## Some S-fractions related to the expansions of $\frac{\sin(ax)}{\cos(bx)}$ and $\frac{\cos(ax)}{\cos(bx)}$

## Peter Bala, May 11 2017

**1.** An S-fraction (also known as a fraction of Stieltjes-type) is a continued fraction of the form

$$S(x) = \frac{1}{1 - \frac{d_1 x}{1 - \frac{d_2 x}{1 - \frac{d_3 x}{1 - \frac{d_3 x}{1 - \dots}}}}$$
(1)

We say that it corresponds to the formal power series

$$f(x) = 1 + c_1 x + c_2 x^2 + \cdots$$
 (2)

if the expansion of its  $n^{th}$  approximant in ascending powers of x agrees with the power series (2) up to and including the term in  $x^{n-1}$ , n = 1, 2, 3, ...

Recall the **even part** of a continued fraction is the continued fraction whose n-th approximant is the 2n-th approximant of the given continued fraction. The even part of the generic S-fraction (1) is given by [2, Chapter 1, Section 4]

$$\frac{1}{1 - d_1 x - \frac{d_1 d_2 x^2}{1 - (d_2 + d_3) x - \frac{d_3 d_4 x^2}{1 - (d_4 + d_5) x - \frac{d_5 d_6 x^2}{1 - (d_6 + d_7) x - \cdots}}}$$
(3)

(3) is an example of a J-fraction (J stands for Jacobi). The general form of a J-fraction is

$$J(x) = \frac{a_0}{1 - b_1 x - \frac{a_1 x^2}{1 - b_2 x - \frac{a_2 x^2}{1 - b_3 x - \ddots}}}$$

We say that the this *J*-fraction is **asociated** with the formal power series (2), denoted by J(x) = f(x), if the expansion of its  $n^{th}$  approximant in ascending powers of x agrees with the power series (2) up to and including the term in  $x^{2n-1}, n = 1, 2, 3, \ldots$ 

By making use of an equivalence transformation it is easy to see that the  $m^{th}$  binomial transform of the generic *J*-fraction J(x) also has the form of a *J*-fraction:

$$\frac{1}{1-mx}J\left(\frac{x}{1-mx}\right) = \frac{a_0}{1-(b_1+m)x - \frac{a_1x^2}{1-(b_2+m)x - \frac{a_2x^2}{1-(b_3+m)x - \ddots}}}$$
(4)

Now in general, a J-fraction will not be the even part of an S-fraction. The purpose of this note is to give some examples of J-fractions whose  $m^{th}$  binomial transform, for particular values of m, is equal to the even part of an S-fraction. In Section 2 we consider the J-fraction associated with the trigonometric function  $\sin(ax)/a\cos(bx)$ . The form of the J-fraction is due to Stieltjes. We show there are two values of m such that the  $m^{th}$  binomial transform of Stieltjes' J-fraction equals the even part of an S-fraction. There are similar results for the trigonometric function  $\cos(ax)/\cos(bx)$ , which we outline in Section 3.

**2.** Consider the exponential generating function  $\sin(ax)/a\cos(bx)$ , with complex parameters *a* and *b*. The Taylor expansion of the function about x = 0 begins

$$\frac{1}{a}\frac{\sin(ax)}{\cos(bx)} = x - (a^2 - 3b^2)\frac{x^3}{3!} + (a^4 - 10a^2b^2 + 25b^4)\frac{x^5}{5!} - (a^6 - 21a^4b^2 + 175a^2b^4 - 427b^6)\frac{x^7}{7!} + \cdots$$
(5)

The coefficients in the expansion are homogeneous polynomials in a and b. See A104033 for information about these polynomials. Let  $A_{a,b}(x)$  denote the ordinary generating function for this sequence of polynomials (taken with an offset of 0):

$$A_{a,b}(x) = 1 - (a^2 - 3b^2) x + (a^4 - 10a^2b^2x + 25b^4) x^2 - (a^6 - 21a^4b^2 + 175a^2b^4 - 427b^6) x^3 + \cdots$$

The *J*-fraction associated with  $A_{a,b}(x)$  is given by

$$A_{a,b}(x) = \frac{1}{1 - b_1 x - \frac{a_1 x^2}{1 - b_2 x - \frac{a_2 x^2}{1 - b_3 x - \ddots}}}$$
(6)

where

$$a_n = 4n^2b^2 (4n^2b^2 - a^2),$$
  

$$b_n = 4 (2n^2 - 2n + 1) b^2 - a^2 - b^2.$$

This is a particular case of a result of Stieltjes (see [1, equation 18, p. 386] with c = 2b).

