A continued fraction expansion related to A002105

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The ordinary generating function of the sequence of reduced tangent numbers A002105 has a corresponding continued fraction of Stieltjes-type with partial numerators given by the triangular numbers. We find a related Stieltjes-type continued fraction for the second binomial transform of the generating function.

A002105 is the sequence of reduced tangent numbers starting $[1, 1, 4, 34, 496, ...]$. The sequence has several combinatorial interpretations. One version of the e.g.f. for the sequence (with interpolated zeros) is

$$\sec^2\left(\frac{x}{\sqrt{2}}\right) \quad = \quad 1 + \frac{x^2}{2!} + \frac{4x^4}{4!} + \frac{34x^6}{6!} + \cdots . \tag{1}$$

It will be convenient in what follows to work with an offset of 0 for the sequence; the o.g.f. $A(x)$ for the sequence thus begins

$$A(x) = 1 + x + 4x^2 + 34x^3 + 496x^4 + \cdots .$$

It is a consequence of (1) and a result of Stieltjes (see [1, Chapter XI, eqn. 53.11 with $k = 2$]) that the Stieltjes-type fraction (S-fraction)

$$\cfrac{1}{1 - \cfrac{x}{1 - \cfrac{3x}{1 - \cfrac{6x}{1 - \cfrac{10x}{1 - \cdots}}}}} \tag{2}$$

corresponds to $A(x)$; that is, the expansion in ascending powers of x of the N-th approximant of this continued fraction agrees with the power series $A(x)$ up to and including the term in $x^{N-1}, N = 1, 2, 3, \ldots$: the coefficients [1,3,6,10,...] in the partial numerators of the continued fraction are the triangular numbers $n(n + 1)/2$.

Sequences of combinatorial interest which have nice S-fraction representations with integer partial numerators are quite rare so it is interesting to note that the generating function of the second binomial transform of A002105 also has an S-fraction development:

$$\frac{1}{1-2x}A\left(\frac{x}{1-2x}\right) = \cfrac{1}{1-\cfrac{3x}{1-\cfrac{x}{1-\cfrac{10x}{1-\cfrac{6x}{1-\cdots}}}}} \tag{3}$$

where now the sequence of coefficients [3,1,6,10,...] in the partial numerators is obtained by swapping adjacent triangular numbers.

The proof of (3) uses the contraction of a continued fraction. Recall the **even part** of a continued fraction is the continued fraction whose N-th approximant is the $2N$-th approximant of the given continued fraction. Consider the continued fraction

$$\cfrac{1}{1-\cfrac{a_1x}{1-\cfrac{a_2x}{1-\cfrac{a_3x}{1-\cdots}}}} \tag{4}$$

of Stieltjes-type. The even part of (4) is given by

$$\cfrac{1}{1-a_1x-\cfrac{a_1a_2x^2}{1-(a_2+a_3)\,x-\cfrac{a_3a_4x^2}{1-(a_4+a_5)\,x-\cfrac{a_5a_6x^2}{1-(a_6+a_7)\,x-\cdots}}}} \tag{5}$$

We show (3) holds by verifying that the even part of the S-fraction on the right side of (3) equals the second binomial transform of the even part of the S-fraction (2).

By (5) we find the even part of the right side of (3) equals

$$\cfrac{1}{1-3x-\cfrac{3x^2}{1-(1+10)x-\cfrac{60x^2}{1-(6+21)x-\cdots-\cfrac{n^2\left(4n^2-1\right)\right)x^2}{1-(4n^2+4n+3)\,x-\cdots}}}} \tag{6}$$

By (5) we find the even part of (2) is

$$\cfrac{1}{1-x-\cfrac{3x^2}{1-9x-\cfrac{60x^2}{1-25x-\cdots-\cfrac{n^2\left(4n^2-1\right)x^2}{1-(2n+1)^2x-\cdots}}}} \tag{7}$$

The second binomial transform of (7) equals

$$\frac{1}{1-2x}\cfrac{1}{1-\frac{x}{1-2x}-\cfrac{3\left(\frac{x}{1-2x}\right)^2}{1-9\frac{x}{1-2x}-\cfrac{60\left(\frac{x}{1-2x}\right)^2}{1-25\frac{x}{1-2x}-\cdots-\cfrac{n^2\left(4n^2-1\right)\left(\frac{x}{1-2x}\right)^2}{1-(2n+1)^2\frac{x}{1-2x}-\cdots}}}} \tag{8}$$

which, with the help of an equivalence transformation, simplifies to

$$\cfrac{1}{1-(2+1)x-\cfrac{3x^2}{1-(2+9)x-\cfrac{60x^2}{1-(2+25)x-\cdots-\cfrac{n^2\left(4n^2-1\right)x^2}{1-\left(2+(2n+1)^2\right)x-\cdots}}}} \tag{9}$$

which agrees with (6). This completes the proof of (3).

Finally, it easily follows from (3) (after making use of an equivalence transformation) that the original generating function $A(x)$ has the continued fraction representation

$$A(x) = \cfrac{1}{1+2x-\cfrac{3x}{1-\cfrac{x}{1+2x-\cfrac{10x}{1-\cfrac{6x}{1+2x-\cdots}}}}} \tag{10}$$

References

[1] H. S. Wall, Analytic Theory of Continued Fractions. Reprinted by AMS Chelsea Publishing, 2000

[2] Wikipedia, Generalized continued fraction