+1})(1-p_{1,k+1})(1-p_{1,1})(1-p_{n-k+1,n-k+1})(1-p_{n-k+1,1})(1-p_{n-k+1,2n-2k+1})$$

and, for $2 \le i \le k$, $(1-p_{i,k+i})(1-p_{i,i})(1-p_{n-k+i,n-k+i})(1-p_{n-k+i,i})(1-p_{n-k+i,2n-2k+i})$, where any term that is not defined is taken to be zero. Let $p = \left\lfloor \frac{k-1}{n-k} \right\rfloor$. There are two cases.

One, suppose $p = (k-1)/(n-k)$ then all but one group is of length $2p+1$ and the other includes the term $1-p_{k,2k-n}$ which when multiplied by p_{kn} reduces to p_{kn}. The expression is therefore

$$E[1-\binom{2p+1}{1}E+\binom{2p}{2}E^2+\ldots(-1)^i\binom{2p-i+1}{i}E^i+\ldots(-1)^p\binom{p}{p}E^p]^k$$

Two, suppose that $p < (k-1)/(n-k)$. Let $j = k - \lfloor (k-1)/(n-k) \rfloor (n-k)$ so that $j > 1$. There are 3 group sizes, but again, one line includes the term $1-p_{k,2k-n}$. This latter term multiplied by p_{kn} reduces to p_{kn}. The whole expression is therefore

$$[1 - \binom{2p+4}{1} E + \ldots (-1)^i \binom{2p-i+5}{i} E^i + \ldots (-1)^{p+2} \binom{p+3}{p+2} E^{p+2}] \times$$

$$[1 - \binom{2p+3}{1} E + \ldots (-1)^i \binom{2p-i+4}{i} E^i + \ldots (-1)^{p+2} \binom{p+2}{p+2} E^{p+2}]^{j-2} \times$$

$$E[1 - \binom{2p}{1} E + \ldots (-1)^i \binom{2p-i+1}{i} E^i + \ldots (-1)^p \binom{p+1}{p} E^p] \times$$

$$[1 - \binom{2p+1}{1} E + \ldots (-1)^i \binom{2p-i+2}{i} E^i + \ldots (-1)^{p+1} \binom{p+1}{p+1} E^{p+1}]^{k-j}$$

If $k = n - 1$, then there is only one group, which gives

$$E\left(1 - \binom{2n-3}{1} E + \binom{2n-4}{2} E^2 + \ldots + (-1)^i \binom{2n-i-2}{i} E^i + \ldots + (-1)^{n-1} \binom{n}{n-1} E^{n-1}\right).$$

If $k = n$, then we are back to derangements. This completes the proof of the theorem.

The methods of the theorem can also be used to find corrected versions of Steen's formulas:

$$c_{n,n-1} = \sum_{k=0}^{n-3} (-1)^k \binom{n-3}{k} (n-k-2)!$$

$$c_{n,n} = (n-2)! + \sum_{k=0}^{n-5} (-1)^{k+1} \left(\binom{n-4}{k} + \binom{n-3}{k+1} \right) (n-k-3)! + 2(-1)^{n-3}$$

The probability that only the card 2 is set aside is

$$\sum_{i=0}^{3} (-1)^i \binom{n-3}{i} \frac{(n-i-1)! - 3(n-i-2)!}{n!}$$

With help from MAPLE we found that this has a closed form. Indeed, the number of permutations which set aside just the card k, $n \geq 2k - 1$, is

$k=1: \quad [\![\frac{1}{e}((n-1)!)]\!]$

$k=2: \quad [\![\frac{1}{e}((n-1)!-(n-2)!-2(n-3)!)]\!]$

$k=3: \quad [\![\frac{1}{e}((n-1)!-2(n-2)!-2(n-3)!+3(n-4)!+2(n-5)!)]\!]$

$k=4: \quad [\![\frac{1}{e}((n-1)!-3(n-2)!-(n-3)!+7(n-4)!+(n-5)!-5(n-6)!-2(n-7)!)]\!]$

$k=5: \quad [\![\frac{1}{e}((n-1)!-4(n-2)!+(n-3)!+11(n-4)!-5(n-5)!-13(n-6)!+ 2(n-7)!+7(n-8)!+2(n-9)!)]\!]$

and more generally there is an expression of the form

$$\left[\!\!\left[\tfrac{1}{e}((n-1)!-(k-1)(n-2)!+(k^2-5k+2)(n-3)!+\ldots+ (-1)^{k-1}(2k-3)(n-2k+2)!+(-1)^{k-1}2(n-2k+1)!)\right]\!\!\right].$$

