A COTANGENT ANALOGUE OF CONTINUED FRACTIONS

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The continued iteration of a rational function $f(x, y)$ of two variables provides an algorithm for the expression of a real number as a sequence of rational numbers. Thus the function

$$f(x_1, f(x_2, f(x_3, \cdots))) \tag{1}$$

becomes an infinite series for $f(x, y) = x + y$ and an infinite product for $f(x, y) = xy$. For $f(x, y) = x + 1/y$ we obtain the regular continued fraction

$$x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \cdots}} = x_1 + \frac{1|}{|x_2} + \frac{1|}{|x_3} + \cdots .$$

By far the most frequently used function is $f(x, y) = x + y/c$, which gives the "power series"

$$x_1 + \cfrac{x_2 + \cfrac{x_3 + \cdots}{c}}{c} = x_1 + \frac{x_2}{c} + \frac{x_3}{c^2} + \cdots ,$$

where the x's are the coefficients, used when $c = 10$ for the decimal representation of real numbers.[1] The algorithm associated with $f(x, y) = x(1 - y)$ has been discussed by T. A. Pierce.[2]

This paper is concerned with the case of

$$f(x, y) = (xy + 1)/(y - x) = \cot(\text{arc cot } x - \text{arc cot } y),$$

so that (1) becomes the function

$$\cot(\text{arc cot } x_1 - \text{arc cot } x_2 + \text{arc cot } x_3 - \cdots).$$

This function, despite its aspect, is no more transcendental than a regular continued fraction and both functions have many properties in common. Furthermore, in order to obtain sequences of rational approximations to a real number, we specialize the x's to be integers, as in the continued fraction, and consider therefore expressions of the form

$$\cot \sum_{\nu=0} (-1)^\nu \text{arc cot } n_\nu , \tag{2}$$

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[1] This use of the function $x + y/c$ is at least 4000 years old. See Amer. Jour. of Semitic Languages and Literature, vol. 36(1920), No. 4. The Babylonians used $c = 60$.

[2] Amer. Math. Monthly, vol. 36(1929), pp. 523–525.

where the n_ν are integers. This expression will be called a "continued cotangent", and we shall use the adjective "finite" or "infinite" according as the series in (2) terminates or not. Although finite and infinite sums of arc cotangents of integers have been considered many times, no systematic treatment of such sums appears to have been given.

Definition of a regular continued cotangent. The continued cotangent (2) will be said to be regular if

(a) n_ν is an integer[3]. $\geqq 0$ for $\nu \geqq 0$.

(b) If (2) is finite and if n_k is the last n, then

$$n_k > n_{k-1}^2 + n_{k-1} + 1. \tag{3}$$

In all other cases

$$n_\nu \geqq n_{\nu-1}^2 + n_{\nu-1} + 1. \tag{4}$$

The principal value of arc cotangent n_ν is understood. In fact, since n_ν is non-negative,

$$0 < \text{arc cot } n_\nu \leqq \tfrac{1}{2}\pi.$$

The inequalities (3) and (4) seem at first sight unnatural. They are, however, the analogues of the inequalities

$$q_k > 1, \tag{3'}$$

$$q_\nu \geqq 1 \tag{4'}$$

for the incomplete quotients of the continued fraction

$$q_0 + \frac{1|}{|q_1} + \frac{1|}{|q_2} + \cdots,$$

which terminates with $\cdots + \frac{1|}{|q_k}$ or is infinite. The reason for insisting on the stronger inequality (3) in the case of a finite continued cotangent is the same as the reason for (3′) in the continued fraction: to insure for every rational number a unique expansion. As a matter of fact, if (4) held for n_k but not (3), so that

$$n_k = n_{k-1}^2 + n_{k-1} + 1, \tag{5}$$

then the last two terms of (2) could be replaced by a single term, since

$$\text{arc cot } n_{k-1} - \text{arc cot } (n_{k-1}^2 + n_{k-1} + 1) = \text{arc cot } (n_{k-1} + 1),$$

just as in continued fractions we write

$$\frac{1|}{|q_{k-1}} + \frac{1|}{|1} = \frac{1|}{|q_{k-1} + 1}.$$

[3] As in continued fractions we might allow n_0 to be negative. However, this extra generality is non-essential for our purposes.

Hence (3) may as well be assumed. It is perhaps worth noting that this contraction of the last two terms cannot be repeated in the continued cotangent any more than in continued fractions. In fact we would need to have as a counterpart of (5)

$$n_{k-1} + 1 = n_{k-2}^2 + n_{k-2} + 1.$$

This violates (4).

