ON THE FUNCTION $\chi(n)$.

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Definitions of $\chi(n)$, §§ 1 – 3.

§ 1. If an uneven number n be the sum of two squares, one of them must be uneven and one even; and if

$$n = a_1^2 + b_1^2 = a_2^2 + b_2^2 = \ldots,$$

where $a_1, a_2, \ldots$ are uneven, then $\chi(n)$ is defined to be the quantity

$$(-1)^{\frac{1}{2}(a_1+b_1-1)}\, 2a_1 + (-1)^{\frac{1}{2}(a_2+b_2-1)}\, 2a_2 + \ldots.$$

Every resolution a^2+b^2 of n as a sum of two squares thus gives rise to a term $(-1)^{\frac{1}{2}(a+b-1)}\, 2a$ in $\chi(n)$; but if n be itself a square $= a^2$, the term due to the resolution $n = a^2$ is $(-1)^{\frac{1}{2}(a-1)}\, a$ (in which the coefficient 2 does not occur).

Thus, for example,

(i) $$25 = 3^2 + 4^2 = 5^2,$$

and therefore

$$\chi(25) = (-1)^{\frac{1}{2}(3+4-1)} \times 6 + (-1)^{\frac{1}{2}(5-1)} \times 5 = -6 + 5 = -1;$$

(ii) $$65 = 1^2 + 8^2 = 7^2 + 4^2,$$

and therefore

$$\chi(65) = (-1)^{\frac{1}{2}(1+8-1)} \times 2 + (-1)^{\frac{1}{2}(7+4-1)} \times 14 = 2 - 14 = -12;$$

(iii) $$169 = 5^2 + 12^2 = 13^2,$$

and therefore

$$\chi(169) = (-1)^{\frac{1}{2}(5+12-1)} \times 10 + (-1)^{\frac{1}{2}(13-1)} \times 13 = 10 + 12 = 23.$$

§ 2. If $a + ib$ be any complex number, the real number $a^2 + b^2$ was termed by Gauss its norm.

The four associated numbers

$$a + ib,\quad a - ib,\quad -a + ib,\quad -a - ib$$

§ 12. We might have obtained the preceding general expression for $\chi(n)$ by a direct application of the method of §§ 8–10 to the general case in which $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots$, *i.e.* by considering all the numbers which have n as their norm, and by deriving directly from them the value of $\chi(n)$. If this course were followed the formula of § 6, viz.

$$\chi(n) = \chi(n_1)\,\chi(n_2)\,\chi(n_3) \ldots,$$

could be deduced at once from the expression for $\chi(n)$.

Relation connecting $\chi(p^n)$ *and* $\chi(p^{n-2})$, § 13.

§ 13. It is evident from § 8 that the numbers which have p^n as norm consist of (i) those which have p^{n-2} as norm multiplied by p and (ii) the numbers $(a+ib)^n$ and $(a-ib)^n$.

We thus obtain the formulæ

$$\chi(p^n) = p\chi(p^{n-2}) + (a+ib)^n + (a-ib)^n,$$

if n be even; and

$$\chi(p^n) = p\chi(p^{n-2}) + k\{(a+ib)^n + (a-ib)^n\},$$

if n be uneven, where as before k denotes $(-1)^{\frac{1}{4}(a+b-1)}$.

The value of $(a+ib)^n + (a-ib)^n$ expressed in a real form is

$$2\left\{a^n - \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)(n-3)}{4!}a^{n-4}b^4 - \ldots\right\},$$

and if we denote this expression by $\psi_n(a, b)$ we have, for all values of n,

$$\chi(p^n) = (a^2+b^2)\,\chi(p^{n-2}) + (-1)^{\frac{1}{4}(a+b-1)n}\,\psi_n(a, b).$$

This formula is almost as convenient as the formula in § 10, not only for the actual numerical calculation of $\chi(p^2)$, $\chi(p^3)$, &c. when p is a given prime number, but also for the calculation of the expressions for these quantities in terms of a and b.

Algebraical definitions of $\chi(n)$, § 14.

§ 14. The arithmetical definition of $\chi(n)$ given in § 1 is equivalent to defining $\chi(4n+1)$ as the coefficient of x^{4n+1} in the product

$$(x - 3x^9 + 5x^{25} - 7x^{49} + \&c.)\,(1 - 2x^4 + 2x^{16} - 2x^{36} + 2x^{64} - \&c.),$$

and the definition of $\chi(n)$ adopted in the preceding paper (and quoted in § 3 of this paper), is equivalent to defining $\chi(4n+1)$ as the coefficient of x^{8n+2} in the product

$$(x - 3x^9 + 5x^{25} - 7x^{49} + \&c.)\,(x + x^9 + x^{25} + x^{49} + \&c.).$$

We thus have

$$\chi(1)\,x + \chi(5)\,x^5 + \chi(9)\,x^9 + \chi(13)\,x^{13} + \&c.$$
$$= (x - 3x^9 + 5x^{25} - 7x^{49} + \&c.)(1 - 2x^4 + 2x^{16} - 2x^{36} + 2x^{64} - \&c.),$$

and

$$\chi(1)\,x^2 + \chi(5)\,x^{10} + \chi(9)\,x^{18} + \chi(13)\,x^{26} + \&c.$$
$$= (x - 3x^9 + 5x^{25} - 7x^{49} + \&c.)(x + x^9 + x^{25} + x^{49} + \&c.).$$

From these two equations it follows that

$$(x - 3x^9 + 5x^{25} - 7x^{49} + \&c.)(x + x^9 + x^{25} + x^{49} + \&c.)$$
$$= (x^2 - 3x^{18} + 5x^{50} - 7x^{98} + \&c.)(1 - 2x^8 + 2x^{32} - 2x^{72} + \&c.)\,;$$

a result which may be readily verified by Elliptic Functions. It will be observed, that an arithmetical proof of the equivalence of the two definitions in §§ 1 and 3 affords a proof of this identity.

The function $E(n)$, §§ 15–17.

§ 15. There is an intimate connexion, and also in many respects a close resemblance, between $\chi(n)$ and the function $E(n)$ which denotes the excess of the number of divisors of n of the form $4m+1$ over the number of divisors of the form $4m+3$.

This function was referred to in § 13 of the preceding paper, and $\chi(n)$ was there expressed in terms of $E(1)$, $E(5)$, ... $E(2n-1)$ by means of the formula

$$\chi(2m+1) = E(1)\,E(4m+1) - E(5)\,E(4m-3) + E(9)\,E(4m-7)$$
$$\ldots + (-)^{m-1}\,E(4m-3)\,E(5) + (-)^m\,E(4m+1)\,E(1).$$

It was also stated that the uneven values for which $E(n)$ vanishes were the same as those for which $\chi(n)$ vanishes.

§ 16. If $n = a^\alpha b^\beta \ldots r^\rho s^\sigma \ldots$, where a, b, ... are different primes of the form $4m+1$, and r, s, ... different primes of the form $4m+3$, then it can be easily shown that $E(n)$ is equal to the value of

$$(1 + a + a^2 \ldots + a^\alpha) \times (1 + b + b^2 \ldots + b^\beta) \ldots$$
$$\times (1 - r + r^2 \ldots \pm r^\rho) \times (1 - s + s^2 \ldots \pm s^\sigma) \ldots,$$

when a, b, ... r, s, ... are all replaced by unity.

Thus $E(n)$ vanishes unless ρ, σ, ... are all even; and if this condition be fulfilled, we have

$$E(n) = \phi(a^{\alpha}b^{\beta}c^{\gamma}...),$$

where $\phi(n)$ denote the number of divisors of n.

It follows also that, if $n = n_1 n_2 n_3 ...$ where n_1, n_2, n_3 ... are prime to each other,

$$E(n) = E(n_1)\, E(n_2)\, E(n_3)\, ...,$$

and if $n = a^{\alpha}b^{\beta}c^{\gamma}$ ..., where a, b, c, ... are any different primes, then

$$E(n) = E(a^{\alpha})\, E(b^{\beta})\, E(c^{\gamma})\, ...$$

Also if p be a prime of the form $4m + 3$,

$$E(p^{2n-1}) = 0, \quad E(p^{2n}) = 1,$$

and if p be a prime of the form $4m + 1$,

$$E(p^{n}) = n + 1.$$

§ 17. Corresponding to the algebraical formulæ which define $\chi(n)$ in § 14, we have the following formulæ involving $E(n)$, viz.

$$E(1)x + E(5)x^5 + E(9)x^9 + E(13)x^{13} + \&c.$$
$$= (x + x^9 + x^{25} + x^{49} + \&c.)(1 + 2x^4 + 2x^{16} + 2x^{36} + 2x^{64} + \&c.),$$

and

$$E(1)x^2 + E(5)x^{10} + E(9)x^{18} + E(13)x^{26} + \&c.$$
$$= (x + x^9 + x^{25} + x^{49} + x^{81} + \&c.)^2.$$

Comparing the first of these formula with the corresponding χ-formula in § 14, we see that $E(n)$ is equal to the number of compositions of n, supposed uneven, as a sum of two squares, *i.e.* $E(n)$ is equal to the number of primary numbers which have n as norm; and if in the expression which defines $\chi(n)$ in § 1 we take all the terms with the positive sign, and substitute unity for a_1, a_2, ..., that expression becomes equal to $E(n)$.

Thus although $E(n)$ admits of being defined in terms of the real divisors of n, it stands in the same relation to the number of primary numbers of which n is the norm as $\chi(n)$ does to their sum, *i.e.* the number of primary divisors of n which have n as norm is $E(n)$, and their sum is $\chi(n)$.

The fact that $E(n)$ is capable of being defined by means of the real as well as of the complex divisors of n gives rise to the known theorem that the number of the compositions of n as the sum of two squares is equal to the excess of the number of $(4m+1)$-divisors of n over the number of $(4m+3)$-divisors.

If, for n uneven, we define $E(n)$ as the number of primary numbers having n as norm, it follows immediately from the considerations in §§ 5 and 8 that $E(pq) = E(p)\ E(q)$, if p and q be prime; and that if p be a prime of the form $4m+1$, then $E(p^n) = n+1$.

A comparison between either of the above E-formulæ and the corresponding χ-formula in § 14 shows that when $E(n)$ vanishes, $\chi(n)$ must also vanish; for, taking for example the first formula, if $E(n) = 0$, the E-formula shows that n is not expressible as a sum of two squares, and therefore necessarily $\chi(n)$ vanishes.

The converse theorem is also true; for $E(n)$ and $\chi(n)$ vanish for exactly the same values of n.

The definition of $E(n)$ as the excess of the number of $(4m+1)$-divisors of n over the number of $(4m+3)$-divisors assigns a meaning to $E(n)$ when n is even, viz: if n be even $=2^r r$ where r is uneven, then $E(n) = E(r)$; and in general whether n be even or uneven, $E(2^r n) = E(n)$. The function $\chi(n)$, however, has been defined in §§ 1–3 only in the case of n uneven.

The function $\psi(n)$, § 18.

