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ON COTESIAN NUMBERS: THEIR HISTORY, COMPUTATION, AND VALUES TO $n=20$.

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1. THE numbers which form the subject of this paper arise in the problem of the Quadrature of Plane Curves, that is, the expression of area in terms of base and certain ordinates. They form a series of which I here present an extension of the known values, prefaced by some account of their history and modes of computation.

Concerning the problem of quadrature, George Atwood, in his "Disquisition on the Stability of Ships,"* says: "The methods of approximation to be used for the quadrature of curvilinear spaces are founded on Sir Isaac Newton's discovery of a theorem by which, from having given any number of points in the same plane, he could ascertain the equation of the curve which would pass through them all, and by means of this equation was enabled to express the ordinate of the curve, corresponding to any abscissa of any given length, as well as the area intercepted by any two of the ordinates. This discovery the author himself considered amongst his happiest inventions."

James Stirling says that Newton in a letter to Oldenburg, dated October 24, 1676, refers to his "expeditious method of passing a parabolic curve through given points,"† and that it was first published as Lemma V., Book III., of the *Principia*. This lemma, which is entitled "Invenire lineam curvam, generi parabolici, quae per data quotcunque puncta transibit," was according to Boole (*Finite Differences*, 2nd ed., p. 61) "the first attempt at finding an interpolation formula, and gives a complete solution of the problem."

In the lemma two cases are considered; first, that in which the abscissae of the given points are in arithmetical progression, the result, or general value of y, being equivalent to the usual interpolation formula in terms of the differences of the given ordinates; in the second or general case, a kind of modified differences are used in a similar manner. Of the areas under the curve, it is only remarked that they can be

* *Phil. Trans.*, vol. 88 (1798), p. 260.

† This letter will be found in the *Commercium Epistolicum*. Newton's exact words are: "Ejus fundamentum est commoda, expedita, generalis solutio hujus Problematis, Curvam Geometricam describere quae per data quotcunque Puncta transibit."

found because the quadrature of a parabolic curve can be effected.

2. Among the "opusculae" published with the *Harmonia Mensurarum* of Roger Cotes is one entitled "De Methodo differentiali Newtoniana." In the introductory paragraph, Cotes refers to Newton's Lemma V., and proposes to supply the demonstrations and to make some additions, which he proceeds to do in a series of propositions. In a postscript he tells us that he gave these propositions in lectures in 1709, not being at the time aware that Newton had given equivalent propositions, as he had since learned from a tract sent to him in 1711 by the editor, William Jones*. Cotes mentions that in this tract Newton refers to the utility of the method in finding areas, and gives in particular the area included between the extremes of four given equidistant ordinates, namely, $\dfrac{A+3B}{8}R$, where A is the sum of the extreme and B that of the mean ordinates, R being the base of the area. This Cotes calls "pulcherrima et utilissima visa regula." He then proceeds to give similar rules for the areas included between the extremes of equidistant given ordinates up to the case of eleven ordinates.

3. If now $n+1$ is the number of points (x_r, y_r) $[r=0, 1, \ldots, n]$ given in rectangular coordinates, the "parabolic" form assumed by Newton for the equation of the curve is

$$y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \;\ldots\ldots\ldots\ldots (1),$$

the conditions are sufficient to determine the coefficients $a_0, \ldots, a_n$ linearly in terms of the known coordinates, and therefore also to determine any area in the form

$$\int_p^q y\,dx = (A_0y_0 + A_1y_1 + \ldots + A_ny_n)\,R \ldots\ldots\ldots (2),$$

where R denotes the base $q-p$, and the coefficients $A_0, \ldots, A_n$ depend on the abscissae. Restricting ourselves however, with Cotes, to the case in which the limits are the extreme abscissae x_0 and x_n, and the abscissae are in arithmetical progression,

* This tract is the *Methodus Differentialis* reprinted in vol. i. of Horseley's edition of Newton's Works.