THEOREM 1. *Every infinite regular continued cotangent converges.*

Proof. We need merely to note that

$$\text{arc cot } n_0 - \text{arc cot } n_1 + \text{arc cot } n_2 - \cdots \tag{6}$$

form an alternating series of terms monotonically decreasing in absolute value in view of (4). Since (6) converges to a positive quantity, the cotangent of (6) exists, and this proves the theorem. In fact, it is easy to see that (6) converges not only absolutely but with tremendous rapidity, more rapidly, indeed, than the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{1}{256} + \frac{1}{65536} + \cdots + \frac{1}{2^{2^n}} + \cdots,$$

in view of (4) and the inequality

$$\text{arc cot } n_\nu < \frac{1}{n_\nu}.$$

This rapidity of convergence is a feature of the continued cotangent not enjoyed by the continued fraction. The least rapidly converging continued fraction may be said to be

$$\frac{\sqrt{5} - 1}{2} = 0 + \frac{1|}{|1} + \frac{1|}{|1} + \frac{1|}{|1} + \cdots, \tag{7}$$

whereas the least rapidly converging continued cotangent is

$$\begin{aligned} \xi = \cot (&\text{arc cot } 0 - \text{arc cot } 1 + \text{arc cot } 3 - \text{arc cot } 13 + \text{arc cot } 183 \\ &- \text{arc cot } 33673 + \text{arc cot } 1133904603 - \cdots), \end{aligned} \tag{8}$$

in which

$$n_{\nu+1} = n_\nu^2 + n_\nu + 1.$$

Uniqueness theorem. Theorem 1 guarantees that every continued cotangent represents a real positive number. Before treating the inverse problem of finding the continued cotangent expansion of a given number, we prove the following uniqueness theorem.

THEOREM 2. *Two regular continued cotangents can be equal only if they are identically equal.*

Proof. Let

$$\cot \sum_{\nu=0} (-1)^\nu \operatorname{arc}\cot n_\nu = \cot \sum_{\nu=0} (-1)^\nu \operatorname{arc}\cot m_\nu \tag{9}$$

be two equal regular continued cotangents, and suppose, if possible, that $n_\nu = m_\nu$ does not hold for all ν. Then there exists a first instance, $\nu = r$, where $n_r \neq m_r$ while $n_\nu = m_\nu$ for $\nu < r$ if $r \neq 0$. Then from (9) we have

$$\sum_{\lambda=0} (-1)^\lambda \operatorname{arc}\cot n_{r+\lambda} = \sum_{\lambda=0} (-1)^\lambda \operatorname{arc}\cot m_{r+\lambda} = S. \tag{10}$$

Since $n_r \neq m_r$, at least one of these sums contains two or more terms. Let this sum be the left one, so that

$$\begin{aligned} S = \sum_{\lambda=0} (-1)^\lambda \operatorname{arc}\cot n_{r+\lambda} &\geqq \operatorname{arc}\cot n_r - \operatorname{arc}\cot n_{r+1} \\ &= \operatorname{arc}\cot \left(n_r + \frac{n_r^2+1}{n_{r+1}-n_r} \right) \geqq \operatorname{arc}\cot (n_r + 1). \end{aligned} \tag{11}$$

In fact, the first $\geqq$ sign reads $=$ only if $\operatorname{arc}\cot n_{r+1}$ is the last term of the left member of (10). In this case, however, (3) applies, so that

$$n_{r+1} > n_r^2 + n_r + 1, \quad \text{or} \quad \frac{n_r^2+1}{n_{r+1}-n_r} < 1.$$

Therefore the second $\geqq$ sign in (11) reads $>$ in case the first reads $=$. That is,

$$S > \operatorname{arc}\cot (n_r + 1). \tag{12}$$

But since the left member of (10) contains at least two terms,

$$S < \operatorname{arc}\cot n_r. \tag{13}$$

We may now show that the right member of (10) contains two or more terms; otherwise we could write from (10), (12) and (13)

$$\operatorname{arc}\cot (n_r + 1) < S = \operatorname{arc}\cot m_r < \operatorname{arc}\cot n_r .$$

That is, $n_r + 1 > m_r > n_r$. But this is impossible, since these letters are integers.[4] We conclude, therefore, that both members of (10) contain two or more terms. Hence not only is $S < \operatorname{arc}\cot m_r$, so that

$$m_r < n_r + 1, \tag{14}$$

but also, since the reasoning used to establish (12) may be now applied to the m's, $S > \operatorname{arc}\cot (m_r + 1)$. Combining this with (13), we have $n_r < m_r + 1$. Finally in view of (14) we may write $n_r - 1 < m_r < n_r + 1$. But this contradicts $m_r \neq n_r$. Hence the theorem is proved.

[4] This is the first place that this part of the definition of the regular continued cotangent is used.

Arc cotangent algorithm. We now describe an algorithm, analogous to that of Euclid, for generating from a given real positive number its regular continued cotangent expansion.

Let x be the given positive number. We define two sets of numbers x_ν and n_ν ($\nu = 0, 1, 2, \cdots$) called respectively the ν-th complete and incomplete cotangent of x as follows.[5]

$$(15)\qquad \begin{aligned} x_0 &= x, & n_0 &= [x_0], \\ x_1 &= \frac{x_0 n_0 + 1}{x_0 - n_0}, & n_1 &= [x_1], \\ x_2 &= \frac{x_1 n_1 + 1}{x_1 - n_1}, & n_2 &= [x_2], \\ &\cdots\cdots\cdots\cdots & & \\ x_{\nu+1} &= \frac{x_\nu n_\nu + 1}{x_\nu - n_\nu}, & n_{\nu+1} &= [x_{\nu+1}]. \end{aligned}$$

This algorithm is to be continued as long as $x_{\nu+1}$ exists, that is, as long as x_ν is not an integer $n_\nu = [x_\nu]$. We next prove

THEOREM 3. *The continued cotangent*

$$(16)\qquad \cot \sum_{\nu=0} (-1)^\nu \operatorname{arc\,cot} n_\nu,$$

where the sum extends over all the incomplete cotangents n_ν of x, is regular.