§ 18. The function $\psi(n)$ which denotes the sum of the real divisors of n, and which occurred in conjunction with $\chi(n)$ in the statement of several of the theorems in the previous paper, admits of expression in terms of $E(1)$, $E(5)$, ... $E(2n-1)$ by a formula which differs from that for $\chi(n)$ in § 15 only in having all its terms positive, viz. we have

$$\psi(2m+1) = E(1)\,E(4m+1) + E(5)\,E(4m-3) + E(9)\,E(4m-7)$$
$$\ldots + E(4m-3)\,E(5) + E(4m+1)\,E(1).$$

Thus, by addition and subtraction,

$$\psi(2m+1)+\chi(2m+1) = 2\,\{E(1)\,E(4m+1)+E(9)\ E(4m-7)\ +\ldots\},$$

$$\psi(2m+1)-\chi(2m+1) = 2\,\{E(5)\,E(4m-3)+E(13)\,E(4m-11)+\ldots\}.$$

The quantities $\psi(2m+1) \pm \chi(2m+1)$ occur in the expression of Theorem III. p. 88 of the preceding paper.

When the argument is of the form $4m+1$ there is always a middle term, and if we begin with this middle term the formulæ may be written

$$\chi(4m+1) = (-)^m\{E(4m+1)\,E(4m+1) - 2E(4m-3)\,E(4m+5) + 2E(4m-7)\,E(4m+9) \ldots + (-)^m E(1)\,E(8m+1)\},$$

$$\psi(4m+1) = \{E(4m+1)\,E(4m+1) + 2E(4m-3)\,E(4m+5) + 2E(4m-7)\,E(4m+9) \ldots + E(1)\,E(8m+1)\}.$$

When the argument is of the form $4m+3$ there are two middle terms, and the formula for $\chi(4m+3)$ vanishes identically as it should do, the terms cancelling one another. The formula for $\psi(4m+3)$ may be written

$$\psi(4m+3) = 2E(4m+1)\,E(4m+5) + 2E(4m-3)\,E(4m+9) \ldots + 2E(1)\,E(8m+5).$$

Thus, for example,

$$\chi(9) = E(9)\,E(9) - 2E(5)\,E(13) + 2E(1)\,E(17),$$

$$\psi(9) = E(9)\,E(9) + 2E(5)\,E(13) + 2E(1)\,E(17),$$

and

$$\psi(11) = 2E(9)\,E(13) + 2E(5)\,E(17) + 2E(1)\,E(21).$$

The formulæ for $\chi(2m+1)$ and $\psi(2m+1)$ in terms of $E(1)$, $E(5)$, ...$E(4m+1)$ were obtained from the formulæ

$$\chi(1)\,x^2 + \chi(5)\,x^{10} + \chi(9)\,x^{18} + \chi(13)\,x^{26} + \&c.$$
$$= \{E(1)\,x + E(5)\,x^5 + E(9)\,x^9 + E(13)\,x^{13} + \&c.\}$$
$$\times \{E(1)\,x - E(5)\,x^5 + E(9)\,x^9 - E(13)\,x^{13} + \&c.\},$$

$$\psi(1)\,x^2 + \psi(3)\,x^6 + \psi(5)\,x^{10} + \psi(7)\,x^{14} + \&c.$$
$$= \{E(1)\,x + E(5)\,x^5 + E(9)\,x^9 + E(13)\,x^{13} + \&c.\}$$
$$\times \{E(1)\,x + E(5)\,x^5 + E(9)\,x^9 + E(13)\,x^{13} + \&c.\}$$

It follows also from these equations that

$$\frac{\chi(1) + \chi(5)\,x^2 + \chi(9)\,x^4 + \&c.}{\psi(1) + \psi(3)\,x + \psi(5)\,x^2 + \&c.} = \frac{E(1) - E(5)\,x + E(9)\,x^2 - \&c.}{E(1) + E(5)\,x + E(9)\,x^2 + \&c.},$$

whence, by equating coefficients, we find

$$E(4m+1)\,\xi(1) - E(4m-3)\,\xi(3) + E(4m-7)\,\xi(5) \ldots + (-)^m E(1)\,\xi(2m+1) = 0,$$

where $\xi(a)$ denotes $\psi(a) - \chi(a)$ or $\psi(a) + \chi(a)$, according as m is even or uneven.

This formula is included as (xvi) in the group in § 33.

It will be seen in the following sections that the function $\psi(n)$ is very similar in its properties to $\chi(n)$, and that many of the formulæ which they satisfy are nearly identical in form.

Applications of $\chi(n)$ *in Elliptic Functions*, §§ 19–24.

§ 19. Denoting $\dfrac{2K}{\pi}$ by ρ, we have in Elliptic Functions the formulæ

$$\rho^{\frac{1}{2}} = 1 + 2q + 2q^4 + 2q^9 + 2q^{25} + \&c.,$$
$$k'^{\frac{1}{2}}\rho^{\frac{1}{2}} = 1 - 2q + 2q^4 - 2q^9 + 2q^{25} - \&c.,$$
$$k^{\frac{1}{2}}\rho^{\frac{1}{2}} = 2q^{\frac{1}{4}} + 2q^{\frac{9}{4}} + 2q^{\frac{25}{4}} + 2q^{\frac{49}{4}} + \&c.,$$
$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{3}{2}} = 2q^{\frac{1}{4}} - 6q^{\frac{9}{4}} + 10q^{\frac{25}{4}} - 14q^{\frac{49}{4}} + \&c.$$

By multiplication we deduce

$$k^{\frac{1}{2}}\rho^{\frac{1}{2}} \times k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{3}{2}} = 4\,(q^{\frac{1}{4}} + q^{\frac{9}{4}} + q^{\frac{25}{4}} + \&c.)(q^{\frac{1}{4}} - 3q^{\frac{9}{4}} + 5q^{\frac{25}{4}} - \&c.)$$
$$= 4\,\{\chi(1)\,q^{\frac{1}{2}} + \chi(5)\,q^{\frac{5}{2}} + \chi(9)\,q^{\frac{9}{2}} + \chi(13)\,q^{\frac{13}{2}} + \&c.\},$$
$$k'^{\frac{1}{2}}\rho^{\frac{1}{2}} \times k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{3}{2}} = 2\,(1 - 2q + 2q^4 - 2q^9 + \&c.)(q^{\frac{1}{4}} - 3q^{\frac{9}{4}} + 5q^{\frac{25}{4}} - \&c.)$$
$$= 2\,\{\chi(1)\,q^{\frac{1}{4}} + \chi(5)\,q^{\frac{5}{4}} + \chi(9)\,q^{\frac{9}{4}} + \chi(13)\,q^{\frac{13}{4}} + \&c.\},$$
$$\rho^{\frac{1}{2}} \times k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{3}{2}} = 2\,(1 + 2q + 2q^4 + 2q^9 + \&c.)(q^{\frac{1}{4}} - 3q^{\frac{9}{4}} + 5q^{\frac{25}{4}} - \&c.)$$
$$= 2\,\{\chi(1)\,q^{\frac{1}{4}} - \chi(5)\,q^{\frac{5}{4}} + \chi(9)\,q^{\frac{9}{4}} - \chi(13)\,q^{\frac{13}{4}} + \&c.\},$$

and we thus obtain the formulæ

$$kk'^{\frac{1}{2}}\rho^2 = 4\Sigma_0^\infty\ \chi(4n+1)\,q^{\frac{1}{2}(4n+1)},$$
$$k^{\frac{1}{2}}k'\rho^2 = 2\Sigma_0^\infty\ \chi(4n+1)\,q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^2 = 2\Sigma_0^\infty\,(-)^n\,\chi(4n+1)\,q^{\frac{1}{4}(4n+1)}.$$

§ 20. It can also be shown that we have in Elliptic Functions the formulæ

$$k^2\rho^2 = 16\Sigma_0^\infty\ \psi(2n+1)\,q^{2n+1},$$
$$k\rho^2 = 4\Sigma_0^\infty\ \psi(2n+1)\,q^{\frac{1}{2}(2n+1)},$$
$$kk'\rho^2 = 4\Sigma_0^\infty\,(-)^n\,\psi(2n+1)\,q^{\frac{1}{2}(2n+1)},$$

and

$$k\rho = 4\Sigma_0^\infty\ E(4n+1)\,q^{\frac{1}{2}(4n+1)},$$
$$k^{\frac{1}{2}}\rho = 2\Sigma_0^\infty\ E(4n+1)\,q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho = 2\Sigma_0^\infty\,(-)^n\,E(4n+1)\,q^{\frac{1}{4}(4n+1)}.$$

It thus appears that the function $\chi(2n+1)$ occurs as coefficient in certain q-series proceeding by ascending powers of q, which bear a close analogy to the known formulæ in which $\psi(2n+1)$ and $E(2n+1)$ occur as coefficients.

The preceding formulæ involving $\psi(2n+1)$ and $E(2n+1)$ may be derived directly from the q-series for sn u, &c. by means of the equations

$$\frac{x}{1-x^2} - \frac{x^3}{1-x^6} + \frac{x^5}{1-x^{10}} - \frac{x^7}{1-x^{14}} + \&c.$$
$$= E(1)\,x + E(3)\,x^3 + E(5)\,x^5 + \&c.$$

$$\frac{x}{1-x^2} + \frac{3x^3}{1-x^6} + \frac{5x^5}{1-x^{10}} + \frac{7x^7}{1-x^{14}} + \&c.$$
$$= \psi(1)\,x + \psi(3)\,x^3 + \psi(5)\,x^5 + \&c.$$

but as $\chi(n)$ apparently depends only on the complex divisors of n, there is no series, corresponding to the left-hand members of these equations, which is such that the coefficient of x^{2n+1} in its development in powers of x is $\chi(2n+1)$.

The elliptic-function formulæ found in the last section seem to be of interest, as giving the q-series for such fundamental quantities as $kk'^{\frac{1}{2}}\rho^2$, &c. It is remarkable that the coefficients in these series should depend upon the complex divisors of the exponents; and that $\chi(n)$ and $\psi(n)$ should be the corresponding coefficients in formulæ which present so many points of similarity to each other.

§ 21. From the second and third equations of the first group of formulæ in the last section we may deduce the four formulæ

$$k^{\frac{1}{2}}\rho^2 = 2\Sigma_0^\infty \qquad \psi(4n+1)\,q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^2 = 2\Sigma_0^\infty (-)^n\,\psi(4n+1)\,q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{3}{2}}\rho^2 = 2\Sigma_0^\infty \qquad \psi(4n+3)\,q^{\frac{1}{4}(4n+3)},$$
$$k^{\frac{3}{2}}k'^{\frac{1}{2}}\rho^2 = 2\Sigma_0^\infty (-)^n\,\psi(4n+3)\,q^{\frac{1}{4}(4n+3)}.$$

§ 22. It was remarked in § 17, that the definition of $E(n)$ as the excess of the number of $(4m+1)$-divisors over the number of $(4m+3)$-divisors applies also to the case when n is even; and it may be noticed, that if $\psi(n)$ were defined as the sum of the divisors only in the case of n uneven, the extension to the case of n even might be made in several ways, for not only does $\psi(n)$, the sum of all the divisors of n, occur as coefficient in elliptic-function formulæ, but so also do the three quantities $\Delta(n)$, $\Delta'(n)$, $\zeta(n)$, where

$\Delta(n) =$ the sum of the uneven divisors of n,

$\Delta'(n) =$ „ „ divisors of n having uneven conjugates,

$$\zeta(n) = \begin{cases} \quad \text{„ „ uneven divisors of } n \\ - \text{ „ „ even „ „ }, \end{cases}$$

each of which becomes equal to $\psi(n)$ when n is uneven.