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the quantity in parentheses becomes a mean value of the ordinate *independent of the base.* Denoting this mean by $\bar{y}$, we have

$$\bar{y} = A_0 y_0 + A_1 y_1 + \ldots + A_r y_r + \ldots + A_n y_n \ldots\ldots (3),$$

where the coefficient of y_r is a function of n and r, which, when it is desired to put the value of n in evidence, may be written ${}^nA_r$. Since the given points may be reversed in order, we must have ${}^nA_r = {}^nA_{n-r}$; and, from the nature of a mean value, for each value of n, we must have

$$\sum_{r=0}^{r=n} {}^nA_r = 1.$$

The numbers ${}^nA_r$ are properly called Cotes's numbers, since he first recognized their existence as a series, and gave their values for all values of n up to $n = 10$.

The values for $n = 1$ are of course embodied in the "Trapezoidal Rule" and those for $n = 2$ in the "Parabolic Rule," usually called Simpson's Rule, having been, as Atwood says, "demonstrated by Simpson from the properties of the Conic Parabola." This rule was, however, given in effect by James Gregory in his *Exercitationes Geometricae*, published in 1668.

4. In the subsequent history of these numbers, Cotes's work was for many years lost sight of. The *Harmonia Mensurarum* was not published until 1722, (six years after Cotes's death), by Robert Smith, Cotes's successor as Plumian Professor at Cambridge.

Meantime, James Stirling in 1718 presented to the Royal Society a paper entitled: "Methodus differentialis Newtoniana Illustrata," *Phil. Trans.*, vol. xxx. (1719), pp. 1050–1070. In this paper, Stirling refers to the areas when the given ordinates are equidistant, and says, p. 1063, "Sed quoniam laborosum nimis esset semper recurrere ad Parabolam, computavi Tabulam sequentem, quâ Areae directe exhibentur ex datis Ordinatis." He then gives the areas for *even* values of n up to $n = 10$, that is to the case of eleven ordinates. His reason for ignoring the cases of uneven values of n is curious. He says: "Tabulam pro pare numero Ordinatorum non apposui, quoniam Area, ceteris paribus, ex impare earum numero accuratius* definitur." Other passages also indicate

* The accuracy of the rules naturally increases with the number of ordinates used. Thus it is *not* true that the rule for $n = 2$ will give a more accurate result than that for $n = 3$, when applied to a given area as a whole. But *when the same ordinates*

that Stirling was interested in the rules rather from the practical point of view than from the mathematical interest of the coefficients as a series of related numbers.

In 1730, Stirling published a separate treatise entitled *Methodus Differentialis* which is a considerable expansion of the *Phil. Trans.* paper. In repeating his table of areas, however, he curiously enough omits the case of $n = 10$, so that subsequent writers have usually referred to his table as corresponding to even values of n up to $n = 8$.

5. Thomas Simpson, in his *Mathematical Dissertations on a Variety of Physical and Analytical subjects* published in 1743, gives the rules up to $n = 6$*. He was ignorant of the work of Cotes in this line, though he elsewhere refers to Cotes's Property of the Circle.

Simpson seems to have been the first to suggest using, on account of their simplicity, a repetition of the coefficients for $n = 2$, in the case of any even number of intervals. This extension in fact constitutes what is usually known as Simpson's First Rule. I have not seen, however, any evidence that he recommended the similar extension of the rule for $n = 3$, which has generally been called Simpson's Second Rule.