Proof. Obviously (a) is satisfied. To show that (b) is satisfied we set $x_\nu = n_\nu + \epsilon_\nu$, where $0 < \epsilon_\nu < 1$. Then (15) becomes

$$(17)\qquad x_{\nu+1} = \frac{n_\nu^2 + 1}{\epsilon_\nu} + n_\nu > n_\nu^2 + n_\nu + 1.$$

Hence

$$[x_{\nu+1}] = n_{\nu+1} \geqq n_\nu^2 + n_\nu + 1,$$

so that (4) is satisfied for $\nu \neq k - 1$. For $\nu = k - 1$ we have from (17)

$$x_k = n_k > n_{k-1}^2 + n_{k-1} + 1,$$

which is (3). Hence the theorem is true.

THEOREM 4. *If $n_0, n_1, n_2, \cdots$ are generated by x, then*

$$(18)\qquad \sum_{\nu=0}^{\mu-1} (-1)^\nu \operatorname{arc\,cot} n_\nu = \operatorname{arc\,cot} x - (-1)^\mu \operatorname{arc\,cot} x_\mu.$$

Remark. This theorem justifies the name "complete cotangent" for x_μ.

Proof. Since

$$x_{\nu+1} = \frac{x_\nu n_\nu + 1}{x_\nu - n_\nu},$$

[5] Here, as usual, $[x]$ means the greatest integer $\leqq x$.

we have

$$(-1)^\nu \text{ arc cot } n_\nu = (-1)^\nu (\text{arc cot } x_{\nu+1} + \text{arc cot } x_\nu).$$

Setting $\nu = 0, 1, 2, \cdots, \mu - 1$ and adding, we get the theorem.

THEOREM 5.

$$x = \cot \sum_{\nu=0} (-1)^\nu \text{ arc cot } n_\nu, \tag{19}$$

where the sum extends over all incomplete cotangents n_ν generated by x.

Proof. In case there exists only a finite number of n's, the last being n_k, we may set $\mu = k$ in (18) and transpose the term $(-1)^k$ arc cot x_k. Taking the cotangent of both sides we obtain (19).

In case an infinite number of n's are generated by x we can write in view of (18) and (4),

$$\lim_{\mu=\infty} \sum_{\nu=0}^{\mu-1} (-1)^\nu \text{ arc cot } n_\nu = \text{arc cot } x - \lim_{\mu=\infty} (-1)^\mu \text{ arc cot } x_\mu = \text{arc cot } x.$$

Hence in this case also

$$x = \sum_{\nu=0}^{\infty} (-1)^\nu \text{ arc cot } n_\nu .$$

THEOREM 6. *Every positive number has a unique regular continued cotangent expansion.*

Proof. The existence of such an expansion follows from the arc cotangent algorithm and Theorem 5, while the uniqueness is provided by Theorem 2.

THEOREM 7. *The number x is rational or irrational according as its continued cotangent expansion* (19) *is finite or not.*

Proof. If (19) is finite, it follows from the addition theorem of the cotangent function that x is rational. This may be seen otherwise. In fact, if x were irrational, so also would be $x_1, x_2, \cdots$. Hence there could not exist a k for which x_k is an integer to terminate the algorithm.

If (19) is infinite, then x is irrational. In fact, suppose that $x = p/q$, where p and q are integers. It follows that $x_\nu = p_\nu/q_\nu$ is also rational for every ν. From (15)

$$x_{\nu+1} = \frac{p_{\nu+1}}{q_{\nu+1}} = \frac{p_\nu n_\nu + q_\nu}{p_\nu - n_\nu q_\nu} = \frac{p_\nu n_\nu + q_\nu}{r_\nu},$$

where, since $n_\nu = [x_\nu] = [p_\nu/q_\nu]$, the denominator r_ν is the remainder on division of p_ν by q_ν, so that $r_\nu < q_\nu$. Since we may suppose that the fraction $p_{\nu+1}/q_{\nu+1}$ is in its lowest terms, we have the inequality

$$q_{\nu+1} \leqq r_\nu < q_\nu$$

for every ν. But this implies the existence of an infinite sequence $q_1, q_2, \cdots$ of strictly decreasing positive integers, and this is absurd. Hence x is irrational.