**Proposition 1.** Let  $a, b \in \mathbb{C}$  and let  $m = (b - a)^2$ . Let  $A_{a,b}(x)$  be the *J*-fraction given by (6). Then the  $m^{th}$  binomial transform of  $A_{a,b}(x)$  is the even part of the S-fraction

$$\frac{1}{1 - \frac{2b(2b-a)x}{1 - \frac{2b(2b+a)x}{1 - \frac{4b(4b-a)x}{1 - \frac{4b(4b+a)x}{1 - \cdots}}}}$$
(7)

**Proof.** By (3), the even part of (7) is the *J*-fraction

$$\frac{1}{1 - (4b^2 - 2ab)x - \frac{2b(2b - a)2b(2b + a)x^2}{1 - (2b(2b + a) + 4b(4b - a))x - \frac{4b(4b - a)(4b + a)x^2}{1 - (4b(4b + a) + 6b(6b - a))x - \cdots}}}$$
(8)

where the  $n^{th}$  numerator equals

$$2nb(2nb-a)2nb(2nb+a) = 4n^2b^2(4n^2b^2-a^2)$$

and the  $n^{th}$  denominator is given by

$$2nb(2nb+a) + 2(n+1)b(2(n+1)b-a) = 4(2n^2 + 2n + 1)b^2 - 2ab.$$

On the other hand, by applying (4) to (6), we find the  $m^{th}$  binomial transform of  $A_{a,b}(x)$  has the *J*-fraction representation

$$\frac{1}{1-mx}A_{a,b}\left(\frac{x}{1-mx}\right) = \frac{1}{1-(4b^2-2ab)x-\frac{a_1x^2}{1-(b_2+m)x-\frac{a_2x^2}{1-(b_3+m)x-\ddots}}}$$
(9)

where the  $n^{th}$  numerator is given by

$$a_n = 4n^2b^2(4n^2b^2 - a^2),$$

and the  $n^{th}$  denominator is given by

$$b_{n+1} + m = 4\left(2n^2 + 2n + 1\right)b^2 - 2ab$$

Thus (9) equals the even part of (7) as claimed.  $\Box$ 

The function  $\sin(ax)/a\cos(bx)$  is unchanged on replacing a with -a. Hence the associated ordinary generating function  $A_{a,b}(x)$  satisfies  $A_{-a,b}(x) = A_{a,b}(x)$ . As an immediate consequence we have the following companion result to Proposition 1.

**Proposition 2.** Let  $a, b \in \mathbb{C}$  and let  $M = (b+a)^2$ . Let  $A_{a,b}(x)$  be the *J*-fraction given by (6). Then the  $M^{th}$  binomial transform of  $A_{a,b}(x)$  is the even part of the S-fraction

$$\frac{1}{1 - \frac{2b(2b+a)x}{1 - \frac{2b(2b-a)x}{1 - \frac{4b(4b+a)x}{1 - \frac{4b(4b-a)x}{1 - \cdots}}}}$$
(10)

**Corollary 1.** The following continued fraction identity holds:

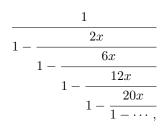
1	_		1
$\frac{2b(2b+a)x}{2b(2b+a)x}$		1 - 4abx -	2b(2b-a)x
$\frac{1}{1-\frac{2b(2b-a)x}{2b(2b-a)x}}$		1 - 4u0x -	$1 - \frac{2b(2b+a)x}{2b(2b+a)x}$
$1 - \frac{4b(4b+a)x}{1-a}$			$1 - 4abx - \frac{4b(4b-a)x}{2}$
$1 - \frac{4b(4b-a)x}{1-a}$			$1 - \frac{4b(4b+a)x}{a}$
1 - · · ·			$1  1 - 4abx - \cdots,$

or equivalently, changing a to -a,

$$\frac{1}{1 - \frac{2b(2b-a)x}{1 - \frac{2b(2b+a)x}{1 - \frac{4b(4b-a)x}{1 - \frac{4b(4b+a)x}{1 - \cdots}}} = \frac{1}{1 + 4abx - \frac{2b(2b+a)x}{1 - \frac{2b(2b-a)x}{1 + 4abx - \frac{4b(4b+a)x}{1 - \frac{4b(4b-a)x}{1 -$$