6. Atwood again had no knowledge of Cotes's work. In the disquisition quoted from at the beginning of this paper he gives Stirling's Table of Areas up to $n = 8$, and supplies from his own computation the entries for uneven values of n. It is curious that in his values for $n = 7$ he fails to remove the common factor 49 from the numerators and the common

are used, as for example, when the base is divided into six parts and three partial areas are computed by the first rule, and again two partial areas are computed by the second rule, then the former or rule for $n = 2$ *does* give a more accurate result than the latter or rule for $n = 3$. This was proved by C. W. Merrifield in 1865 (*Trans. of the Naval Institute*, vol. vi., p. 40), the contrary having previously been supposed to be the case. Merrifield later found the same advantage of the even number to hold in the comparison of the rules for $n = 4$ and $n = 5$, and he conjectured the same thing to be generally true (*Report of the British Association for* 1880, p. 338 and p. 340). It follows that, if in the passage quoted we could hold that "ceteris paribus" requires the same ordinates to be used, while "pare" and "impare" refer only to the numbers used in the rules employed, we should have to credit Stirling with having anticipated Merrifield in this discovery.

* Merrifield, in the paper quoted in the preceding footnote, being at the time unacquainted with Cotes's work, speaks of the "true rule for six intervals" as "given by Stirling first and afterward appropriated by Simpson" This is absolutely unjust to Simpson, who in his preface gives due credit to Stirling and others In fact, in his text, he gives the interpolation formula in differences, and integrates it, exactly in the same manner, for example, by Boole, and then proceeds to give the results of substitution in terms of the ordinates, as far as $n = 6$, by way of examples.

denominator. In Merrifield's report on the subject of Interpolation and Quadrature (*British Association Report* for 1880, p. 321) a note occurs (p. 337) in which, having previously stated that Simpson had given values to $n = 8$, Merrifield refers to this error of Atwood's, and adds "he is endeavoring to correct Simpson." I have not, however, found that Simpson gave values beyond $n = 6$; and it seems, moreover, very unlikely that Atwood should not have detected his error if he had had before him any previous computation of the values for $n = 7$.

Merrifield in this report gives all the values up to $n = 10$, as taken from Cotes's *Harmonia Mensurarum*, and this is in fact the earliest reference which I have found to Cotes's numerical work.

7. With regard to methods of computation, the most obvious one is to integrate equation (1), and then eliminate the coefficients $a_0, a_1, \ldots, a_n$ by means of the $n+1$ conditions expressing that the curve passes through the given points. This is done in many text books for the cases $n=2$ and $n=3$; and Merrifield, in the report mentioned above, using undetermined multipliers, applies the method to the general case, using separate treatments for even and uneven values of n, since it is convenient to place the origin at the middle point of the base. No general expressions for ${}^nA_r$ result from this method.

8. Since the numbers are independent of the base, the common interval between consecutive abscissae, as well as the origin, may be assumed at pleasure. The usual method of computation employs the interpolation formula in terms of differences of the ordinates, thus placing the origin at the left-hand extremity of the base, and taking the common interval as the unit. By integrating this from 0 to n, we have a general expression for the area, in which the differences beyond the nth must be assumed to vanish. It remains, for each value of n, to eliminate the differences $\Delta y, \Delta^2 y, \ldots, \Delta^n y$ by means of their values in terms of $y_0, y_1, \ldots, y_n$. The general expression in terms of differences is given in Boole's *Finite Differences* as far as $\Delta^6 y$, and in Weddle's paper in the *Cambridge and Dublin Mathematical Journal*, vol. ix., p. 79, as far as $\Delta^8 y$. This method is probably equivalent to that employed by Cotes himself. It again fails to give a general expression for ${}^nA_r$.

9. Such general expressions, however, result from the integration of Lagrange's Interpolation Formula, by which the equation of the curve is given directly in terms of the ordinates. In its general form this is

$$y = \sum_{r=0}^{r=n} \frac{(x-x_0)(x-x_1)\ldots(x-x_{r-1})(x-x_{r+1})\ldots(x-x_n)}{(x_r-x_0)(x_r-x_1)\ldots(x_r-x_{r-1})(x_r-x_{r+1})\ldots(x_r-x_n)}\, y_r$$

$$= \sum_{r=0}^{r=n} \frac{X_r}{M_r}\, y_r{}^* \,\ldots\ldots\ldots(4),$$

where M_r represents what the numerator becomes when $x = x_r$. The coefficient of y_r in the expression for any area is therefore