The values of the q-series in which $\psi(n)$, $\Delta'(n)$ and $\zeta(n)$ occur as coefficients involve E as well as K, viz. we have, for example,

$$24\Sigma_1^\infty \psi(n) q^{2n} = 1 + \frac{4(1+k'^2)K^2}{\pi^2} - \frac{12KE}{\pi^2},$$

$$32\Sigma_1^\infty \Delta'(n) q^{2n} = \frac{4(1+k'^2)K^2}{\pi^2} - \frac{4KE}{\pi^2},$$

$$8\Sigma_1^\infty \zeta(n) q^{2n} = -1 + \frac{4KE}{\pi^2},$$

but the series involving $\Delta(n)$ can be expressed in terms of K alone, viz. we have

$$24\Sigma_1^\infty \Delta(n) q^{2n} = -1 + \frac{2(1+k'^2)K^2}{\pi^2}.$$

§ 23. We can express ρ^2, $k'^2\rho^2$ and $k'\rho^2$ as q-series involving $\Delta(n)$ by means of the formulæ

$$\rho^2 = 1 + 8\Sigma_1^\infty \{2 + (-1)^n \quad\} \Delta(n) q^n,$$

$$k'^2\rho^2 = 1 + 8\Sigma_1^\infty \{1 + (-1)^n 2\} \Delta(n) q^n,$$

$$k'\rho^2 = 1 + 8\Sigma_1^\infty \{1 + (-1)^n 2\} \Delta(n) q^{2n},$$

and $k'^{\frac12}\rho^2$ and $k'^{\frac32}\rho^2$ by means of the formulæ

$$k'^{\frac12}\rho^2 = 1 + 4\Sigma_1^\infty (\ \ 1, 0, -1, -2, \quad 1, 0, -1, 6) \Delta(n) q^n,$$

$$k'^{\frac32}\rho^2 = 1 + 4\Sigma_1^\infty (-1, 0, \quad 1, -2, -1, 0, \quad 1, 6) \Delta(n) q^n,$$

where the meaning of the notation $\Sigma_1^\infty (a_1, a_2, a_3, \ldots a_8) \Delta(n) q^n$ is that the coefficient of q^n is $a_1 \times \Delta(n)$, $a_2 \times \Delta(n)$, ... or $a_8 \Delta(n)$ according as $n \equiv 1, 2, \ldots$ or 8, mod. 8.

(v)

$$E(0)\,\psi(r)-E(1)\,\psi(r-4)\ldots\pm E(n)\,\psi(3)$$
$$=\quad E(1)\,\chi(p)-E(3)\,\chi(p-4)\ldots\pm E(m)\,\chi(1).$$

(vi)

$$E(0)\,\psi(r)+E(1)\,\psi(r-4)\ldots+E(n)\,\psi(3)$$
$$=\quad E(1)\,\psi(p)+E(3)\,\psi(p-4)\ldots+E(m)\,\psi(1).$$

(vii)

$$4\{E(0)\,\chi(p)-E(1)\,\chi(p-4)\ldots\pm E(n)\,\chi(1)\}$$
$$=\quad E(1)\,\chi(s)+E(5)\,\chi(s-4)\ldots+E(s)\,\chi(1).$$

(viii)

$$4\{E(0)\,\psi(m)-E(1)\,\psi(m-4)+\ldots\ldots\ldots\ldots\}$$
$$=(-)^n\{E(1)\,\chi(p)-E(5)\,\chi(p-4)\ldots\pm E(p)\,\chi(1)\}.$$

(ix)

$$E(0)\,\psi(r)-E(1)\,\psi(r-8)+\ldots\ldots\ldots\ldots\ldots$$
$$=(-)^n\{E(1)\,\chi(p)+E(5)\,\chi(p-8)+\ldots\ldots\ldots\ldots\ldots\}.$$

The last term is omitted in the second member of (viii) and in both members of (ix) as it depends upon the form of n. According as n is even or uneven, it is $E(\frac{1}{2}n)\,\psi(1)$ or $E\{\frac{1}{2}(n-1)\}\,\psi(3)$ in (viii), $E(\frac{1}{2}n)\,\psi(3)$ or $E\{\frac{1}{2}(n-1)\}\,\psi(7)$ in the left-hand member of (ix) and $E(2n+1)\,\chi(1)$ or $E(2n-1)\,\chi(5)$ in the right-hand member.

§51. It may be remarked that we may deduce from these formulæ the equalities:

(i)

$$E(1)\,\chi(s)+E(5)\,\chi(s-4)\ldots+E(s)\,\chi(1)$$
$$=\quad 4\{E(0)\,\chi(p)-E(1)\,\chi(p-4)\ldots\pm E(n)\,\chi(1)\}$$
$$=\quad 4\{E(0)\,\psi(p)-E(1)\,\psi(p-2)\ldots+E(2n)\,\psi(1)\}.$$

(ii)

$$E(1)\,\chi(s)-E(5)\,\chi(s-4)\ldots+E(s)\,\chi(1)$$
$$=\quad E(1)\,\psi(s)-E(5)\,\psi(s-4)\ldots+E(s)\,\psi(1)$$
$$=\quad 4\{E(0)\,\chi(p)+E(2)\,\chi(p-4)\ldots+E(2n)\,\chi(1)\}.$$

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The function $E_2(n)$, §§ 52, 53.

§ 52. If we denote by $E_2(n)$ the excess of the sum of the squares of the $(4m+1)$-divisors of n over the sum of the squares of the $(4m+3)$-divisors, it can be shown that, if

$$n = 2^p a^\alpha b^\beta c^\gamma \dots r^\rho s^\sigma t^\tau \dots,$$

where $a, b, c, \dots$ are primes of the form $4m+1$ and $r, s, t, \dots$ are primes of the form $4m+3$, then

$$E_2(n) = (-)^{\rho+\sigma+\dots} \frac{a^{2\alpha+2}-1}{a^2-1} \times \frac{b^{2\beta+2}-1}{b^2-1} \dots$$

$$\times \frac{r^{2\rho+2}+(-1)^\rho}{\rho^2+1} \times \frac{s^{2\sigma+2}+(-1)^\sigma}{s^2+1} \dots$$

If therefore $n = 2^p r$, where r is uneven, $E_2(n)$ is positive or negative according as r is of the form $4m+1$ or $4m+3$.

We see also that if $n = a^\alpha b^\beta c^\gamma \dots$ where $a, b, c, \dots$ are any different primes,

$$E_2(n) = E_2(a^\alpha) \, E_2(b^\beta) \, E_2(c^\gamma) \dots,$$

and that if $n = n_1 n_2 n_3 \dots$ where $n_1, n_2, n_3, \dots$ are relatively prime to each other,

$$E_2(n) = E_2(n_1) \, E_2(n_2) \, E_2(n_3) \dots.$$

§ 53. The expression $E_2(n)$ occurs as coefficient in the q-series for certain expressions involving ρ^3, viz. we have

$$k\rho^3 = 4q^{\frac{1}{2}} \Sigma_0^\infty (-)^n E_2(2n+1) q^n,$$

$$kk'^2\rho^3 = 4q^{\frac{1}{2}} \Sigma_0^\infty E_2(2n+1) q^n,$$

$$k^{\frac{1}{2}}\rho^3 = - q^{\frac{1}{4}} \Sigma_0^\infty E_2(4n+3) q^n,$$

$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^3 = - q^{\frac{1}{4}} \Sigma_0^\infty (-)^n E_2(4n+3) q^n,$$

$$k^3\rho^3 = -8q^{\frac{3}{2}} \Sigma_0^\infty E_2(4n+3) q^{2n}.$$

We find also the formulæ

$$k'\rho^3 = 1 + 4\Sigma_1^\infty (-)^{n-1} E_2(n) q^n,$$

$$k'^2\rho^3 = 1 - 4\Sigma_1^\infty E_2(n) q^n,$$

which, if we define $E_2(0)$ to denote $-\frac{1}{4}$, may be written

$$k'\rho^3 = -4\Sigma_0^\infty (-)^n E_2(n) q^n,$$

$$k'^2\rho^3 = -4\Sigma_0^\infty E_2(n) q^n.$$

Expressions for $E_2(m)$ and $E_2(r)$ in terms of E and ψ, §§ 54, 55.

54. By combining the first group of formulæ in the preceding section with the formulæ in § 31, and equating coefficients we obtain the following expressions for $E_2(m)$ and $E_2(r)$ in terms of the E and ψ functions, the arguments being uneven:

If $m = 2n+1, \quad p = 4n+1, \quad r = 4n+3,$

then

(i)

$$(-)^n E_2(m) = E(p)\,\psi(1) + E(p-4)\,\psi(5) \ldots + E(1)\,\psi(p).$$

(ii)

$$-\tfrac{1}{8}E_2(r) = E(p)\,\psi(1) + E(p-4)\,\psi(3) \ldots + E(1)\,\psi(m),$$

$$= \psi(p)\,E(1) + \psi(p-4)\,E(3) \ldots + \psi(1)\,E(m).$$

For example, let $n = 2$, and the formulæ give

$$E_2(5) = E(9)\,\psi(1) + E(5)\,\psi(5) + E(1)\,\psi(9) = 26;$$

$$-\tfrac{1}{8}E_2(11) = E(9)\,\psi(1) + E(5)\,\psi(3) + E(1)\,\psi(5) = 15,$$

$$= \psi(9)\,E(1) + \psi(5)\,E(3) + \psi(1)\,E(5) = 15.$$

§ 55. By means of the formulæ for ρ, &c. in § 46 we may also obtain the following formulæ in which the E's of even arguments are involved and $E(0) = \tfrac{1}{4}$.

(iii)

$$(-)^n E_2(m) = 4\,\{E(0)\,\psi(m) + E(1)\,\psi(m-2) \ldots + E(n)\,\psi(1)\},$$

(iv)

$$\tfrac{1}{8}(-)^{n-1}E_2(r) = E(0)\,\psi(r) + E(1)\,\psi(r-4) \ldots + E(n)\,\psi(3).$$

Putting as before $n = 2$, these formulæ give

$$E_2(5) = 4\,\{E(0)\,\psi(5) + E(1)\,\psi(3) + E(2)\,\psi(1)\} = 26,$$

$$-\tfrac{1}{8}E_2(11) = E(0)\,\psi(11) + E(1)\,\psi(7) + E(2)\,\psi(3) = 15.$$

The formulæ in §§ 31 and 46 do not afford expressions for $E_2(m)$ and $E_2(r)$ in which the function χ is involved.

