is: the output value of the

$\ldots - 1)w_c > 0$ (85)

$\ldots - 1)w_c < 0,$

input vector $(j = 1, \cdots, 2^n)$

icture of a majority decision
to the linear programming
$_1 + \cdots + w_n + w_c$ is to be
iequalities (85) where equali-
fter knowing the minimum of
the maximum of those of the

he inequalities in (85) do not
small $w_i$'s and $w_c$ can be solu-
minimization problem. Then
rst inequalities and the maxi-
ties ...uld be known or pre-
performance of the element.
$\epsilon_2|$ and write it as $\epsilon$, that is,
(85) can be made to include
ialities by dividing these by $\epsilon$.
of Problem 1 and the simplex
the same processes as those
easily seen. This form of the
working directly on $w_c$ without

to see that if we introduce a
y decision element given by
the constraints in Problem 3
he objective function also will
des formally with Problem 1
non-integral coupling numbers
s to solve Problem 1 formally.
blem 1 the relation between $T$
zation of the threshold $T$ only,
ng into account relation (3),
s of coupling numbers which

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TABLE II.—*Numbers of Functions.*

| $n$ | $2^{2^n}$ | Number of Positive Functions of up to $n$ Variables | Number of Majority Decision Functions of $n$ Variables without Negation: Representative Functions* | Number of Majority Decision Functions of $n$ Variables without Negation: Functions Counted with Permutation | Number of Majority Decision Functions of up to $n$ Variables without Negation | Number $N(n)$ of Majority Decision Functions of up to $n$ Variables with Permutation and Negation | Theoretical Evaluation of $N(n)$: Lower Bound | Theoretical Evaluation of $N(n)$: Upper Bound |
|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 3 | 1 | 1 | 3 | 4 | 4 | / |
| 2 | 16 | 6 | 2 | 2 | 6 | 14 | 14 | 27 |
| 3 | 256 | 20 | 5 | 9 | 20 | 104 | 104 | 875 |
| 4 | 65536 | 168 | 17 | 96 | 150 | 1882 | 1882 | $3.6\times10^4$ |
| 5 | 4294967296 | 7581 | 92 | 2690 | 3287 | 94572 | 32172 | $1.1\times10^7$ |
| 6 | 18446744073709551616 | 7828354 | 994 | 226360 | 244158 | 15028134 | / | $3.7\times10^{10}$ |

* Functions identical by permutations and negations of variables are counted into a single representative function.

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### 6. MAJORITY DECISION FUNCTIONS OF UP TO SIX VARIABLES

In order to check various properties of the majority decision functions described so far, the Parametron Computer MUSASINO-1 (15) was used to find all majority decision functions of up to six variables and the structures of elements to realize these, based on Problems 1 and 2.

*Majority Decision Functions of a Few Variables and Their Number*

All majority decision functions of up to six variables are classified by permutations of variables, and negations of the variables. A representative function in each class such that $w_1 \geqq w_2 \geqq w_3 \geqq w_4 \geqq w_5 \geqq w_6 \geqq 0$ are obtained (16, 17). The number of these functions is too great to be shown here.

These functions were checked both by the simplex method and the combination method, where the functions are discovered by giving all possible integer values to the inputs.

Table II lists the numbers of functions enumerated from several different viewpoints (18).

(i) The number of general Boolean functions of up to $n$ variables, $2^{2^n}$, is shown for reference in the second column of Table II.

(ii) The number of the positive functions of up to $n$ variables, those that can be realized without negation of the variables. These may generally not be realizable by a single majority decision element. The numbers were quoted from Birkhoff's book (19). The majority decision functions constitute only a subset of these.

(iii) The number of the majority decision functions of exactly $n$ variables without negations of the variables. This is divided into two cases. The first case shows the number of the majority decision functions, representatives of equivalent classes, each of which consists of functions identical by permutations and negations of the variables. For example, $x_1x_2 + x_3$, $x_1x_3 + x_2$ and $\bar{x}_1 \cdot \bar{x}_2 + x_3$ belong to the same class represented by $x_1 + x_2x_3$. The second case shows the number of the functions in the first case, taking into account only permutations of variables. The functions in the second case are not enumerable unless the forms of the functions representing the above classes are known, because there are partially or totally symmetric functions. For example, for $n = 2$, we have two, counting $x_1 + x_2$ and $x_1x_2$.

(iv) The number of the majority decision functions of up to $n$ variables, without negations of variables. For example, for $n = 2$ we have six, counting 0, 1, $x_1$, $x_2$, $x_1 + x_2$ and $x_1x_2$.

(v) The number $N(n)$ of the majority decision functions of up to $n$ variables, taking into account permutations and negations of variables. If this is compared with $2^{2^n}$, we can see how small a part of $2^{2^n}$ this occupies.

(vi) Theoretical upper and lower bounds on $N(n)$ shown in Section 4. The lower bound shows the number of functions constructed similarly to