$$\int_p^q \frac{X_r}{M_r}\, dx.$$

Bertrand in his *Calcul Intégral*, 1870, p. 333, gives the value of ${}^nA_r$ in the form of a definite integral whose limits are 0 and 1. Since the base is thus made unity, the coefficient of y_r in the expression for the area serves also for the mean ordinate $\bar{y}$; whence

$${}^nA_r = \int_0^1 \frac{X_r}{M_r}\, dx, \quad \text{where } x_r = \frac{r}{n}.$$

Bertrand proceeds to give the numerical values up to $n = 10$ in the form of fractions each reduced to its lowest terms. Although this section of the *Calcul Intégral* is headed "Méthode d'interpolation de Cotes," it seems certain that Bertrand was not aware that Cotes had given numerical results. Carr in his *Synopsis of Mathematics* gives Bertrand's formula, and quotes the numerical results which he expressly states that he has verified by independent calculation, correcting two misprints. He certainly was not acquainted with Cotes's work, otherwise he would have mentioned that the values were all correctly given in the *Harmonia Mensurarum*.

10. In computing the values of ${}^nA_r$ from $n = 11$ to $n = 20$, I have found it more convenient to take the origin at the middle of the base, so that in the integration of the expansion of X_r even powers of x only need be considered. This, like

* This is verified at sight as the equation of the parabolic curve of the nth degree which passes through the given points $(x_0, y_0), \ldots, (x_n, y_n)$.

Merrifield's method, requires the use of separate formulæ for even and uneven values of n. Those I have employed are developed below, that for even values is given in a slightly different form in Boole's *Finite Differences*, 2nd edition, p. 50.

Putting in this case $n=2m$, we take unity for the common interval, so that the base is n. The abscissae corresponding to

$$r=0, \quad 1, \quad \ldots, \quad m, \; m+1, \; \ldots, \; 2m,$$

are

$$x_r=-m, \; -(m-1), \; \ldots, \; 0, \quad 1, \quad \ldots, \quad m;$$

and our general formula for even values of n is

$$^{2m}A_r=\frac{1}{2m}\int_{-m}^{m}\frac{X_r}{M_r}\,dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The series of values of A_r for a given value of n were computed in conjunction, and verified by the formula $\Sigma A_r=1$.

Consider first the denominators M_r. Referring to equation (4), these are

$$M_0=(2m)!=n!, \quad M_1=-(n-1)!,$$

$$M_2=(n-2)!\,2!, \quad M_3=-(n-3)!\;3!, \text{ etc.}$$

Hence, if we reduce the series to the common denominator $n!$, these denominators contribute to the numerators, or values of $\frac{1}{n}\int_{-m}^{m} X_r dx$, the factors

$$1, \quad -n, \quad \frac{n(n-1)}{2}, \quad -\frac{n(n-1)(n-2)}{2.3}, \text{ etc.,}$$

that is to say, the coefficients of the expansion of $(1-z)^n$. These binomial coefficients were placed at the feet of the columns in which were collected the terms of the expanded numerators for the several values of r.

11. The integrand X_r, from which the factor $x-x_r$ is absent, may be written

$$X_r=(x^2-m^2)\{x^2-(m-1)^2\}\ldots(x+x_r)\ldots(x^2-2^2)(x^2-1^2)\,x\ldots (6),$$

in which all the factors, except x^2-xx_r, are pure quadratics (except in the case of $r=m$, when x is the omitted factor and there is no complementary factor $x+r$). Since the integrals of uneven powers of x vanish for the given limits,

we may drop the term xx_r, and use a modified integrand derived from the expression

$$(x^2-m^2)\{x^2-(m-1)^2\}\ldots(x^2-2^2)(x^2-1^2)\,x^2,$$

by omitting, for each value of r, one quadratic factor, x^2 being the omitted factor in the middle case $r=m$, and the integrands for $r=m+q$ and $r=m-q$ being the same.