Formulæ involving the functions χ *and* E_2, § 56.

§ 56. We may obtain by means of the formulæ in § 53 numerous equations connecting the functions χ, E, ψ, E_2, the arguments being uneven: I confine myself however to the only two formulæ in which χ is involved, viz.

(i)

$$\chi(p)\,\psi(1)-\chi(p-4)\,\psi(3)\ldots\pm\chi(1)\,\psi(m)$$
$$=\quad E(p)\,E_2(1)+E(p-4)\,E_2(3)\ldots+E(1)\,E_2(m).$$

(ii)

$$\chi(p)\,\psi(3)-\chi(p-4)\,\psi(7)\ldots\pm\chi(1)\,\psi(r)$$
$$=-\tfrac{1}{2}\{E(p)\,E_2(3)-E(p-4)\,E_2(7)\ldots\pm E(1)\,E_2(r)\}.$$

We may also notice the following formula in which even as well as uneven arguments of E_2 occur, the value of $E_2(0)$ being $-\frac{1}{4}$:

(iii)

$$E_2(0)\,\chi(p)-E_2(1)\,\chi(p-4)\ldots\pm E_2(n)\,\chi(1)$$
$$=E_2(0)\,\psi(p)+E_2(1)\,\psi(p-4)\ldots+E_2(n)\,\psi(1).$$

The function $\lambda(n)$, §§ 57–61.

§ 57. In the previous sections of this paper the function $E(n)$, denoting the number of primary numbers having n as norm, and the function $\chi(n)$, denoting the sum of the primary numbers having n as norm, have been considered; and it has been shown that the q-series for certain quantities involving ρ depend upon $E(n)$, and that the q-series for certain quantities involving ρ^2 depend upon $\chi(n)$.

The function which denotes the sum of the squares of the primary numbers having n as norm will now be considered, and it will be shown that it serves to express the coefficients in the q-series for certain expressions involving ρ^3.

§ 58. If we denote by $\lambda(n)$ the sum of the squares of the primary numbers having n as norm, we see as in § 7 that if p be a prime of the form $4m+3$, then

$$\lambda(p^{2n-1})=0,\qquad \lambda(p^{2n})=p^{2n}.$$

If p be a prime of the form $4m+1$, then it follows from § 10 (p. 104) that if $p = a^2 + b^2$, where a is uneven, then

$$\lambda(p^n) = \frac{(a+ib)^{2n+2} - (a-ib)^{2n+2}}{(a+ib)^2 - (a-ib)^2} = \frac{(a+ib)^{2n+2} - (a-ib)^{2n+2}}{4iab}.$$

Now in § 10 it was shown that, if n be uneven,

$$\chi(p^n) = k\frac{(a+ib)^{n+1} - (a-ib)^{n+1}}{2ib},$$

where k denotes $(-1)^{\frac{1}{4}(a+b-1)}$.

Thus we find

$$\lambda(p^n) = \frac{\chi(p^{2n+1})}{2ka} = \frac{\chi(p^{2n+1})}{\chi(p)}.$$

§ 59. This remarkable formula renders it unnecessary to give any special formulæ for the calculation of the λ-function corresponding to those for $\chi(n)$ in § 10. The general expression for $\lambda(n)$ corresponding to that for $\chi(n)$ in § 11 (p. 105) is, in the notation of that section,

$$\lambda(n) = \frac{(a_1+ib_1)^{2a_1+2} - (a_1-ib_1)^{2a_1+2}}{4ia_1b_1}$$
$$\times \frac{(a_2+ib_2)^{2a_2+2} - (a_2-ib_2)^{2a_2+2}}{4ia_2b_2} \times \ldots \times s_1^{\sigma_1}s_2^{\sigma_2}\ldots.$$

It will be noticed that, like the function $\chi(n)$, the function $\lambda(n)$ has been defined only in the case of n uneven.

§ 60. The function $\lambda(n)$ vanishes for the same values as $E(n)$ and $\chi(n)$, *i.e.* $\lambda(n)$ vanishes unless every prime factor of n of the form $4m+3$ occurs with an even exponent. It is also evident that if $a, b, c, \ldots$ be any different uneven primes

$$\lambda(a^\alpha b^\beta c^\gamma \ldots) = \lambda(a^\alpha)\,\lambda(b^\beta)\,\lambda(c^\gamma)\ldots,$$

and that if $n = n_1 n_2 n_3 \ldots$, where $n_1, n_2, n_3, \ldots$ are prime to one another, then

$$\lambda(n) = \lambda(n_1)\,\lambda(n_2)\,\lambda(n_3)\ldots.$$

If n be any number having no prime factor of the form $4m+3$, and if $a, b, c, \ldots$ be the prime factors of n, then

$$\lambda(n) = \frac{\chi(n^2abc\ldots)}{\chi(abc\ldots)} = \frac{\chi(n^2\delta)}{\chi(\delta)},$$

where δ is the greatest divisor of n which contains no square factor.

61. If $$n = a_1^2 + b_1^2 = a_2^2 + b_2^2 = \ldots;$$
where a_1, a_2, ... are uneven, then
$$\lambda(n) = 2(a_1^2 - b_1^2) + 2(a_2^2 - b_2^2) + \ldots.$$

The resolutions of $2n$ as a sum of two squares are
$$2n = (a_1 + b_1)^2 + (a_1 - b_1)^2 = (a_2 + b_2)^2 + (a_2 - b_2)^2 = \ldots\ldots\ldots,$$
and, with regard to the form of these squares, it is evident that if $a_1 > b_1$, then $a_1 + b_1$ and $a_1 - b_1$ are positive, and therefore they are both of the form $4m + 1$ or both of the form $4m + 3$; but if $a_1 < b_1$, then $a_1 + b_1$ and the (positive) numerical value of $a_1 - b_1$ are of different forms. Thus when the numerical value of $2(a_1^2 - b_1^2)$ is positive, the two squares $(a_1 + b_1)^2$ and $(a_1 - b_1)^2$ are both of the form $(4m + 1)^2$ or both of the form $(4m + 3)^2$, and when it is negative one square is of the form $(4m + 1)^2$ and the other is of the form $(4m + 3)^2$.

It follows therefore that if
$$2n = \alpha_1^2 + \beta_1^2 = \alpha_2^2 + \beta_2^2 = \ldots,$$
where α_1, β_1, α_2, β_2, ... are positive numbers, which are necessarily uneven, then $\lambda(n) =$
$$2\times(-1)^{\frac{1}{2}(\alpha_1-1)}\alpha_1\times(-1)^{\frac{1}{2}(\beta_1-1)}\beta_1 + 2\times(-1)^{\frac{1}{2}(\alpha_2-1)}\alpha_2\times(-1)^{\frac{1}{2}(\beta_2-1)}\beta_2 + \ldots,$$
viz. the terms are of the form $2\alpha\beta$ and the positive or negative sign is to be prefixed according as α and β are of the same form or of different forms. If $n = \alpha^2$ the corresponding term in $\lambda(n)$ is α^2 (*i.e.* without the factor 2).

For example, let $n = 65$, then $2n = 3^2 + 11^2 = 7^2 + 9^2$,

and $\lambda(n) = 2 \times -3 \times -11 + 2 \times -7 \times 9 = 66 - 126 = -60$.

Elliptic-Function formulæ involving $\lambda(n)$, § 62.

§ 62. The expression given in the last section for $\lambda(2n)$ in terms of the complex numbers having $2n$ as norm shows that $\lambda(4n + 1)$ is equal to the coefficient of x^{8n+2} in the expansion of
$$(x - 3x^9 + 5x^{25} - 7x^{49} + 9x^{81} - \&c.)^2.*$$

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* Each member of the equation which forms the theorem in the "Note on the compositions of a number as a sum of two and four uneven squares" (*Quarterly Journal*, vol. xix. pp. 212—215) is equal to $\lambda(N)$.

It follows from the last formula in that 'Note' that
$$\Sigma_0^\infty \lambda(4n + 1)\, x^{16n+4} = \{\Sigma_0^\infty x^{(2n+1)^2}\}^3 \times \Sigma_0^\infty (-)^n (2n + 1)\, x^{(2n+1)^2}.$$

§ 66. We may deduce from these formulæ results similar to those given in §§ 41–45, but I only give here the formula which corresponds to I. of § 41, viz.

$$\begin{aligned} &\lambda(p) \\ &+2\lambda(p-4)+2\lambda(p-8) \\ &+3\lambda(p-12)+3\lambda(p-16)+3\lambda(p-20) \\ &+\ldots\ldots\ldots\ldots \\ &= \\ &\chi(p) \\ &-2\chi(p-4)-2\chi(p-8) \\ &+3\chi(p-12)+3\chi(p-16)+3\chi(p-20) \\ &-\ldots\ldots\ldots\ldots \end{aligned}$$

This formula corresponds exactly to I. of § 41, λ and χ replacing χ and E. These two formulæ are perhaps the most curious and interesting of those given in this paper. Corresponding to II.* of § 41 we have an exactly similar formula, in which χ and E are replaced by λ and ψ respectively.

§ 67. The product-formulæ (in which each term contains two factors) are very numerous, but there are only two in which E, χ, and λ are alone involved, viz.:

(i)

$$E(p)\lambda(1)+E(p-4)\lambda(5)\ldots+E(1)\lambda(p)$$
$$=\chi(p)\chi(1)+\chi(p-4)\chi(5)\ldots+\chi(1)\chi(p).$$

(ii)

$$E(p)\lambda(1)+E(p-8)\lambda(5)+\ldots\ldots\ldots$$
$$=(-)^n\{\chi(p)\chi(1)+\chi(p-8)\chi(5)+\ldots\ldots\ldots\}.$$

Tables of $\chi(n)$, § 68.

§ 68. The contents of the Tables are as follows:

Table I. gives the value of $\chi(n)$ for all values of n up to $n=1000$ for which $\chi(n)$ is not zero.

* See the erratum at the end of the paper.

The function $\chi(n)$ is defined only for the case of n uneven, and, if n be of the form $4m+3$, $\chi(n)=0$. All the arguments are therefore of the form $4m+1$.

Table II. gives the value of $\chi(n)$ for every prime number of the form $4m+1$ (*i.e.* for every prime number for which $\chi(n)$ is not zero) up to $n=13{,}000$.

Table III. gives the values of $\chi(n)$ for powers of primes up to $n=13{,}000$.

Table I.