For purposes of calculation, it may be more convenient to write these as the product of a factorial and a polynomial, e.g., for $k=2$, $[\![\frac{1}{e}(n-3)!(n^2-4n+2)]\!]$, and more generally

$$[\![\frac{1}{e}(n-2k+1)!\left(n^{2k-2}-2k(k-1)n^{2k-3}+\ldots\right)]\!].$$

In this way we obtain the following extension to Table 3.

n	$k=$ 1	2	3	4	5
11	1334961	1171799	1044395	932645	834341
12	14684570	13082617	11795863	10650463	9628825
13	176214841	158860349	144605933	131765675	120182567
14	2290792932	2085208951	1913265985	1756864189	1614448051
15	32071101049	29427878435	27183809135	25125937217	23237307353
16	481066515734	444413828821	412900741435	383803666315	356920864909
17	7697064251745	7151855533913	6678013826657	6237929276087	5828999672555

A third question arose during our investigations about which we also know very little. Consider a permutation for which every number is set aside. The list of numbers in the order that they were set aside is another permutation. Any permutation obtained in this way we call a **reformed** permutation.

2469

3. Characterize the reformed permutations.

Not all permutations are reformed permutations. For example, permutations on n objects are not reformed permutations if they start with n; or with $x, n-1, y$ where $y \neq n \neq x$; or with $k, j, n-(k+j+1), i$ where $i < k$ and $k < n-1$. On the other hand, the identity permutation is always a reformed permutation; the permutations yielding this are 1, 12, 132, 1423, 13254, 142563, 1527436, 16245378, 142863795, ...

The permutation 4213 is a reformed permutation which gives rise to the permutation 2134; this in turn gives 3214 which is not a reformed permutation.

4. For a given n, what is the longest sequence of reformed permutations?

For $n = 3$ there are $132 \to 123$ and $321 \to 213$: of length 1. For $n = 4$ there are $4213 \to 2134 \to 3214$ and $1432 \to 1423 \to 1234$ of length 2. An example of length 3 is $165342 \to 132564 \to 125346 \to 136524$. Of course, longer sequences contain shorter ones. Table 4, whose column sums are $n!$, gives the numbers of permutations yielding sequences of length l.

Table 4. Numbers of sequences of reformed permutations.

l	$n =$ 1	2	3	4	5	6	7	8	9
0	1	1	4	18	105	636	4710	38508	352902
1	-	1	2	4	14	72	316	1730	9728
2				2	1	11	14	81	242
3						1		1	8

Are there sequences of arbitrary length? Are there any cycles other than

$$1 \to 1 \to 1 \to 1 \ldots \quad \text{and} \quad 12 \to 12 \to 12 \to 12 \ldots ?$$

Modular Mousetrap. We can play Mousetrap, but instead of counting n, $n+1$, ... , we can start again, ... , n, 1, 2, ... Now at least as many cards get set aside. In fact if n is prime, then either the initial deck is a derangement, or all cards get set aside, so every sequence cycles or terminates in a derangement. The identity permutation $123\ldots n$ will always form a 1-cycle and now there are also examples of nontrivial cycles.

For $n = 2$, $12 \to 12 \to 12 \to \ldots$ cycles and 21 terminates.

For $n = 3$, $132 \to 123 \to 123 \to \ldots$ cycle, while $321 \to 213 \to 312$ and 231 terminate.

If n is composite, the number of cards set aside may be strictly between 0 and n. As before, exactly $n-1$ cards cannot be set aside; and it's easy to see that neither can just one card. For example, with $n = 4$ there are 9 derangements with the permutations 2431 & 4132 at distance 1 from two of them; 7 permutations which set aside just 2 cards and 4213 at distance 1 from one of them; and the 1-cycle 1234 with two permutations at distance 1 from it and 1243 & 1432 at distance 2.

For $n = 5$ there is the 1-cycle, 12345; nine permutations at distance 1 from it; eight at distance 2; four at distance 3; and two (12354 & 15432) at distance 4. There is a 2-cycle, 21345 & 32145; six permutations at distance 1 from it; and two (41352 & 43215) at distance 2. There are, of course, 44 derangements; and there are 23 permutations at distance 1 from some of these; 12 at distance 2; six at distance 3; and a path of length 4:

$$54321 \to 34215 \to 52143 \to 21435 \to 51423.$$

For $n = 6$, the 5! permutations starting with 1, and which thus set aside 1, consist of the 1-cycle 123456, thirteen permutations which feed into it, the longest sequence of which is

$$125436 \to 132645 \to 124653 \to 134562 \to 123456,$$

10 permutations which set aside just one other card (2, 4 or 6), 37 which set aside just three cards, 17 which set aside just 4 cards, and 42 which set aside the whole deck, the longest sequences of which begin with 126534, 153642 and 165342.