Thus, for the middle case, the expanded value of the integrand is

$$x^{2m}-S_1x^{2m-2}+S_2x^{2m-4}-\ldots+(-1)^m S_m,$$

where S_1 denotes the sum of the elements $1^2, 2^2, 3^2, \ldots, m^2$, S_2 the sum of the products of pairs, S_3 the sum of the products of triads, and so on; S_m being the product of all the elements.

For any other value of r, the modified integrand in expanded form is the similar expression

$$x^{2m}-S'_1x^{2m-2}+S'_2x^{2m-4}-\ldots-(-1)^m S'_{m-1}x^2,$$

where the accents refer to the omission of one element in forming the S''s, and S'_{m-1} is the product of the $m-1$ elements used.

12. Integrating this expression, and dividing by the base $2m=n$, we have, for the numerators mentioned in § 10,

$$\frac{1}{2m}\int_{-m}^{m} X_r dx=\frac{m^{2m}}{2m+1}-S_1'\frac{m^{2m-2}}{2m-1}+\ldots$$

$$-(-1)^m S'_{m-1}\frac{m^2}{3}+(-1)^m S'_m \ldots\ldots (7),$$

the last term vanishing except when $r=m$. These expressions, corresponding to the several values of r, must be multiplied by the binomial coefficients, referred to as foot-numbers in § 10, in order to produce the numerators of $^{2m}A_r$ when reduced to the common denominator $n!$.

13. The first step in the computation of the numbers for a given value of m was the formation of a table of the values of S_s', in which the rows corresponded to values of s and the columns to values of r, the top row, for $s=0$, being a row of

units. A second table, in which the entries for the several rows were divided by the odd numbers which appear in equation (7), was then formed, so that its entries were the coefficients of sucessive powers of m^2.

An example of this "divided table" constructed for the value $n = 10$ is subjoined. [The factor $m^2 = 25$ has been removed from the powers of m^2 on the right, except in the case of the last row, where it could be removed from the single tabular entry as well as from the common denominator].

$n = 10$.

$s \backslash r$	0	1	2	3	4	5	
0	$+\frac{1}{11}$	$+\frac{1}{11}$	$+\frac{1}{11}$	$+\frac{1}{11}$	$+\frac{1}{11}$	$+\frac{1}{11}$	25^4
1	$-3\frac{1}{3}$	$-4\frac{1}{3}$	$-5\frac{1}{9}$	$-5\frac{2}{3}$	-6	$-6\frac{1}{9}$	25^3
2	$+39$	$+57$	$+87$	$+117$	$+138\frac{3}{7}$	$+146\frac{1}{7}$	25^2
3	-164	$-252\frac{1}{5}$	$-432\frac{4}{5}$	$-873\frac{4}{5}$	$-1335\frac{1}{5}$	-1529	25
4	$+192$	$+300$	$+533\frac{1}{3}$	$+1200$	$+4800$	$+7025\frac{1}{3}$	1
5						-576	1
	$+1$	-10	$+45$	-120	$+210$	-252	

By the aid of these divided tables, the summations for the several columns could now be made by alternate multiplication by m^2 and addition of a tabular entry, and final multiplication by the foot-number.

14. The results thus found were in the form of mixed numbers, whose algebraic sum should equal the common denominator at this stage. It remained to reduce the values of A_r, now in the form of complex fractions, by multiplication by such of the odd denominators as had not disappeared in the process.