Values of $\chi(n)$ for all (uneven) numbers up to $n=1000$ for which $\chi(n)$ is not zero.

n	$\chi(n)$	n	$\chi(n)$	n	$\chi(n)$	n	$\chi(n)$
1	1	229	+ 30	481	− 12	757	− 18
5	− 2	233	+ 26	485	− 36	761	− 38
9	− 3	241	− 30	493	− 20	765	+ 12
13	+ 6	245	+ 14	505	+ 4	769	+ 50
17	+ 2	257	+ 2	509	− 10	773	− 34
25	− 1	261	+ 30	521	− 22	785	− 44
29	− 10	265	− 28	529	− 23	793	− 60
37	− 2	269	− 26	533	+ 60	797	+ 22
41	+ 10	277	− 18	541	− 42	801	− 30
45	+ 6	281	+ 10	545	− 12	809	+ 10
49	− 7	289	− 13	549	+ 30	821	− 50
53	+ 14	293	− 34	557	+ 38	829	+ 54
61	− 10	305	+ 20	565	+ 28	833	− 14
65	− 12	313	+ 26	569	+ 26	841	+ 71
73	− 6	317	+ 22	577	+ 2	845	− 46
81	+ 9	325	− 6	585	+ 36	853	+ 46
85	− 4	333	+ 6	593	− 46	857	+ 58
89	+ 10	337	+ 18	601	+ 10	865	+ 52
97	+ 18	349	− 10	605	+ 22	873	− 54
101	− 2	353	+ 34	613	− 34	877	− 58
109	+ 6	361	− 19	617	− 38	881	+ 50
113	− 14	365	+ 12	625	− 19	901	+ 28
117	− 18	369	− 30	629	− 4	905	+ 36
121	− 11	373	+ 14	637	− 42	909	+ 6
125	+ 12	377	− 60	641	+ 50	925	+ 2
137	− 22	389	− 34	653	− 26	929	− 46
145	+ 20	397	+ 38	657	+ 18	937	− 38
149	+ 14	401	+ 2	661	− 50	941	− 58
153	− 6	405	− 18	673	− 46	949	− 36
157	+ 22	409	− 6	677	− 2	953	+ 26
169	+ 23	421	+ 30	685	+ 44	961	− 31
173	− 26	425	− 2	689	+ 84	965	+ 28
181	− 18	433	+ 34	697	+ 20	977	− 62
185	+ 4	441	+ 21	701	− 10	981	− 18
193	− 14	445	− 20	709	+ 30	985	+ 4
197	− 2	449	− 14	725	+ 10	997	+ 62
205	− 20	457	+ 42	729	− 27		
221	+ 12	461	+ 38	733	+ 54		
225	+ 3	477	− 42	745	− 28		

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Table II.

Values of $\chi(n)$ for primes of the form $4m+1$ up to $n=13,000$.

n	$\chi(n)$	n	$\chi(n)$	n	$\chi(n)$	n	$\chi(n)$	n	$\chi(n)$
5	− 2	673	− 46	1489	+ 66	2377	+ 42	3301	− 98
13	+ 6	677	− 2	1493	+ 14	2381	+ 70	3313	+ 114
17	+ 2	701	− 10	1549	+ 70	2389	− 50	3329	+ 50
29	− 10	709	+ 30	1553	− 46	2393	+ 74	3361	− 30
37	− 2	733	+ 54	1597	− 42	2417	+ 98	3373	+ 6
41	+ 10	757	− 18	1601	+ 2	2437	− 98	3389	− 10
53	+ 14	761	− 38	1609	− 6	2441	+ 58	3413	+ 14
61	− 10	769	+ 50	1613	− 26	2473	+ 26	3433	− 54
73	− 6	773	− 34	1621	+ 78	2477	+ 38	3449	− 86
89	+ 10	797	+ 22	1637	+ 62	2521	− 70	3457	− 78
97	+ 18	809	+ 10	1657	− 38	2549	+ 14	3461	+ 62
101	− 2	821	− 50	1669	+ 30	2557	− 42	3469	− 90
109	+ 6	829	+ 54	1693	− 74	2593	+ 34	3517	+ 118
113	− 14	853	+ 46	1697	+ 82	2609	− 94	3529	− 70
137	− 22	857	+ 58	1709	+ 70	2617	− 102	3533	− 26
149	+ 14	877	− 58	1721	− 22	2621	+ 22	3541	− 50
157	+ 22	881	+ 50	1733	− 34	2633	− 86	3557	− 98
173	− 26	929	− 46	1744	− 58	2657	+ 98	3581	+ 118
181	− 18	937	− 38	1753	− 54	2677	+ 78	3593	+ 106
193	− 14	941	− 58	1777	− 78	2689	+ 66	3613	+ 86
197	− 2	953	+ 26	1789	− 10	2693	+ 94	3617	+ 82
229	+ 30	977	− 62	1801	− 70	2713	− 6	3637	+ 78
233	+ 26	997	+ 62	1861	+ 62	2729	+ 10	3673	+ 74
241	− 30	1009	− 30	1873	+ 66	2741	− 50	3677	+ 118
257	+ 2	1013	+ 46	1877	− 82	2749	+ 86	3697	+ 98
269	− 26	1021	+ 22	1889	+ 34	2753	− 14	3701	+ 110
277	− 18	1033	− 6	1901	+ 70	2777	+ 58	3709	− 106
281	+ 10	1049	+ 10	1913	− 86	2789	− 34	3733	− 114
293	− 34	1061	+ 62	1933	− 26	2797	+ 102	3761	+ 50
313	+ 26	1069	− 26	1949	+ 86	2801	+ 98	3769	+ 26
317	+ 22	1093	− 66	1973	+ 46	2833	− 46	3793	+ 66
337	+ 18	1097	+ 58	1993	− 86	2837	− 82	3797	− 82
349	− 10	1109	− 50	1997	− 58	2857	− 102	3821	− 122
353	+ 34	1117	− 42	2017	+ 18	2861	+ 38	3833	+ 106
373	+ 14	1129	− 54	2029	− 90	2897	− 62	3853	+ 6
389	− 34	1153	+ 66	2053	− 34	2909	− 106	3877	+ 62
397	+ 38	1181	− 10	2069	− 50	2917	− 2	3881	− 118
401	+ 2	1193	+ 26	2081	+ 82	2953	+ 106	3889	+ 34
409	− 6	1201	+ 50	2089	+ 90	2957	− 58	3917	− 122
421	+ 30	1213	+ 54	2113	+ 66	2969	+ 74	3929	− 70
433	+ 34	1217	− 62	2129	− 46	3001	− 102	3989	− 50
449	− 14	1229	+ 70	2137	+ 58	3037	+ 22	4001	+ 98
457	+ 42	1237	− 18	2141	− 10	3041	− 110	4013	− 26
461	+ 38	1249	− 30	2153	+ 74	3049	+ 90	4021	+ 78
509	− 10	1277	+ 22	2161	− 30	3061	+ 110	4049	− 110
521	− 22	1289	− 70	2213	+ 94	3089	− 110	4057	− 118
541	− 42	1297	+ 2	2221	− 90	3109	+ 94	4073	+ 74
557	+ 38	1301	− 50	2237	+ 22	3121	− 78	4093	+ 54
569	+ 26	1321	+ 10	2269	− 74	3137	+ 2	4129	− 46
577	+ 2	1361	− 62	2273	− 94	3169	− 110	4133	− 34
593	− 46	1373	− 74	2281	+ 90	3181	− 90	4153	− 86
601	+ 10	1381	+ 30	2293	+ 46	3209	+ 106	4157	+ 118
613	− 34	1409	+ 50	2297	− 38	3217	+ 18	4177	+ 18
617	− 38	1429	+ 46	2309	+ 94	3221	+ 110	4201	− 102
641	+ 50	1433	+ 74	2333	+ 86	3229	+ 54	4217	− 22
653	− 26	1453	+ 6	2341	+ 30	3253	− 114	4229	− 130
661	− 50	1481	− 70	2357	− 82	3257	− 22	4241	+ 130

n	χ(n)	n	χ(n)	n	χ(n)	n	χ(n)	n	χ(n)
4253	− 106	5381	− 130	6421	+ 78	7541	+ 142	8629	+ 46
4261	− 130	5393	+ 146	6449	− 14	7549	− 170	8641	− 142
4273	+ 114	5413	+ 126	6469	+ 126	7561	− 150	8669	− 170
4289	+ 130	5417	− 118	6473	− 86	7573	+ 174	8677	− 162
4297	+ 122	5437	− 138	6481	+ 18	7577	− 118	8681	− 182
4337	+ 98	5441	− 142	6521	− 22	7589	− 130	8689	− 30
4349	+ 86	5449	− 86	6529	+ 130	7621	+ 30	8693	− 146
4357	− 2	5477	− 2	6553	+ 74	7649	− 110	8713	+ 186
4373	+ 46	5501	− 10	6569	+ 26	7669	+ 174	8737	+ 82
4397	− 122	5521	+ 130	6577	+ 162	7673	− 166	8741	+ 158
4409	+ 106	5557	− 18	6581	− 82	7681	+ 50	8753	+ 34
4421	− 130	5569	− 126	6637	− 122	7717	− 162	8761	− 150
4441	+ 58	5573	+ 94	6653	− 106	7741	+ 150	8821	− 178
4457	− 38	5581	+ 70	6661	− 162	7753	− 6	8837	− 2
4481	+ 130	5641	− 150	6673	− 126	7757	+ 38	8849	+ 130
4493	+ 134	5653	− 146	6689	+ 34	7789	+ 166	8861	− 10
4513	− 94	5657	+ 122	6701	+ 70	7793	− 14	8893	− 106
4517	− 98	5669	− 130	6709	− 50	7817	+ 122	8929	+ 146
4549	− 130	5689	− 150	6733	+ 6	7829	− 146	8933	+ 94
4561	− 62	5693	+ 86	6737	− 62	7841	− 158	8941	− 58
4597	− 82	5701	+ 30	6761	− 38	7853	+ 134	8969	− 70
4621	− 122	5717	+ 142	6781	+ 150	7873	+ 114	9001	− 102
4637	+ 118	5737	− 102	6793	− 134	7877	− 98	9013	+ 174
4649	+ 10	5741	− 58	6829	− 154	7901	− 170	9029	+ 190
4657	− 78	5749	− 114	6833	− 94	7933	+ 86	9041	− 190
4673	− 14	5801	+ 10	6841	+ 42	7937	+ 178	9049	+ 186
4721	+ 50	5813	− 146	6857	+ 122	7949	+ 70	9109	+ 110
4729	+ 90	5821	+ 150	6869	+ 110	7993	+ 106	9133	− 186
4733	− 74	5849	− 70	6917	+ 158	8009	+ 170	9137	− 142
4789	+ 110	5857	+ 18	6949	+ 30	8017	− 62	9157	+ 158
4793	+ 26	5861	+ 62	6961	+ 162	8053	+ 174	9161	− 170
4801	+ 130	5869	− 90	6977	− 142	8069	− 130	9173	− 146
4813	+ 134	5881	− 150	6997	+ 78	8081	+ 82	9181	+ 182
4817	+ 82	5897	− 22	7001	− 70	8089	− 134	9209	+ 106
4861	− 138	5953	+ 114	7013	− 34	8093	− 74	9221	+ 190
4877	− 122	5981	+ 118	7057	+ 2	8101	− 2	9241	+ 10
4889	− 134	6029	− 154	7069	+ 150	8117	− 178	9257	− 118
4909	+ 6	6037	− 82	7109	+ 94	8161	+ 162	9277	− 42
4933	− 66	6053	+ 94	7121	− 110	8209	− 110	9281	− 190
4937	+ 58	6073	+ 154	7129	− 54	8221	+ 22	9293	− 154
4957	− 138	6089	− 134	7177	− 22	8233	+ 154	9337	− 22
4969	+ 74	6101	− 50	7193	− 134	8237	− 58	9341	− 170
4973	+ 134	6113	+ 146	7213	+ 166	8269	− 26	9349	+ 190
4993	− 126	6121	+ 90	7229	− 170	8273	− 46	9377	− 158
5009	+ 130	6133	+ 14	7237	− 162	8293	+ 94	9397	+ 142
5021	+ 22	6173	− 106	7253	+ 46	8297	− 182	9413	− 194
5077	+ 142	6197	+ 142	7297	− 78	8317	+ 182	9421	− 90
5081	− 118	6217	+ 42	7309	+ 70	8329	− 150	9433	+ 186
5101	+ 102	6221	− 122	7321	+ 122	8353	− 174	9437	+ 182
5113	+ 106	6229	− 146	7333	+ 126	8369	+ 50	9461	− 50
5153	− 46	6257	− 158	7349	− 50	8377	− 102	9473	+ 194
5189	− 34	6269	− 74	7369	+ 170	8389	+ 34	9497	+ 122
5197	− 58	6277	+ 158	7393	− 94	8429	− 154	9521	+ 178
5209	+ 10	6301	+ 150	7417	− 38	8461	+ 38	9533	− 106
5233	− 14	6317	− 58	7433	+ 106	8501	+ 110	9601	− 190
5237	+ 142	6329	+ 154	7457	+ 82	8513	− 14	9613	+ 6
5261	+ 140	6337	− 142	7477	− 18	8521	+ 170	9629	− 10
5273	− 134	6353	+ 146	7481	+ 170	8537	− 182	9649	+ 114
5281	+ 82	6361	+ 138	7489	+ 66	8573	+ 86	9661	− 138
5297	− 142	6373	− 34	7517	+ 22	8581	− 130	9677	− 58
5309	− 106	6389	+ 110	7529	+ 154	8597	− 178	9689	− 70
5333	− 146	6397	+ 118	7537	− 158	8609	− 94	9697	+ 162