Are there k-cycles for every k ? What is the least value of n which yields a k-cycle?

We are grateful to Sherwood Washburn for bringing Mousetrap to our attention and for supplying copies of the early literature.

References

[1] W. Ahrens, *Mathematische Unterhaltungen und Spiele*, vol. 2 (2-aufl., Leipzig, 1918) 73–79.

[2] Paul Armsen, Veralgemeinerungen von Sequenzen in Permutationen, *J. reine angew. Math.*, **189**(1952) 77-99; *MR* **13**, 616g.

[3] Paul Armsen & Hans Rohrbach, Sequenzen in Permutationen, *Archiv Math.*, **1**(1948) 106-112; *MR* **10**, 347b (& see 347a).

[4] Kenneth P. Bogart & Peter G. Doyle, Nonsexist solution of the ménage problem, *Amer. Math. Monthly*, **93**(1986) 514–519; *MR* **87h**:05007.

[5] L. Carlitz, Congruences for the ménage polynomials, *Duke Math. J.*, **19**(1952) 549–552; *MR* **14**, 346.

[6] L. Carlitz, Congruences connected with three-line Latin rectangles, *Proc. Amer. Math. Soc.*, **4**(1953) 9–11; *MR* **14**, 726.

[7] L. Carlitz, Congruence properties of the ménage polynomials, *Scripta Math.*, **20**(1954) 51–57; *MR* **16**, 113.

[8] A. Cayley, A Problem in Permutations, *Quart. Math. J.*, I(1857), 79.

[9] A. Cayley, On the game of Mousetrap, *Quart. J. Pure Appl. Math.*, **XV**(1877), 8-10.

[10] A. Cayley, A problem on arrangements, *Proc. Roy. Soc. Edinburgh*, **9**(1878) 338–342.

[11] A. Cayley, Note on Mr. Muir's solution of a problem of arrangement, *Proc. Roy. Soc. Edinburgh*, **9**(1878) 388–391.

[12] Louis Comtet, Advanced Combinatorics, D. Reidel, 1974, 180–185.

[13] H. Dörrie, *Triumph der Mathematik*, 2 aufl., Breslau, 1940, 25-32.

[14] Jacques Dutka, On the problème des ménages, *Math. Intelligencer*, **8**(1986) 18–25,33; *MR* **87i**:05005.

[15] P. Erdős & I. Kaplansky, The asymptotic number of Latin rectangles, *Amer. J. Math.*, **68**(1946) 230–236; *MR* **7**, 407.

[16] E. N. Gilbert, Knots and classes of ménage permutations, *Scripta Math.*, **22**(1956) 228–233; *MR* **19**, 831c.

[17] Hugo Hadwiger, Eine bemerkung über zufälligkeit Anordnungen der natürlichen Zahlen, *Mitt. Verein. Schweiz. Versich.-Math.*, **46**(1946) 105-109; *MR* **7**, 406i.

[18] D. Hanson, K. Seyffarth & J. H. Weston, Matchings, Derangements, Rencontres, *Math. Mag.*, **56**(1983) 224–229.

[19] Anthonie van Heemert, Cyclic permutations with sequences and related problems, *J. reine angew. Math.*, **198**(1956) 77-99; *MR* **19**, 627b.

[20] S. M. Jacob, The enumeration of the Latin rectangle of depth three, *Proc. London Math. Soc.*, **31**(1930) 329–354.

[21] A. W. Joseph, A problem in derangements, *J. Inst. Actuaries Students Soc.*, (1946) 14–22.

[22] I. Kaplansky, Solution of the "Problème des ménages", *Bull. Amer. Math. Soc.*, **49**(1943) 784–785; *MR* **5**, 86.

[23] I. Kaplansky, Symbolic solution of certain problems in permutations, *Bull. Amer. Math. Soc.*, **50**(1944) 906–914; *MR* **6**, 159c.

[24] I. Kaplansky & J. Riordan, The Problème des Ménages, *Scripta Math.*, **12**(1946) 113-124; *MR* **8**, 365a.

[25] Sulaiman M. Kerawala, The enumeration of the Latin rectangle of depth three by means of a difference equation, *Bull. Calcutta Math. Soc.*, **33**(1941) 119–127; *MR* **4**, 69.

[26] Sulaiman M. Kerawala, The asymptotic number of three-deep Latin rectangles, *Bull. Calcutta Math. Soc.*, **39**(1947) 71–72; *MR* **9**, 404.