**Proof.** Comparing Proposition 1 with Proposition 2, we see that the S-fraction (10) is the  $(M - m)^{th} = (a + b)^2 - (a - b)^2 = (4ab)^{th}$  binomial transform of the S-fraction (7). Making use of (4) we arrive at the desired result.  $\Box$ 

**Example 1.** A000182 is the sequence of tangent numbers [1, 2, 16, 272, 7936, ...]. The e.g.f. is  $\tan(x) = \sin(x)/\cos(x)$ . Applying Proposition 1 with a = 1 and b = 1 gives the (well-known) result that the o.g.f. for the tangent numbers corresponds to the S-fraction



where the (unsigned) partial numerators are given by n(n+1), n = 1, 2, ...

By Proposition 2, the  $4^{th}$  binomial transform of the o.g.f. for the tangent numbers corresponds to the S-fraction

$$\frac{1}{1 - \frac{6x}{1 - \frac{2x}{1 - \frac{20x}{1 - \frac{12x}{1 - \cdots}}}}}$$

Corollary 1 gives the continued fraction identity

$$\frac{1}{1 - \frac{2x}{1 - \frac{6x}{1 - \frac{6x}{1 - \frac{12x}{1 - \frac{20x}{1 - \cdots}}}}} = \frac{1}{1 + 4x - \frac{6x}{1 - \frac{2x}{1 + 4x - \frac{20x}{1 - \frac{12x}{1 + 4x - \cdots}}}}$$

**Example 2.** A002439 is the sequence of Glaisher's T-numbers [1, 23, 1681, 257543, ...]. The e.g.f. is  $1/2 \times \sin(2x)/\cos(3x)$ . By (6), the *J*-fraction associated with the generating function of this sequence begins

$$J(x) = \frac{1}{1 - 23x - \frac{(1 \times 2)(24x)^2}{1 - 167x - \frac{(5 \times 7)(24x)^2}{1 - 455x - \ddots}}}$$

Applying Proposition 1 with a = 2 and b = 3 we find that the binomial transform of the ordinary generating function of A002439 corresponds to the elegant S-fraction

		1
1		$1 \times 24x$
1	1	$2 \times 24x$
	1 -	$5 \times 24x$
	$1 - \frac{1}{7 \times 24x}$	
		$1 = \frac{1}{1 - \cdots},$

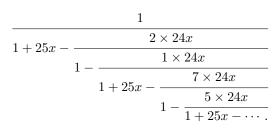
where the multiplicands 1,2,5,7,... in the partial numerators are the generalized pentagonal numbers A001318. Thus the o.g.f. of Glaisher's T-numbers has the continued fraction representation

		1
1 + ~		$1 \times 24x$
1 + x -	1	$2 \times 24x$
	1 -	$5 \times 24x$
	$1+x-\frac{1}{7\times 24x}$	
		$1 - \frac{1}{1 + x - \cdots}$

By Proposition 2, the  $25^{th}$  binomial transform of the ordinary generating function of A002439 corresponds to the S-fraction

		1
1		$2 \times 24x$
1	1	$1 \times 24x$
	1 -	$7 \times 24x$
	$1 - \frac{1}{1 - \frac{5 \times 24x}{1 - \cdots}}$	

where now where the multiplicands 2,1,7,5,... in the partial numerators are obtained by swapping adjacent generalized pentagonal numbers. Thus we have an alternative representation for the o.g.f. of Glaisher's T-numbers as



Corollary 1 in this case leads to the continued fraction identity

$$\frac{1}{1 - \frac{x}{1 - \frac{2x}{1 - \frac{5x}{1 - \frac{7x}{1 - \cdots}}}}} = \frac{1}{1 + x - \frac{2x}{1 - \frac{2x}{1 - \frac{x}{1 + x - \frac{7x}{1 - \frac{5x}{1 + x - \cdots}}}}}$$
(11)

**Remark 1.** It follows from a formula of Zagier for the terms of A079144 that the S-fraction on the left-hand side of (11) is a generating function for the number of labeled interval orders on n elements (the number of (2+2)-free posets).