n	$\chi(n)$	n	$\chi(n)$	n	$\chi(n)$	n	$\chi(n)$	n	n
9721	− 150	10289	+ 34	10837	− 178	11353	+ 186	11941	+ 190
9733	− 194	10301	− 202	10853	− 194	11369	+ 74	11953	+ 34
9749	+ 110	10313	− 86	10861	− 90	11393	− 206	11969	+ 130
9769	+ 90	10321	− 190	10889	− 134	11437	+ 102	11981	− 218
9781	− 82	10333	+ 54	10909	− 106	11489	− 110	12037	− 162
9817	− 198	10337	− 158	10937	− 22	11497	+ 202	12041	− 70
9829	+ 30	10357	+ 78	10949	− 130	11549	+ 214	12049	+ 210
9833	+ 74	10369	− 126	10957	+ 198	11593	− 214	12073	− 166
9857	+ 178	10429	− 10	10973	− 74	11597	+ 38	12097	− 142
9901	+ 198	10433	+ 194	10993	+ 114	11617	+ 98	12101	− 2
9929	+ 170	10453	+ 14	11057	+ 178	11621	− 130	12109	+ 6
9941	+ 142	10457	+ 202	11069	− 170	11633	− 206	12113	+ 194
9949	+ 86	10477	+ 198	11093	+ 206	11657	+ 58	12149	+ 14
9973	− 114	10501	− 98	11113	+ 154	11677	− 42	12157	− 138
10009	− 6	10513	+ 146	11117	− 122	11681	+ 82	12161	− 190
10037	− 178	10529	− 46	11149	− 186	11689	+ 10	12197	+ 62
10061	+ 70	10589	− 170	11161	+ 138	11701	− 210	12241	− 110
10069	+ 174	10597	+ 158	11173	− 194	11717	+ 158	12253	+ 86
10093	− 186	10601	+ 202	11177	− 38	11777	− 62	12269	− 26
10133	+ 46	10613	+ 206	11197	+ 182	11789	+ 166	12277	− 178
10141	− 170	10657	+ 162	11213	+ 134	11801	+ 202	12281	+ 218
10169	+ 26	10709	+ 206	11257	+ 42	11813	+ 94	12289	+ 50
10177	− 62	10729	− 54	11261	− 10	11821	− 122	12301	+ 198
10181	+ 190	10733	+ 166	11273	+ 106	11833	+ 26	12329	+ 154
10193	+ 194	10753	− 206	11317	− 18	11897	+ 218	12373	+ 206
10253	+ 166	10781	+ 182	11321	+ 170	11909	− 194	12377	− 182
10273	− 174	10789	+ 190	11329	− 190	11933	+ 214		

Table III.

Values of $\chi(n)$, up to $n = 13,000$, for squares and higher powers of primes as arguments.

n	$\chi(n)$	n	$\chi(n)$
5^2	− 1	37^2	− 33
5^3	+ 12	41^2	+ 59
5^4	− 19	43^2	− 43
5^5	− 22	47^2	− 47
7^2	− 7	53^2	+ 143
7^3	0	59^2	− 59
7^4	+ 49	61^2	+ 39
11^2	− 11	67^2	− 67
11^3	0	71^2	− 71
11^4	+ 121	73^2	− 37
13^2	+ 23	79^2	− 79
13^3	+ 60	83^2	− 83
17^2	− 13	89^2	+ 11
17^3	− 60	97^2	+ 227
19^2	− 19	101^2	− 97
19^3	0	103^2	− 103
23^2	− 23	107^2	− 107
23^3	0	109^2	− 73
29^2	+ 71	113^2	+ 83
31^2	− 31		

Calculation of the Tables, § 69.

§ 69. Table I. was calculated by multiplying the expression

$$1 + x + x^3 + x^6 + x^{10} + x^{15} + x^{21} + \&c.$$

by $$1 - 3x + 5x^3 - 7x^6 + 9x^{10} - 11x^{15} + \&c.,$$

the coefficient of x^n in the product being $\chi(4n+1)$. This table was calculated before I had obtained the formula

$$\chi(n) = \chi(n_1)\,\chi(n_2)\,\chi(n_3) \ldots .$$

It was subsequently verified by means of this formula.

Table II. was deduced from Reuschle's table* of decompositions of primes into the form $a^2 + b^2$.

Table III. was calculated by means of the formulæ in §§ 7 and 10.

Remarks on the formulæ in the paper, § 70.

§ 70. The results given in this paper relate to the five functions $E(n)$, $\chi(n)$, $\psi(n)$, $E_2(n)$, $\lambda(n)$; but the formulæ which are deducible from the q-series in Elliptic Functions are so numerous that it is only possible to consider in detail, without great expenditure of space, those relating to a very restricted group of the functions occurring as coefficients. In the preceding sections the relations connecting $\chi(n)$, $E(n)$, and $\psi(n)$ are those which have been most fully considered; and they afford a good example of the complete system of such formulæ. The most interesting of the functions, however, appear to be the three $E(n)$, $\chi(n)$, $\lambda(n)$, as these quantities depend in so simple a manner upon the complex numbers of which n is norm, and are so closely allied to one another by their definitions. It is an interesting fact also that the coefficients in the q-series for such simple quantities as $k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho$, ..., $k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^2$, ..., $kk'\rho^3$, ... depend upon the complex divisors of the exponent.

The next seven sections relate to certain groups of relations which seem to be especially deserving of notice.

* "Mathematische Abhandlung des Professors Reuschle, enthaltend neue zahlentheoretische Tabellen," pp. 32—38. Reuschle has omitted the decompositions of 197, 11173, 12269, 12301, and 12373.

Formulæ involving ψ, E_2, and λ, § 71.

§ 71. We find

$$k'^{\frac{1}{2}} = \frac{\Sigma_0^\infty (-)^n \psi(4n+1) q^n}{\Sigma_0^\infty \psi(4n+1) q^n}$$

$$= \frac{\Sigma_0^\infty (-)^n E_2(4n+3) q^n}{\Sigma_0^\infty E_2(4n+3) q^n}$$

$$= \frac{\Sigma_0^\infty \lambda(4n+1) q^n}{\Sigma_0^\infty (-)^n \lambda(4n+1) q^n};$$

and from these equalities we may deduce that, if

$$t = 8n+5, \qquad v = 8n+7,$$

then

(i)

$$\psi(t)\lambda(1) + \psi(t-4)\lambda(5) \ldots + \psi(1)\lambda(t) = 0.$$

(ii)

$$E_2(v)\lambda(1) + E_2(v-4)\lambda(5) \ldots + E_2(3)\lambda(s) = 0,$$

(iii)

$$E_2(v)\psi(1) - E_2(v-4)\psi(5) \ldots \pm E_2(3)\psi(s) = 0.$$

Formulæ involving E, ψ, E_2, χ, §§ 72, 73.

§ 72. We find also

$$k' = \frac{\Sigma_0^\infty (-)^n \psi(2n+1) q^n}{\Sigma_0^\infty \psi(2n+1) q^n}$$

$$= \frac{\Sigma_0^\infty (-)^n E(n) q^n}{\Sigma_0^\infty E(n) q^n}$$

$$= \frac{\Sigma_0^\infty (-)^n E_2(n) q^n}{\Sigma_0^\infty E_2(n) q^n};$$

whence we deduce that, if

$$m = 2n + 1, \qquad r = 4n + 3,$$

then

(i)

$$E(0)\,\psi(r) - E(1)\,\psi(r-2)\ldots \pm E(m)\,\psi(1) = 0,$$

(ii)

$$E_2(0)\,\psi(r) + E(1)\,\psi(r-2)\ldots + E_2(m)\,\psi(1) = 0,$$

(iii)

$$E(0)\,E_2(m) + E(1)\,E_2(m-1)\ldots + E(m)\,E_2(0) = 0,$$

the values of $E(0)$ and $E_2(0)$ being $\frac{1}{4}$ and $-\frac{1}{4}$ respectively.

§ 73. Since

$$k' = \frac{\Sigma_0^\infty\,\chi(4n+1)\,q^n}{\Sigma_0^\infty\,\psi(4n+1)\,q^n},$$

we find also that, if

$$m = 2n + 1, \qquad p = 4n + 1,$$

then

(i)

$$E(0)\,\chi(p) + E(1)\,\chi(p-4)\ldots + E(m)\,\chi(1)$$
$$= E(0)\,\psi(p) - E(1)\,\psi(p-4)\ldots \pm E(m)\,\psi(1).$$

(ii)

$$E_2(0)\,\chi(p) - E_2(1)\,\chi(p-4)\ldots \pm E_2(m)\,\chi(1)$$
$$= E_2(0)\,\psi(p) + E_2(1)\,\psi(p-4)\ldots + E_2(m)\,\psi(1).$$

(iii)

$$\psi(1)\,\chi(p) + \psi(3)\,\chi(p-4)\ldots + \psi(m)\,\chi(1)$$
$$= \psi(1)\,\psi(p) - \psi(3)\,\psi(p-4)\ldots \pm \psi(m)\,\psi(1).$$

The second and third equations have been given already as (iii) of § 56 (p. 145) and (x) of § 35 (p. 126).