If x is a rational number p/q, the successive numerators p_ν and the denominators q_ν of x, can be found as in the greatest common divisor process as follows:

$$
\begin{array}{lll}
p = n_0 q + q_1 & (0 \leqq q_1 < q), & p n_0 + q = p_1 , \\
p_1 = n_1 q_1 + q_2 & (0 \leqq q_2 < q_1), & p_1 n_1 + q_1 = p_2 , \\
p_2 = n_2 q_2 + q_3 & (0 \leqq q_3 < q_2), & p_2 n_2 + q_2 = p_3 , \\
\cdots & \cdots & \cdots \\
p_\nu = n_\nu q_\nu + q_{\nu+1} & (0 \leqq q_{\nu+1} < q_\nu), & p_\nu n_\nu + q_\nu = p_{\nu+1} , \\
\cdots & & \\
p_k = n_k q_k . & &
\end{array}
$$

In general, p_ν will not be prime to q_ν. In fact, any factor which they may have in common will be a common factor of $p_{\nu+1}$ and $q_{\nu+1}$ and hence of all further p's and q's. For example, for $x = 65/37$, we find the following values of p_ν, q_ν, n_ν, and the greatest common divisor δ_ν of p_ν and q_ν.

ν	0	1	2	3
p_ν	65	102	334	6030
q_ν	37	28	18	10
n_ν	1	3	18	603
δ_ν	1	2	2	10

Hence $65/37 = \cot (\text{arc cot } 1 - \text{arc cot } 3 + \text{arc cot } 18 - \text{arc cot } 603)$.

Convergents. Let $n_0 , n_1 , n_2 , \cdots$ be the incomplete cotangents generated by x. Then the curtate expansion of μ terms

$$\sigma_\mu(x) = \cot \sum_{\nu=0}^{\mu-1} (-1)^\nu \text{ arc cot } n_\nu$$

is called the μ-th convergent of x. It is clearly a rational number depending only on μ and x. The following expression relates x, $\sigma_\mu(x)$, and the complete cotangent x_μ by (18):

$$\sigma_\mu(x) = \cot (\text{arc cot } x - (-1)^\mu \text{ arc cot } x_\mu) = \frac{(-1)^\mu x_\mu x + 1}{(-1)^\mu x_\mu - x}. \tag{20}$$

THEOREM 8. *If the integers A_ν and B_ν are defined by*

$$
\begin{array}{ll}
A_0 = 1, & A_{\nu+1} = A_\nu n_\nu - (-1)^\nu B_\nu , \\
B_0 = 0, & B_{\nu+1} = B_\nu n_\nu + (-1)^\nu A_\nu ,
\end{array}
\tag{21}
$$

then the μ-th complete cotangent is given by

$$x_\mu = (-1)^\mu \frac{A_\mu x + B_\mu}{A_\mu - B_\mu x}, \tag{22}$$

and the μ-th convergent $\sigma_\mu(x)$ is given by

$$\sigma_\mu(x) = A_\mu / B_\mu . \tag{23}$$

Proof. Formula (22) is easily established by induction. In fact (22) holds for $\mu = 0$, since $A_0 = 1$, $B_0 = 0$, $x_0 = x$. If it is true for $\mu = \nu$, we may write by (15) and (21)

$$x_{\nu+1} = \frac{(-1)^\nu(A_\nu x + B_\nu)n_\nu + A_\nu - B_\nu x}{(-1)^\nu(A_\nu x + B_\nu) - n_\nu(A_\nu - B_\nu x)}$$

$$= (-1)^{\nu+1}\frac{(A_\nu n_\nu - (-1)^\nu B_\nu)x + B_\nu n_\nu + (-1)^\nu A_\nu}{A_\nu n_\nu - (-1)^\nu B_\nu - (B_\nu n_\nu + (-1)^\nu A_\nu)x}$$

$$= (-1)^{\nu+1}\frac{A_{\nu+1}x + B_{\nu+1}}{A_{\nu+1} - B_{\nu+1}x},$$

so that the induction is complete. Having established (22), we see that (23) follows from (20). In fact,

$$\sigma_\mu(x) = \left\{\frac{A_\mu x^2 + B_\mu x}{A_\mu - B_\mu x} + 1\right\} \Big/ \left\{\frac{A_\mu x + B_\mu}{A_\mu - B_\mu x} - x\right\} = A_\mu/B_\mu .$$

The numbers A_μ and B_μ are, of course, the analogues of the numerator and denominator of the μ-th convergent of the regular continued fraction. However, the recurrence formulas (21) are of a different nature, A_μ or B_μ depending not on the preceding A's or B's, but on the preceding A and B. This fact allows one to give an explicit formula for A_μ and B_μ in terms of the first μ incomplete cotangents $n_0, n_1, \cdots, n_{\mu-1}$.

$$A_0 = 1, \qquad B_0 = 0,$$
$$A_1 = n_0, \qquad B_1 = 1,$$
$$A_2 = n_0n_1 + 1, \qquad B_2 = n_1 - n_0,$$
$$A_3 = n_0n_1n_2 + n_0 - n_1 + n_2, \qquad B_3 = n_0n_1 - n_0n_2 + n_1n_2 + 1,$$
$$A_4 = n_0n_1n_2n_3 + n_0n_1 + n_1n_2 + n_1n_3 + n_2n_3 - n_1n_3 - n_0n_2 + 1,$$
$$B_4 = n_0n_1n_3 - n_0n_2n_3 + n_1n_2n_3 - n_0n_1n_2 + n_1 + n_3 - n_0 - n_2 .$$