It is noteworthy, however, that with the exception of the greatest, $n+1$, every other denominator greater than $\frac{1}{2}n+1$ (or at least a factor of it) invariably disappeared. This was due to the fact that, in the table of S's, the values of S'_s in any row were all divisible by this denominator $(2m+1-2s)$, or a factor of it, until we reached the point where it became a factor of the foot-numbers, generally remaining such until the end of the row. For instance, in the table given above, the absence of fractions in the row for $s = 2$ shows that the

corresponding S''s were all divisible by 7, except in the last two columns where the foot number is divisible by 7. So in the next row the S''s in the first and last columns were divisible by 5, and the foot numbers in the other columns are divisible by 5. Thus 11 and 3 were the only denominators, in this case, which appeared in the mixed numbers. It was further found that in every case after clearing of these denominators a high power of 2 could be removed. Again m or one of its factors could in many cases be divided out. Thus the denominators are considerably smaller than might have been expected.

15. In the case of uneven values of n, the common interval between the abscissae was taken as two units. The abscissae corresponding to

$$r = 0, \quad 1, \quad 2, \quad \ldots, \tfrac{1}{2}(n-1), \tfrac{1}{2}(n+1), \ldots, n-1, n,$$

are therefore

$$x_r = -n, -(n-2), -(n-4), \ldots, \quad -1, \quad +1, \quad \ldots, n-2, n,$$

and the base is $2n$. We have now for the general expression, when n is uneven,

$$^nA_r = \frac{1}{2n}\int_{-n}^{n} \frac{X_r}{M_r}\,dx \ldots\ldots\ldots\ldots\ldots\ldots (8);$$

and for the denominators

$$M_0 = -2^n n!, \quad M_1 = 2^n(n-1)!, \quad M_2 = -2^n(n-2)!2!, \text{ etc.},$$

so that, reducing to the common denominator $2^n n!$, they contribute to the numerators the factors

$$-1,\ n,\ -\frac{n(n-1)}{2},\ \frac{n(n-1)(n-2)}{2.3},\ \text{etc.}$$

The value of X_r, equation (4), is now

$$X_r = (x^2-n^2)\{x^2-(n-2)^2\}\ldots(x+x_r)\ldots(x^2-3^2)(x^2-1^2)\ldots(9).$$

In this case, the fact that uneven powers of x disappear in the integration allows us to reject the term x from the factor $x + x_r$. The remaining factors x_r were united with the binomial coefficients above, thus giving [after removing the common factor n, which reduces the denominator to $2^n(n-1)!$] the foot-numbers

$$1,\ -(n-2),\ \frac{(n-1)(n-4)}{2},\ \frac{-(n-1)(n-2)(n-6)}{2.3},\ \text{etc.}$$

16. In the values of the numerators $\frac{1}{2n}\int_{-n}^{n} X_r\,dx$, we may now use a modified integrand, derived from the expression

$$(x^2-n^2)\{x^2-(n-2)^2\}\ldots(x^2-3^2)(x^2-1^2),$$

by omitting, for each value of r, one of the quadratic factors. The expanded value of any integrand is thus of the form

$$x^{n-1}-S_1'x^{n-3}+S_2'x^{n-5}-\ldots+(-1)^{\frac{1}{2}(n-1)}S'_{\frac{1}{2}(n-1)}$$

in which the S''s are formed from the elements $1^2, 3^2, 5^2, \ldots, n^2$ by the omission, in each case, of one element.

Integrating, and dividing by the base $2n$, we have

$$\frac{1}{2n}\int_{-n}^{n} X_r\,dx = n^{n-1} - S_1'\frac{n^{n-3}}{n-2} + S_2'\frac{n^{n-5}}{n-4} - \ldots$$

$$+(-1)^{\frac{1}{2}(n-1)}S'_{\frac{1}{2}(n-1)}\ldots\ldots(10).$$

To form the numerators of ${}^nA_r$, as reduced to the common denominator $2^n(n-1)!$, these expressions must be multiplied by the foot-numbers given in § 15.

17. The computation took the same form as before, and again it was remarkable that, of the odd divisors $n-2$, $n-4$, etc., all the greater ones, in fact all primes greater than $\frac{1}{2}(n+1)$, disappeared by virtue of the divisibility of the corresponding S''s, except when the foot-number was itself so divisible; and this exception held even when the divisibility of the foot-number was due to the factor x_r carried into the foot-number as noticed in § 15, so that the sequence of divisible numbers in the S' table was an interrupted one.