System of formulæ for $\Delta'_3(n)$, §§ 74, 75.

§ 74. If we denote by $\Delta'_3(n)$ the sum of the cubes of those divisors of n which have uneven conjugates, we have in Elliptic Functions the formula

$$k^2k'^2\rho^4 = 16\Sigma_1^\infty(-)^{n-1}\Delta'_3(n)q^n;$$

and, since

$$\begin{aligned} k^2k'^2\rho^4 &= kk'\rho^2 \times kk'\rho^2 \\ &= k^{\frac12}k'^{\frac12}\rho^2 \times k^{\frac32}k'^{\frac32}\rho^2 \\ &= k\rho \times kk'^2\rho^3 \\ &= k^{\frac12}k'^{\frac12}\rho \times k^{\frac32}k^{\frac32}\rho^3, \end{aligned}$$

we thus obtain (changing the sign of q) the following system of algebraical equalities:

$$\begin{aligned} &\Sigma_0^\infty \Delta'_3(n+1)q^n \\ = &\ \Sigma_0^\infty \psi(2n+1)q^n \times \Sigma_0^\infty \psi(2n+1)q^n \\ = &\ \tfrac14\Sigma_0^\infty \psi(4n+1)q^n \times \Sigma_0^\infty \psi(4n+3)q^n \\ = &\ \Sigma_0^\infty E(2n+1)q^n \times \Sigma_0^\infty (-)^n E_2(2n+1)q^n \\ = &-\tfrac18\Sigma_0^\infty E(4n+1)q^n \times \Sigma_0^\infty E_2(4n+3)q^n. \end{aligned}$$

§ 75. By equating the coefficients of q^n, we find that the following four expressions in which

$$m = 2n+1, \qquad r = 4n+3,$$

are all equal to $\Delta'_3(n+1)$:

(i)

$$\psi(1)\psi(m) + \psi(3)\psi(m-2) \ldots + \psi(m)\psi(1),$$

(ii)

$$\tfrac14\{\psi(1)\psi(r) + \psi(5)\psi(r-4) \ldots + \psi(p)\psi(3)\},$$

(iii)

$$(-)^n\{E(1)E_2(m) + E(5)E_2(m-4) + \ldots\ldots\ldots\ldots\},$$

(iv)

$$-\tfrac18\{E(1)E_2(r) + E(5)E_2(r-4) \ldots + E(p)E_2(3)\}.$$

System of six equal formulæ, §§ 76, 77.

§ 76. Since

$$\begin{aligned} kk'\rho^4 &= k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^2 \times k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^2 \\ &= k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho \times k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^3 \\ &= k^{\frac{1}{2}}\rho^2 \times k^{\frac{1}{2}}k'\rho^2 \\ &= k'\rho \times k\rho^3 \\ &= k\rho \times k'\rho^3 \\ &= \rho \times kk'\rho^3, \end{aligned}$$

it follows that the following six products are all equal to one another:

(i)

$$\Sigma_0^\infty \; \chi(4n+1)q^n \times \Sigma_0^\infty \; \chi(4n+1)q^n,$$

(ii)

$$\Sigma_0^\infty \; E(4n+1)q^n \times \Sigma_0^\infty \; \lambda(4n+1)q^n,$$

(iii)

$$\Sigma_0^\infty(-)^n\psi(4n+1)q^n \times \Sigma_0^\infty(-)^n\chi(4n+1)q^n,$$

(iv)

$$4\,\Sigma_0^\infty \; E(n)q^n \times \Sigma_0^\infty \; E_2(2n+1)q^n,$$

(v)

$$-4\,\Sigma_0^\infty \; E_2(n)q^n \times \Sigma_0^\infty \; E(2n+1)q^n,$$

(vi)

$$4\,\Sigma_0^\infty(-)^n E(n)q^n \times \Sigma_0^\infty \; \lambda(4n+1)q^n.$$

§ 77. By equating coefficients in these expressions, we find that the six quantities

(i)

$$\chi(1)\chi(p) + \chi(5)\chi(p-4) \ldots + \chi(p)\chi(1),$$

(ii)

$$E(1)\lambda(p) + E(5)\lambda(p-4) \ldots + E(p)\lambda(1),$$

(iii)

$$(-)^n\{\psi(1)\,\chi(p)+\psi(5)\,\chi(p-4)\ldots+\psi(p)\,\chi(1)\},$$

(iv)

$$4\{E(0)\,E_2(m)+E(1)\,E_2(m-2)\ldots+E(n)\,E_2(1)\},$$

(v)

$$-4\{E_2(0)\,E(m)+E_2(1)\,E(m-2)\ldots+E_2(n)\,E(1)\},$$

(vi)

$$4\{E(0)\,\lambda(m)-E(1)\,\lambda(m-2)\ldots\pm E(n)\,\lambda(1)\},$$

are all equal.

Each of these quantities is equal to $\frac{1}{4}\times(-1)^n\times$ coefficient of $q^{n+\frac{1}{2}}$ in the q-series for $kk'\rho^4$, but I know of no single function (corresponding to Δ'_3 in §75), which serves to express this coefficient.

On results involving $E(n)$ and $\psi(n)$ only, §78.

§78. As the subject of the present paper is $\chi(n)$, I have as a rule, omitted results which involve $E(n)$ and $\psi(n)$ only. Several formulæ involving $E(n)$, in which the terms follow laws of the same kind as those which occur in §§ 39–45, are contained in a paper* communicated to the London Mathematical Society on February 14, 1884; and a collection of formulæ involving $\psi(n)$ is given in a paper† which was communicated to the Cambridge Philosophical Society on January 28, 1884, and is now in course of publication in their *Transactions*.‡

The former paper contains a table of $E(n)$ up to $n=1000$; and the latter contains a table of $\psi(n)$ up to $n=3000$.

The five functions, §§ 79, 80.

§79. In the three following sections I have collected together for reference the definitions of the five functions and the groups of q-series which have been used in deriving the

* "On the difference between the number of $(4m+1)$-divisors and the number of $(4m+3)$-divisors of a number."

† "Tables of the number of numbers not greater than a given number and prime to it, and of the number and sum of the divisors of a number, with the corresponding inverse tables, up to 3000."

‡ These papers contain also corresponding formulæ relating to the function $\zeta(n)$, which denotes the excess of the sum of the uneven divisors of n over the sum of the even divisors.

formulæ contained in the paper; and I have also added a table giving the values of the five functions up to $n = 100$. This table I found very useful in verifying the formulæ.

A table of contents, consisting of the sectional headings, with references to the pages, is appended (p. 164).

§ 80. The definitions of the five functions are:

$$E(n) = \left.\begin{array}{l}\text{number of } (4m+1)\text{-divisors of } n \\ -\ \ ,,\ \ ,,\ \ (4m+3)\text{-divisors}\ ,,\end{array}\right\},$$

$= $ number of primary numbers having n as norm, if n be uneven.

$\psi(n) = $ sum of the divisors of n,

$\chi(n) = $ sum of primary numbers having n as norm,

$$E_2(n) = \left.\begin{array}{l}\text{sum of squares of } (4m+1)\text{-divisors of } n \\ -\ \ ,,\ \ ,,\ \ (4m+3)\text{-divisors}\ ,,\end{array}\right\},$$

$\lambda(n) = $ sum of squares of primary numbers having n as norm.

The definitions of $\chi(n)$ and $\lambda(n)$ apply only to the case of n uneven.

In general $\chi(n) = 0$, unless $n = uv^2$, where all the prime factors of u are of the form $4m+1$ and all the prime factors of v are of the form $4m+3$, (the case $v^2 = 1$ being included), and then

$$\chi(n) = (-)^r v\chi(u),$$

where r denotes the sum of the exponents of the prime factors in v, viz. the sum of the exponents when v is resolved into its prime factors (see § 11).

Similarly $\lambda(n) = 0$, unless $n = uv^2$, and then

$$\lambda(n) = v^2\lambda(u),$$

and $E(n) = 0$, unless $n = 2^p uv^2$, and then

$$E(n) = E(u).$$

In the case of $E(n)$ and $E_2(n)$ the presence of a power of 2 in the argument does not affect the value of the function, *i.e.* if $n = 2^p r$,

$$E(n) = E(r), \qquad E_2(n) = E_2(r).$$

List of q-series, § 81.

§ 81. The following is a complete list of the q-series which have been used in this paper:

(i)

$$
\begin{aligned}
k^{\frac{1}{2}}\rho &= 2q^{\frac{1}{4}}\,\Sigma_0^\infty \qquad\;\; E(4n+1)\,q^n,\\
k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho &= 2q^{\frac{1}{4}}\,\Sigma_0^\infty\,(-)^n\,E(4n+1)\,q^n,\\
k\rho &= 4q^{\frac{1}{2}}\,\Sigma_0^\infty \qquad\;\; E(4n+1)\,q^{2n}\\
&= 4q^{\frac{1}{2}}\,\Sigma_0^\infty \qquad\;\; E(2n+1)\,q^n \;;
\end{aligned}
$$

(ii)

$$
\begin{aligned}
k\rho^2 &= 4q^{\frac{1}{2}}\,\Sigma_0^\infty \qquad\;\; \psi(2n+1)\,q^n,\\
kk'\rho^2 &= 4q^{\frac{1}{2}}\,\Sigma_0^\infty\,(-)^n\,\psi(2n+1)\,q^n,\\
k^2\rho^2 &= 16q\;\Sigma_0^\infty \qquad\;\; \psi(2n+1)\,q^{2n}\;;
\end{aligned}
$$

(iii)

$$
\begin{aligned}
k^{\frac{1}{2}}\rho^2 &= 2q^{\frac{1}{4}}\,\Sigma_0^\infty \qquad\;\; \psi(4n+1)\,q^n,\\
k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^2 &= 2q^{\frac{1}{4}}\,\Sigma_0^\infty\,(-)^n\,\psi(4n+1)\,q^n,\\
k^{\frac{3}{2}}\rho^2 &= 2q^{\frac{3}{4}}\,\Sigma_0^\infty \qquad\;\; \psi(4n+3)\,q^n,\\
k^{\frac{3}{2}}k'^{\frac{1}{2}}\rho^2 &= 2q^{\frac{3}{4}}\,\Sigma_0^\infty\,(-)^n\,\psi(4n+3)\,q^n\;;
\end{aligned}
$$

(iv)

$$
\begin{aligned}
k^{\frac{1}{2}}k'\rho^2 &= 2q^{\frac{1}{4}}\,\Sigma_0^\infty \qquad\;\; \chi(4n+1)\,q^n,\\
k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^2 &= 2q^{\frac{1}{4}}\,\Sigma_0^\infty\,(-)^n\,\chi(4n+1)\,q^n,\\
kk'^{\frac{1}{2}}\rho^2 &= 4q^{\frac{1}{2}}\,\Sigma_0^\infty \qquad\;\; \chi(4n+1)\,q^{2n}\\
&= 4q^{\frac{1}{2}}\,\Sigma_0^\infty \qquad\;\; \chi(2n+1)\,q^n\,,
\end{aligned}
$$