The general formula for the A_μ and the B_μ is given by

THEOREM 9.

$$A_\mu + iB_\mu = (n_0 + i)(n_1 - i)(n_2 + i)(n_3 - i) \cdots (n_{\mu-1} + (-1)^{\mu-1}i)$$
$$(24) \qquad = \prod_{\nu=0}^{\mu-1}(n_\nu + (-1)^\nu i) \qquad (i^2 = -1).$$

In other words, if S_ν *denotes the sum of the products of* $(-1)^t n_t$ *taken* ν *at a time, then for* $\mu > 0$

$$(-1)^{[\frac{1}{2}\mu]}A_\mu = S_\mu - S_{\mu-2} + S_{\mu-4} - \cdots ,$$
$$(-1)^{[\frac{1}{2}\mu]}B_\mu = S_{\mu-1} - S_{\mu-3} + S_{\mu-5} - \cdots .$$

Proof. Formula (24) is easily established by induction, if we use (21). It also follows readily from

$$\operatorname{arc\,cot} u = \frac{1}{2i} \log \frac{u+i}{u-i}.$$

THEOREM 10.

$$(25) \qquad \begin{vmatrix} A_\mu & B_\mu \\ A_{\mu+1} & B_{\mu+1} \end{vmatrix} = (-1)^\mu (A_\mu^2 + B_\mu^2) = (-1)^\mu \prod_{\nu=0}^{\mu-1} (n_\nu^2 + 1).$$

Proof. The first equality follows at once from (21) while the second equality is obtained by taking the squares of the absolute values of both sides of (24).

THEOREM 11.

$$(26) \quad A_\mu A_{\mu+1} + B_\mu B_{\mu+1} = \begin{vmatrix} A_\mu & iB_\mu \\ iB_{\mu+1} & A_{\mu+1} \end{vmatrix} = n_\mu (A_\mu^2 + B_\mu^2) = n_\mu \prod_{\nu=0}^{\mu-1} (n_\nu^2 + 1).$$

Proof. The theorem follows at once from (21) and (25).

For example, the values of A_ν, B_ν for $x = 65/37$ are given in the following table.

ν	0	1	2	3	4
n_ν	1	3	18	603	
A_ν	1	1	4	70	42250
B_ν	0	1	2	40	24050

Here we find that $A_4/B_4 = 65/37$ and that A_4 and B_4 have the common factor $650 = (n_0^2 + 1)(n_2^2 + 1)$.

As a second example, we give the elements for $x = 6954069/2559142$.

ν	0	1	2	3	4
p_ν	6954069	16467280	133574025	9886258850	
q_ν	2559142	1835785	1781000	1780025	
n_ν	2	8	74	5554	
A_ν	1	2	17	1252	6954069
B_ν	0	1	6	461	2559142

In this example A_ν and B_ν have no common factor. It is clear from (25) that any factor common to A_ν and B_ν will divide $(n_0^2 + 1)(n_1^2 + 1) \cdots (n_{\nu-1}^2 + 1)$, and this factor will also be common to $(A_{\nu+1}, B_{\nu+1})$ by (21) and hence to all the further pairs (A, B).

THEOREM 12. *The convergents $\sigma_\nu(x)$ approach x with errors which are alternately positive and negative, but whose absolute values tend steadily to zero and are less than*

$$(x\sigma_\nu + 1) \tan \varphi_\nu,$$

where φ_ν is the smaller of $[x]^{-2\nu}$ and $3^{-2\nu-2}$.

Proof. By definition of σ_ν,

(27) $$\text{arc cot } x = \text{arc cot } \sigma_\nu + (-1)^\nu \{\text{arc cot } n_\nu - \text{arc cot } n_{\nu+1} + \cdots\}.$$

Since arc cot u is a decreasing function of u, we have

(28) $$(-1)^{\nu+1}(x - \sigma_\nu) \geqq 0,$$

which implies the first statement of the theorem. Moreover, by (27) and (4),

$$|\text{arc cot } x - \text{arc cot } \sigma_\nu| \leqq \text{arc cot } n_\nu < n_\nu^{-1} < n_{\nu-1}^{-2} < \cdots < n_2^{-2^{\nu-2}} < n_0^{-2^\nu}.$$

Hence if $n_0 = [x] > 1$, we may write

$$|\text{arc cot } x - \text{arc cot } \sigma_\nu| < [x]^{-2^\nu}.$$

If $n_0 = [x] \leqq 1$, then, by (4), $n_1 \geqq 1$, $n_2 \geqq 3$. Therefore in this case

$$|\text{arc cot } x - \text{arc cot } \sigma_\nu| < 3^{-2^{\nu-2}}.$$

Hence in either case

$$|\text{arc cot } x - \text{arc cot } \sigma_\nu| < \varphi_\nu,$$

and the final statement of the theorem follows by taking the tangent of both sides of this inequality. It remains to show that the absolute value of the error tends steadily to zero. Denoting this absolute value by Δ_ν, we have by (28) and (20)