[27] Sulaiman M. Kerawala, Asymptotic solution of the "problème des ménages", *Bull. Calcutta Math. Soc.*, **39**(1947) 82–84; *MR* **9**, 405.

[28] E. Lucas, *Théorie des Nombres*, Paris, 1891, 491–495.

[29] P. A. MacMahon, *Combinatory Analysis*, vol. 1, Cambridge, 1915, 253–263.

[30] N. S. Mendelsohn, Symbolic solution of card matching problems, *Bull. Amer. Math. Soc.*, **50**(1946) 918-924; *MR* **8**, 365c.

[31] N. S. Mendelsohn, The asymptotic series for a certain class of permutation problems, *Canad. J. Math.*, **8**(1956) 234–245; *MR* **17**, 935a.

[32] W. O. J. Moser, The number of very reduced $4 \times n$ Latin rectangles, *Canad. J. Math.*, **19**(1967) 1011–1017; *MR* **36** #61.

[33] T. Muir, On Professor Tait's problem of arrangement, *Proc. Royal Soc. Edinburgh*, **9**(1878) 382–387.

[34] T. Muir, Additional note on a problem of arrangement, *Proc. Royal Soc. Edinburgh*, **11**(1882) 187–190.

[35] E. Netto, *Lehrbuch der Combinatorik*, 2nd edition, Berlin, 1927, 75–80.

[36] J. Riordan, Three-line Latin rectangles, *Amer. Math. Monthly*, **51**(1944) 450–452;*MR* **6**, 113.

[37] J. Riordan, Three-line Latin rectangles – II, *Amer. Math. Monthly*, **53**(1946)18–20; *MR* **7**, 233.

[38] J. Riordan, Discordant permutations, *Scripta Math.*, **20**(1954) 14–23; *MR* **16**, 104.

[39] J. Riordan, A recurrence relation for three-line Latin rectangles, *Amer. Math. Monthly*, **59**(1952) 159–162; *MR* **13**, 813.

[40] J. Riordan, The arithmetic of the ménage numbers, *Duke Math. J.*, **19**(1952) 27–30; *MR* **13**, 616c.

[41] John Riordan, *An Introduction to Combinatorial Analysis*, John Wiley, 1958, 163–267.

[42] Waldemar Schöbe, Das Lucassche Ehepaarproblem, *Math. Z.*, **48**(1943) 781–784; *MR* **5**, 29.

[43] Peter J. Slater How few n-permutations contain all possible k-permutations? *Amer. Math. Monthly*, **90**(1983) 461.

[44] N. J. A. Sloane, *A Handbook of Integer Sequences*, Academic Press, 1973.

[45] Roland Percival Sprague, Über ein Anordnungsproblem, *Math. Ann.*, **121**(1949) 52–53; *MR* **10**, 670i.

[46] A. Steen, Some formulae respecting the game of Mousetrap, *Quart. J. Pure Appl. Math.*, **XV**(1878), 230-241.

[47] Peter Guthrie Tait, *Scientific Papers*, vol. 1, Cambridge, 1898, 287.

[48] Jacques Touchard, Sur une problème de permutations, *C. R. Acad. Sci. Paris*, **198**(1934) 631-633; *Zbl* **9**, 3.

[49] Jacques Touchard, Permutations discordant with two given permutations, *Scripta Math.*, **19**(1953) 118; *MR* **15**, 387d.

[50] W. A. Whitworth, *Choice and Chance,* 5th edition, Cambridge, 1901, 102.

[51] Max Wyman & L. Moser, On the Problème des Ménages, *Canad. J. Math.*, **10**(1958) 468-480; *MR* **20**, #1633.

[52] Koichi Yamamoto, Latin Kukei no zenkinsu to symbolic method, *Sugaku*, **2**(1944) 159–162.

[53] Koichi Yamamoto, An asymptotic series for the number of three-line Latin rectangles, *J. Math. Soc. Japan*, **1**(1950) 226–241; *MR* **12**, 494g.

[54] Koichi Yamamoto, On the asymptotic number of Latin rectangles, *Japan. J. Math.*, **21**(1951) 113–119; *MR* **14**, 442.

[55] Koichi Yamamoto, Symbolic methods in the problem of three-line Latin rectangles, *J. Math. Soc. Japan*, **5**(1953) 13–23; *MR* **15**, 3.

[56] Koichi Yamamoto, Structure polynomial of Latin rectangles and its application to a combinatorial problem, *Mem. Fac. Sci. Kyusyu Univ. Ser. A*, **10**(1956) 1–13; *MR* **17**, 1174a.