(v)

$$
\begin{aligned}
k\rho^3 &= 4q^{\frac{1}{2}}\Sigma_0^\infty\,(-)^n\,E_2(2n+1)\,q^n,\\
kk'^2\rho^3 &= 4q^{\frac{1}{2}}\Sigma_0^\infty \qquad\;\; E_2(2n+1)\,q^n\;;
\end{aligned}
$$

(vi)

$$
\begin{aligned}
k^{\frac{3}{2}}\rho^3 &= -\;q^{\frac{3}{4}}\Sigma_0^\infty \qquad\;\; E_2(4n+3)\,q^n,\\
k^{\frac{3}{2}}k'^{\frac{1}{2}}\rho^3 &= -\;q^{\frac{3}{4}}\Sigma_0^\infty\,(-)^n\,E_2(4n+3)\,q^n,\\
k^3\rho^3 &= -\,8q^{\frac{3}{2}}\Sigma_0^\infty \qquad\;\; E_2(4n+3)\,q^{2n}\;;
\end{aligned}
$$

(vii)

$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{2} = 2q^{\frac{1}{4}}\Sigma_0^{\infty} \quad \lambda(4n+1)q^{n},$$
$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{2} = 2q^{\frac{1}{4}}\Sigma_0^{\infty}(-)^{n}\lambda(4n+1)q^{n},$$
$$\left.\begin{aligned} kk'\rho^{2} &= 4q^{\frac{1}{2}}\Sigma_0^{\infty} \quad \lambda(4n+1)q^{2n} \\ &= 4q^{\frac{1}{2}}\Sigma_0^{\infty} \quad \lambda(2n+1)q^{n} \end{aligned}\right\},$$
$$k^{2}k'^{\frac{1}{2}}\rho^{2} = 16q\,\Sigma_0^{\infty} \quad \lambda(4n+1)q^{4n};$$

(viii)

$$\rho = 4\Sigma_0^{\infty} \quad E(n)q^{n},$$
$$k'\rho = 4\Sigma_0^{\infty}(-)^{n}E(n)q^{n},$$
$$k'^{\frac{1}{2}}\rho = 4\Sigma_0^{\infty}(-)^{n}E(n)q^{2n},$$
$$E(0) = \tfrac{1}{4};$$

(ix)

$$k'\rho^{2} = -4\Sigma_0^{\infty}(-)^{n}E_3(n)q^{n},$$
$$k'^{2}\rho^{2} = -4\Sigma_0^{\infty} \quad E_3(n)q^{n},$$
$$E_3(0) = -\tfrac{1}{4}.$$

The following list shows the sections in which the different groups were given:

(i), § 20 (p. 111), (v), (vi), § 53 (p. 143),
(ii), § 20 (p. 111), (vii), § 62 (p. 148),
(iii), § 21 (p. 112), (viii), § 23 (p. 114), § 46 (p. 137),
(iv), § 19 (p. 111), (ix), § 53 (p. 143).

The first four groups were reproduced in § 31 (p. 119).

The five q-series

$$\rho^{2} = 1 + 8\Sigma_1^{\infty}\{2 + (-1)^{n} \quad \}\Delta(n)q^{n},$$
$$k'^{2}\rho^{2} = 1 + 8\Sigma_1^{\infty}\{1 + (-1)^{n}2\}\Delta(n)q^{n},$$
$$k'\rho^{2} = 1 + 8\Sigma_1^{\infty}\{1 + (-1)^{n}2\}\Delta(n)q^{2n},$$
$$k'^{\frac{1}{2}}\rho^{2} = 1 + 4\Sigma_1^{\infty}(\ \ 1, 0, -1, -2, \ \ 1, 0, -1, 6)\Delta(n)q^{n},$$
$$k'^{\frac{3}{2}}\rho^{2} = 1 + 4\Sigma_1^{\infty}(-1, 0, \ \ 1, -2, -1, 0, \ \ 1, 6)\Delta(n)q^{n},$$

were given in § 23 (p. 113), but no use has been made of them in the paper. The q-series for $\rho^{\frac{1}{2}}$, $k'^{\frac{1}{4}}\rho^{\frac{1}{2}}$, $k^{\frac{1}{4}}\rho^{\frac{1}{2}}$, $k^{\frac{1}{4}}k'^{\frac{1}{4}}\rho^{\frac{1}{2}}$ were given in § 19 (p. 111). The formula $k^{2}k'^{2}\rho^{4} = 16\Sigma_1^{\infty}$, $(-1)^{n-1}\Delta'_3(n)q^{n}$ was used in § 74 (p. 158).

Table of the five functions, § 82.

Values of $\chi(n)$, $\psi(n)$, $E(n)$, $E_2(n)$, $\lambda(n)$ from $n=1$ to $n=100$.

Table IV.

n	$\chi(n)$	$\psi(n)$	$E(n)$	$E_2(n)$	$\lambda(n)$
1	1	1	1	+ 1	1
2	...	3	1	+ 1	...
3	0	4	0	− 8	0
4	...	7	1	+ 1	...
5	− 2	6	2	+ 26	− 6
6	...	12	0	− 8	...
7	0	8	0	− 48	0
8	...	15	1	+ 1	...
9	− 3	13	1	+ 73	+ 9
10	...	18	2	+ 26	...
11	0	12	0	− 120	0
12	...	28	0	− 8	...
13	+ 6	14	2	+ 170	+ 10
14	...	24	0	− 48	...
15	0	24	0	− 208	0
16	...	31	1	+ 1	...
17	+ 2	18	2	+ 290	− 30
18	...	39	1	+ 73	...
19	0	20	0	− 360	0
20	...	42	2	+ 26	...
21	0	32	0	+ 384	0
22	...	36	0	− 120	...
23	0	24	0	− 528	0
24	...	60	0	− 8	...
25	− 1	31	3	+ 651	+ 11
26	...	42	2	+ 170	...
27	0	40	0	− 656	0
28	...	56	0	− 48	...
29	− 10	30	2	+ 842	+ 42
30	...	72	0	− 208	...
31	0	32	0	− 960	0
32	...	63	1	+ 1	...
33	0	48	0	+ 960	0
34	...	54	2	+ 290	..
35	0	48	0	− 1248	0
36	...	91	1	+ 73	...
37	− 2	38	2	+ 1370	− 70
38	...	60	0	− 360	...
39	0	56	0	− 1360	0
40	...	90	2	+ 26	...
41	+ 10	42	2	+ 1682	+ 18
42	...	96	0	+ 384	...
43	0	44	0	− 1848	0
44	...	84	0	− 120	...
45	+ 6	78	2	+ 1898	− 51
46	...	72	0	− 528	...
47	0	48	0	− 2208	0
48	...	124	0	− 8	...
49	− 7	57	1	+ 2353	+ 49
50	...	93	3	+ 651	...

Table IV (continued).

n	$\chi(n)$	$\psi(n)$	$E(n)$	$E_3(n)$	$\lambda(n)$
51	0	72	0	− 2320	0
52	...	98	2	+ 170	...
53	+ 14	54	2	+ 2810	+ 90
54	..	120	0	− 656	...
55	0	72	0	− 3120	0
56	...	120	0	− 48	...
57	0	80	0	+ 2880	0
58	...	90	2	+ 842	...
59	0	60	0	− 3480	0
60	...	168	0	− 208	...
61	− 10	62	2	+ 3722	− 22
62	...	96	0	− 960	...
63	0	104	0	− 3504	0
64	...	127	1	+ 1	...
65	− 12	84	4	+ 4420	− 60
66	...	144	0	+ 960	..
67	0	68	0	− 4488	0
68	...	126	2	+ 290	...
69	0	96	0	+ 4224	0
70	...	144	0	− 1248	...
71	0	72	0	− 5040	0
72	...	195	1	+ 73	...
73	− 6	74	2	+ 5330	− 110
74	...	114	2	+ 1370	...
75	0	124	0	− 5208	0
76	...	140	0	− 360	...
77	0	96	0	+ 5760	0
78	...	168	0	− 1360	...
79	0	80	0	− 6240	0
80	...	186	2	+ 26	...
81	+ 9	121	1	+ 5905	+ 81
82	...	126	2	+ 1682	...
83	0	84	0	− 6888	0
84	...	224	0	+ 384	...
85	− 4	108	4	+ 7540	+ 180
86	...	132	0	− 1848	...
87	0	120	0	− 6736	0
88	...	180	0	− 120	...
89	+ 10	90	2	+ 7922	− 78
90	...	234	2	+ 1898	...
91	0	112	0	− 8160	0
92	...	168	0	− 528	...
93	0	128	0	+ 7680	0
94	...	144	0	− 2208	...
95	0	120	0	− 9360	0
96	...	252	0	− 8	...
97	+ 18	98	2	+ 9410	+ 130
98	...	171	1	+ 2353	...
99	0	156	0	− 8760	0
100	...	217	3	+ 651	...

CONTENTS.

Erratum—In II., p. 131, the sign $(-)^n$ should be applied to one member of the equation, *i.e.* the equation should be

$$\chi(p) + 2\chi(p-8) + 2\chi(p-16) + \ldots = (-)^n \{E(p) - 2E(p-8) - 2E(p-16) + \ldots\}.$$

THE SYMMEDIAN-POINT AXIS OF AN ASSOCIATED SYSTEM OF TRIANGLES.

By R. Tucker, M.A.

THE following notes arose out of a consideration of the question to shew that *the three straight lines joining the mid-point of each side of a triangle to the mid-point of the corresponding perpendicular meet in a point.* My attention was particularly drawn to the subject by a letter I received from Dr. Casey (April 16th, 1884), in which he also states the point to be the "Symmedian-point" of the triangle. He was not aware that he had been partly anticipated in Question 7644 of the *Educational Times* (March, 1884). The earliest publication of this neat result, however, appears to be by Prof. J. Neuberg in his paper "Sur le centre des médianes anti-parallèles," where the Author also shews that the point is the "Symmedian-point" (*point de Grebe*) of ABC.

Let AD, BE, CF be the perpendiculars meeting in the orthocentre P, and let D', E', F'; d, e, f be the mid-points of the sides and of the perpendiculars.

Then since

$$2dE' = DC,\ 2eF' = AE,\ 2fE' = AF,$$

we get for the $\Delta D'E'F'$,

$$dE'.eF'.fD'. = fE'.dF'.eD';$$

whence $D'd$, $E'e$, $F'f$ meet in a point K, the "Symmedian-point" of ABC.

Now on the respective sides of ABC take

$$B\alpha = CD,\ C\beta = AE,\ B\gamma = AF,$$

and we see that $A\alpha$, $B\beta$, $C\gamma$ meet in a point π. Hence, by the above result, π is the "Symmedian-point" of the triangle formed by drawing lines through A, B, C, parallel to the opposite sides; and $P\pi$ is the diameter of the Brocard circle of the same triangle.