(29) $$\Delta_\nu = |x - \sigma_\nu| = (-1)^{\nu+1}(x - \sigma_\nu) = \frac{x^2 + 1}{x_\nu - (-1)^\nu x}.$$

To show that Δ_ν is greater than $\Delta_{\nu+1}$, it suffices to show that

(30) $$(1 + x^2)(\Delta_{\nu+1}^{-1} - \Delta_\nu^{-1}) = x_{\nu+1} - x_\nu + (-1)^\nu 2x$$

is positive. From (15) and (4)

$$x_{\nu+1} > n_\nu x_\nu + 1, \qquad n_\nu \geqq 3 \qquad (\nu \geqq 2).$$

Hence $x_{\nu+1} - x_\nu > x_\nu - x_{\nu-1}$. It follows from (30) that

$$(1 + x^2)(\Delta_{\nu+1}^{-1} - \Delta_\nu^{-1}) > x_2 - x_1 - 2x.$$

To show that the right member is positive we separate two cases. If $x > 1$, then $n_1 \geqq 3$, $x_2 > 3x_1 + 1$, $x_1 > n_0 x + 1 \geqq x + 1$. Hence in this case

$$x_2 - x_1 - 2x > 2(x + 1) + 1 - 2x = 3 > 0.$$

If $x < 1$, let $x^{-1} = \delta > 1$. Then $x_1 = \delta$, $n_1 = \delta - \epsilon$ $(0 < \epsilon < 1)$, $x_2 = \epsilon^{-1}[\delta(\delta - \epsilon) + 1]$. Therefore

$$x_2 - x_1 - 2x = (\delta^2 + 1)(\delta - 2\epsilon)/\delta\epsilon.$$

If $\delta > 2$, this is positive. If $1 < \delta < 2$ so that $n_1 = \delta - \epsilon = 1$, we have

$$x_2 - x_1 - 2x = (\delta^2 + 1)(1 - \epsilon)/\delta\epsilon > 0.$$

This completes the proof of the theorem.

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have

$\geqq 0$,

n. Moreover, by (27) and (4),

$< n_{\nu-1}^{-2} < \cdots < n_2^{-2^{\nu-2}} < n_0^{-2^\nu}$.

$< [x]^{-2^\nu}$.

herefore in this case

$< 3^{-2^{\nu-2}}$.

$| < \varphi_\nu$,

by taking the tangent of both the absolute value of the error value by Δ_ν, we have by (28)

$= \dfrac{\quad + 1}{x_\nu - (-1)^\nu x}$.

to show that

$x_\nu + (-1)^\nu 2x$

$\geqq 3 \qquad (\nu \geqq 2)$.

(30) that

$- x_1 - 2x$.

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$- 2x = 3 > 0$.

$= \delta - \epsilon \ (0 < \epsilon < 1),\ x_2 =$

$- 2\epsilon)/\delta\epsilon$.

$_1 = \delta - \epsilon = 1$, we have

$\epsilon)/\delta\epsilon > 0$.

The expression of a regular continued cotangent as an irregular continued fraction. The partial cotangents n_ν of a number x may be used to represent x by an irregular continued fraction of special type as the following theorem shows.

THEOREM 13. *If $n_0, n_1, \cdots$ are the partial cotangents generated by a real positive number x, then*

$$x = n_0 + \frac{n_0^2 + 1|}{|n_1 - n_0} + \frac{n_1^2 + 1|}{|n_2 - n_1} + \frac{n_2^2 + 1|}{|n_3 - n_2} + \cdots. \tag{31}$$

Proof. Let ϵ_ν be the fractional part of x_ν so that $x_\nu = n_\nu + \epsilon_\nu$. Substituting for $x_{\nu+1}$ and x_ν in (15) and solving for ϵ_ν, we obtain

$$\epsilon_\nu = \frac{n_\nu^2 + 1}{n_{\nu+1} - n_\nu + \epsilon_{\nu+1}}.$$

Setting $\nu = 0, 1, 2, \cdots$ in succession, we see that (31) follows from $x = n_0 + \epsilon_0$. It is clear also that the numbers $A_{\mu+1}$ and $B_{\mu+1}$ are the numerator and denominator of the μ-th convergent of (31).

Regular continued fraction for ξ. The number ξ defined by (8) may be expressed as a regular continued fraction as follows. Let $n_0, n_1, n_2, \cdots$ be the partial cotangents of ξ so that

$$n_{\nu+1} - n_\nu = n_\nu^2 + 1. \tag{32}$$

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We define integers a_ν by

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 2, \quad a_3 = 5, \quad a_4 = 34, \quad a_5 = 985$$

and in general

$$a_{\nu+1} = (n_\nu + n_{\nu-1} + 1)a_{\nu-1} \qquad (\nu \geqq 1), \tag{33}$$

so that

$$a_{\nu+1} = (n_\nu + n_{\nu-1} + 1)(n_{\nu-2} + n_{\nu-3} + 1)(n_{\nu-4} + n_{\nu-5} + 1) \cdots, \tag{34}$$

where the last factor is $n_1 + n_0 + 1 = 2$ or $n_2 + n_1 + 1 = 5$ according as ν is even or odd. Then it is true that

$$a_{\nu+1}a_\nu = n_{\nu+1} - n_\nu = n_\nu^2 + 1. \tag{35}$$

This fact is true for $\nu = 0$, since $n_0^2 + 1 = 1$, while $a_0a_1 = 1$. If it is true for $\nu = k - 1$, it may be shown true for $\nu = k$ as follows:

$$a_{k+1}a_k = \frac{a_{k+1}}{a_{k-1}} a_k a_{k-1} = (n_k + n_{k-1} + 1)(n_k - n_{k-1})$$
$$= n_k^2 + n_k - n_{k-1}^2 - n_{k-1} = n_k^2 + 1.$$