18. With respect to the whole series of Cotesian numbers given below, it is noteworthy that no negative numbers occur for even values of n until $n=8$; and none for odd values until $n=11$. But, from these points on, the first two numbers are invariably positive, and then the values alternate in sign up to the middle number or middle pair of numbers.

As n increases, the denominators increase, but not uniformly; composite values of n tending to diminish them. The numerical values of the numerators tend to increase toward the middle values of r; so much so that the numerical value of ${}^nA_r$ for the most part exceeds unity and in one case rises as high as 90.

It may also be noticed that the value of N_0 is in every case an odd number.

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VALUES OF THE COTESIAN NUMBERS ${}^nA_r = \frac{{}^nN_r}{D_n}$. 2176

Values of ${}^nN_r$ and D_n from $n=1$ to $n=20$.

r	0	1	2	3	4	5	6	7	D_n
$n=1$	1	1							2
$n=2$	1	4	1						6
$n=3$	1	3	3	1					8
$n=4$	7	32	12	32	7				90
$n=5$	19	75	50	50	75	19			288
$n=6$	41	216	27	272	27	216	41		840
$n=7$	751	3577	1323	2989	2989	1323	3577	751	17280

$n=8$.

$N_0 = N_8 = +\ 989$
$N_1 = N_7 = +\ 5888$
$N_2 = N_6 = -\ 928$
$N_3 = N_5 = +\ 10496$
$N_4 = -\ 4540$
$D = 28350$

$n=9$.

N
N
N
N
N

$n=10$.

$N_0 = N_{10} = +\ 16067$
$N_1 = N_9 = +\ 1\,06300$
$N_2 = N_8 = -\ 48525$
$N_3 = N_7 = +\ 2\,72400$
$N_4 = N_6 = -\ 2\,60550$
$N_5 = +\ 4\,27368$
$D = 5\,98752$

$N_0 =$
$N_1 =$
$N_2 =$
$N_3 =$
$N_4 =$
$N_5 =$

$n=12$.

$N_0 = N_{12} = +\ 13\,64651$
$N_1 = N_{11} = +\ 99\,03168$
$N_2 = N_{10} = -\ 75\,87864$
$N_3 = N_9 = +\ 357\,25120$
$N_4 = N_8 = -\ 514\,91295$
$N_5 = N_7 = +\ 875\,16288$
$N_6 = -\ 877\,97136$
$D = 630\,63000$

$N_0 = N_{13} = +\ 61376\,98213$
$N_1 = N_{12} = +\ 4\,21942\,38652$
$N_2 = N_{11} = -\ 2\,33615\,40993$
$N_3 = N_{10} = +\ 11\,67782\,74403$
$N_4 = N_9 = -\ 11\,32197\,77650$
$N_5 = N_8 = +\ 15\,44245\,90209$
$N_6 = N_7 = -\ 3\,20679\,78834$
$D = 30\,17710\,08000$

$n = 14.$

$N_0 = N_{14}$	$= +$	902 41897
$N_1 = N_{13}$	$= +$	7109 86864
$N_2 = N_{12}$	$= -$	7707 20657
$N_3 = N_{11}$	$= +$	35014 42784
$N_4 = N_{10}$	$= -$	66250 93363
$N_5 = N_9$	$= +$	1 26301 21616
$N_6 = N_8$	$= -$	1 68022 70373
N_7	$= +$	1 95344 38464
D	$=$	50038 56000

$n = 15.$

$N_0 = N_{15}$	$= +$	1059 30069
$N_1 = N_{14}$	$= +$	7966 61595
$N_2 = N_{13}$	$= -$	6988 08195
$N_3 = N_{12}$	$= +$	31433 32755
$N_4 = N_{11}$	$= -$	46885 22055
$N_5 = N_{10}$	$= +$	73856 54007
$N_6 = N_9$	$= -$	60009 98415
$N_7 = N_8$	$= +$	30564 22815
D	$=$	61993 45152