**3.** Consider now the exponential generating function  $\cos(ax)/\cos(bx)$ , with a and b constants. The Taylor expansion of the function about x = 0 begins

$$\frac{\cos(ax)}{\cos(bx)} = 1 - (b^2 - a^2) \frac{x^2}{2!} + (5b^4 - 16b^2a^2 + a^4) \frac{x^4}{4!} - (61b^6 - 75b^4a^2 + 15b^2a^4 - a^6) \frac{x^6}{6!} + \cdots$$
(12)

The coefficients in the expansion are homogeneous polynomials in a and b. See A086646 for information about these polynomials. Let  $C_{a,b}(x)$  denote the ordinary generating function for this sequence of polynomials (taken with an offset of 0):

$$C_{a,b}(x) = 1 - (b^2 - a^2) x + (5b^4 - 16b^2a^2 + a^4) x^2 - (61b^6 - 75b^4a^2 + 15b^2a^4 - a^6) x^3 + \cdots$$

The *J*-fraction associated to  $C_{a,b}(x)$ , which turns out to be an *S*-fraction, is essentially due to Stieltjes and can be found by applying Stieltjes' expansion theorem, [2, Chapter 11, Section 53] to an addition formula satisfied by  $\cos(ax)/\cos(bx)$ . The result is

$$C_{a,b}(x) = \frac{1}{1 - \frac{a_1 x}{1 - \frac{a_2 x}{1 - \frac{a_3 x}{1 - \cdots}}}}$$
(13)

where

$$a_{2n} = (2nb)^2,$$
  
 $a_{2n+1} = (2n+1)^2b^2 - a^2.$ 

The same method used in Section 2 to prove Proposition 1 can be used to establish the following result.

**Proposition 3.** Let  $a, b \in \mathbb{C}$ . Let  $C_{a,b}(x)$  be given by (13).

(i) Define  $m = (b - a)^2$ . Then the  $m^{th}$  binomial transform of  $C_{a,b}(x)$  equals the S-fraction

$$\frac{\frac{1}{1-\frac{2b(b-a)x}{1-\frac{2b(b+a)x}{1-\frac{4b(3b-a)x}{1-\frac{4b(3b+a)x}{1-\frac{6b(5b-a)x}{1-\frac{6b(5b+a)x}{1-\frac{6b(5b+a)x}{1-\cdots}}}}}$$
(14)

(ii) Define  $M = (b+a)^2$ . Then the  $M^{th}$  binomial transform of  $C_{a,b}(x)$  equals the S-fraction

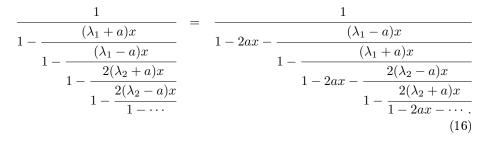
$$\frac{1}{1 - \frac{2b(b+a)x}{1 - \frac{2b(b-a)x}{1 - \frac{4b(3b+a)x}{1 - \frac{4b(3b-a)x}{1 - \frac{6b(5b+a)x}{1 - \frac{6b(5b-a)x}{1 - \frac{6b(5b-a)x}{1 - \cdots}}}}}$$
(15)

Since (15) is the  $(M - m)^{th} = (4ab)^{th}$  binomial transform of (14) then by (4) we have the following identity:

## Corollary 2.

$$\frac{1}{1 - \frac{2b(b+a)x}{1 - \frac{2b(b-a)x}{1 - \frac{4b(3b+a)x}{1 - \frac{4b(3b-a)x}{1 - \cdots}}}} = \frac{1}{1 - 4abx - \frac{2b(b-a)x}{1 - \frac{2b(b+a)x}{1 - 4abx - \frac{4b(3b-a)x}{1 - \frac{4b(3b+a)x}{1 - 4abx - \cdots}}}$$

**Remark 2.** The continued fraction identities in Corollaries 1 and 2 appear to be particular cases of a more general identity



where  $\{\lambda_n\}_{n\geq 1}$  is an arbitrary sequence. Indeed, the 2*n*-th approximants of the left and right sides of (16) seem to be identically equal.

## References

- T. J. Stieltjes, Sur quelques intégrales définies et leur développement en fraction continues, *Oeuvres*, vol. 2, 378-394.
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- [2] H. S. Wall, Analytic Theory of Continued Fractions. Reprinted by AMS Chelsea Publishing, 2000.