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This establishes (35). Returning to (31) and using (32) and (35), we obtain

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$$
(36)\quad
\begin{aligned}
\xi &= 0 + \frac{a_0 a_1|}{|a_0 a_1} + \frac{a_1 a_2|}{|a_1 a_2} + \frac{a_2 a_3|}{|a_2 a_3} + \cdots \\
&= \frac{1|}{|a_0} + \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + \cdots \\
&= \frac{1|}{|1} + \frac{1|}{|1} + \frac{1|}{|2} + \frac{1|}{|5} + \frac{1|}{|34} + \frac{1|}{|985} \\
&\qquad + \frac{1|}{|1151138} + \frac{1|}{|1116929202845} + \cdots .
\end{aligned}
$$

The successive convergents C_ν/D_ν to ξ are

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$$
\frac{1}{1},\ \frac{1}{2},\ \frac{3}{5},\ \frac{16}{27},\ \frac{547}{923},\ \frac{538811}{909182},\ \frac{620245817465}{1046593950039},\ \cdots .
$$

In decimals we have

$$
\xi = \begin{array}{l} .59263\ \ 27182\ \ 01636\ \ 19710\ \ 40786\ \ 04995\ \ 70146\ \ 90842 \\ \ \ 75407\ \ 19716\ \ 10710\ \ 99562\ \ 60815\ \ 82473\ \ 51869\ \ 72201 \cdots . \end{array}
$$

30125

An investigation into the nature of the number ξ. The writer has been unable to discover any simple connection between ξ and other known constants. As to the nature of ξ, it is neither rational nor the root of a quadratic equation with rational coefficients, since its continued fraction is neither finite nor periodic. In what follows we show that ξ is not a root of a cubic equation with rational coefficients. We begin with

THEOREM 14. *Let a_ν and D_ν be the ν-th partial quotient and the denominator of the ν-th convergent of the continued fraction* (36). *Then $a_\nu > D_\nu$ for $\nu \leqq 4$.*

Remark. For $\nu = 1, 2, 3$, we have $a_\nu = D_\nu$.

Proof. The theorem is true for $\nu = 4$ since $a_4 = 34$, and $D_4 = 27$. If the theorem is true for $4 \leqq \nu < k$, we may prove it true for $\nu = k$ by showing that $a_{k-2}^{-1}(a_k - D_k)$ is positive. Using the fundamental recursion formula

$$
(37)\qquad D_{\mu+1} = a_\mu D_\mu + D_{\mu-1}
$$

for $\mu = k-1, k-2, k-3$ we have

$$
(38)\qquad a_{k-2}^{-1}(a_k - D_k) = a_{k-2}^{-1}\left\{a_k - (a_{k-1}a_{k-2} + 1)D_{k-2} - a_{k-1}\frac{D_{k-2} - D_{k-4}}{a_{k-3}}\right\}.
$$

If in (38) we introduce the hypothesis of the induction $D_{k-2} \leqq a_{k-2}$, we get

$$
(39)\qquad a_{k-2}^{-1}(a_k - D_k) \geqq a_k a_{k-2}^{-1} - (a_{k-1}a_{k-2} + 1) - a_{k-1}a_{k-3}^{-1}.
$$

By (33) and (35) we may write (39) in the form

$$a_{k-2}^{-1}(a_k - D_k) \geqq n_{k-1} + n_{k-2} + 1 - (n_{k-1} - n_{k-2} + 1) - (n_{k-2} + n_{k-3} + 1)$$
$$= n_{k-2} - n_{k-3} - 1 = n_{k-3}^2 > 0.$$

Hence the induction is complete.

THEOREM 15. *The number ξ does not satisfy a cubic equation with rational coefficients.*

Proof. By Theorem 14 and the familiar inequality

$$\left|\xi - \frac{C_\nu}{D_\nu}\right| < \frac{1}{D_\nu D_{\nu+1}} = \frac{1}{D_\nu(D_\nu a_\nu + D_{\nu-1})} < \frac{1}{D_\nu^3},$$

it follows that the Diophantine inequality

$$\left|\xi - \frac{x}{y}\right| < \frac{1}{y^3} \tag{40}$$

has infinitely many solutions in integers (x, y). Now if ξ satisfied a cubic equation with rational coefficients, the cubic would be irreducible, since ξ is neither rational nor a root of a quadratic equation. By a theorem of Siegel[6] the inequality (40) would in this case have only a finite number of solutions in integers (x, y), contrary to fact.