$n = 16.$

$N_0 = N_{16}$	$= +$	1 50436 11773
$N_1 = N_{15}$	$= +$	12 76266 06592
$N_2 = N_{14}$	$= -$	17 97311 34720
$N_3 = N_{13}$	$= +$	83 22118 55360
$N_4 = N_{12}$	$= -$	192 94986 07520
$N_5 = N_{11}$	$= +$	417 75888 93696
$N_6 = N_{10}$	$= -$	680 65344 07936
$N_7 = N_9$	$= +$	936 88750 18240
N_8	$= -$	1023 42389 72220
D	$=$	97 69246 98750

$n = 17.$

$N_0 = N_{17}$	$= +$	5529 47208 74657
$N_1 = N_{16}$	$= +$	45018 55154 46285
$N_2 = N_{15}$	$= -$	54202 34370 08852
$N_3 = N_{14}$	$= +$	2 42863 65257 64260
$N_4 = N_{13}$	$= -$	4 76891 68001 23440
$N_5 = N_{12}$	$= +$	8 85541 66486 84984
$N_6 = N_{11}$	$= -$	10 9053 718597 96660
$N_7 = N_{10}$	$= +$	10 06961 57501 32836
$N_8 = N_9$	$= -$	3 75978 59740 54070
D	$=$	3 76610 21798 40000

$n = 18.$

$N_0 = N_{18}$	$= +$	20 37323 52169
$N_1 = N_{17}$	$= +$	184 87302 21900
$N_2 = N_{16}$	$= -$	321 27443 74395
$N_3 = N_{15}$	$= +$	1552 98303 12096
$N_4 = N_{14}$	$= -$	4236 86306 85840
$N_5 = N_{13}$	$= +$	10368 05634 65808
$N_6 = N_{12}$	$= -$	19864 84298 67720
$N_7 = N_{11}$	$= +$	31903 57844 79840
$N_8 = N_{10}$	$= -$	41912 79511 14198
N_9	$= +$	46132 73443 40680
D	$=$	1520 91139 20000

$n = 19.$

$N_0 = N_{19}$	$= +$	69 02876 31556 44023
$N_1 = N_{18}$	$= +$	603 65208 22708 08125
$N_2 = N_{17}$	$= -$	926 84051 57002 22955
$N_3 = N_{16}$	$= +$	4301 58153 84505 00095
$N_4 = N_{15}$	$= -$	10343 69223 42431 92788
$N_5 = N_{14}$	$= +$	22336 42032 84799 61316
$N_6 = N_{13}$	$= -$	35331 88842 11147 81580
$N_7 = N_{12}$	$= +$	43920 76837 05651 35580
$N_8 = N_{11}$	$= -$	37088 37026 13798 51390
$N_9 = N_{10}$	$= +$	15148 33730 59217 59574
D	$=$	5377 99391 28115 20000

$n = 20.$

$N_0 = N_{20}$	$= +$	1947 01402 41329
$N_1 = N_{19}$	$= +$	18792 60903 80000
$N_2 = N_{18}$	$= -$	38935 81941 77500
$N_3 = N_{17}$	$= +$	1 98596 91593 40000
$N_4 = N_{16}$	$= -$	6 20894 88358 89375
$N_5 = N_{15}$	$= +$	17 01938 77765 17504
$N_6 = N_{14}$	$= -$	37 38973 46712 90000
$N_7 = N_{13}$	$= +$	68 86928 75743 20000
$N_8 = N_{12}$	$= -$	105 49901 48137 01250
$N_9 = N_{11}$	$= +$	136 32452 17984 40000
N_{10}	$= -$	148 19252 66072 80936
D	$=$	1 64648 54410 80480