To show that this type of argument cannot be used further to prove that ξ is not an algebraic number of degree > 3, we give

THEOREM 16. *If $\epsilon > 0$ and if c is a positive constant, no matter how large, the Diophantine inequality*

$$\left|\xi - \frac{x}{y}\right| < \frac{c}{y^{3+\epsilon}} \tag{41}$$

has only a finite number of solutions (x, y).

We first prove two other theorems.

THEOREM 17. *For every k the sequence*

$$\frac{D_k}{a_k}, \frac{D_{k+2}}{a_{k+2}}, \frac{D_{k+4}}{a_{k+4}}, \cdots$$

tends to a limit.

Proof. Since

$$\frac{D_{k+2N}}{a_{k+2N}} = \frac{D_k}{a_k} - \sum_{\lambda=0}^{N-1}\left\{\frac{D_{k+2\lambda}}{a_{k+2\lambda}} - \frac{D_{k+2(\lambda+1)}}{a_{k+2(\lambda+1)}}\right\},$$

[6] See Landau, *Vorlesungen über Zahlentheorie*, vol. 3, 1927, pp. 37-65.

it is sufficient to show that this series tends to a limit as $N \to \infty$. To examine its general term we replace $k + 2\lambda$ by ν for simplicity. Now

$$\frac{D_{\nu+2}}{a_{\nu+2}} = \frac{D_{\nu+1}a_{\nu+1} + D_\nu}{a_{\nu+2}} = \frac{D_\nu a_\nu a_{\nu+1} + D_{\nu-1}a_{\nu+1} + D_\nu}{a_{\nu+2}} = \frac{D_\nu(n_\nu^2 + 2)}{a_{\nu+2}} + \frac{D_{\nu-1}a_{\nu+1}}{a_{\nu+2}}$$

$$= \frac{D_\nu}{a_\nu}\frac{a_\nu}{a_{\nu+2}}(n_\nu^2 + 2) + \frac{D_{\nu-1}}{a_{\nu+2}^2}(n_{\nu+1}^2 + 1).$$

Hence the general term of the above series may be written

$$\frac{D_\nu}{a_\nu} - \frac{D_{\nu+2}}{a_{\nu+2}} = \frac{D_\nu}{a_\nu}\left(1 - \frac{a_\nu}{a_{\nu+2}}(n_\nu^2 + 2)\right) - \frac{D_{\nu-1}}{a_{\nu-1}}\frac{a_{\nu-1}}{a_{\nu+2}^2}(n_{\nu+1}^2 + 1).$$

By Theorem 14 it is sufficient to show that as ν runs over all numbers of the same parity as k the two infinite series

$$\sum\left(1 - \frac{a_\nu}{a_{\nu+2}}(n_\nu^2 + 2)\right), \qquad \sum \frac{a_{\nu-1}}{a_{\nu+2}^2}(n_{\nu+1}^2 + 1)$$

converge. The first of these may be written

$$\sum\left(1 - \frac{n_{\nu+1} - n_\nu + 1}{n_{\nu+1} + n_\nu + 1}\right) = \sum \frac{2n_\nu}{n_{\nu+1} + n_\nu + 1} < 2\sum\frac{1}{n_\nu},$$

a rapidly convergent series. As for the second series we have

$$\sum \frac{a_{\nu-1}}{a_\nu^2}\frac{n_{\nu+1}^2 + 1}{(n_{\nu+1} + n_\nu + 1)^2} < \sum\frac{a_{\nu-1}}{a_\nu^2} < \sum\frac{1}{a_\nu},$$

which also converges with rapidity. This completes the proof.

The two sequences

$$\frac{D_1}{a_1}, \frac{D_3}{a_3}, \frac{D_5}{a_5}, \cdots \qquad \text{and} \qquad \frac{D_2}{a_2}, \frac{D_4}{a_4}, \frac{D_6}{a_6}, \cdots$$

tend to different limits. In fact we find

$$\frac{D_6}{a_6} = .7898114735158, \qquad \frac{D_5}{a_5} = .9370558376,$$

$$\frac{D_8}{a_8} = .7898114728192, \qquad \frac{D_7}{a_7} = .9370280114.$$

These two limits we denote by R_0 and R_1. That is,

$$R_0 = \lim_{\nu\to\infty} D_{2\nu}/a_{2\nu} = .78981147 \cdots,$$

$$R_1 = \lim_{\nu\to\infty} D_{2\nu+1}/a_{2\nu+1} = .93702801 \cdots.$$

It can be proved without difficulty that the two sequences above are both strictly decreasing except for the fact that $\dfrac{D_1}{a_1} = \dfrac{D_3}{a_3}$. We are now in a position to prove

THEOREM 18. *If* $\nu \to \infty$, *then*

$$D_{2\nu}^3 \left| \xi - \frac{C_{2\nu}}{D_{2\nu}} \right| \to R_0 \qquad \text{and} \qquad D_{2\nu+1}^3 \left| \xi - \frac{C_{2\nu+1}}{D_{2\nu+1}} \right| \to R_1.$$

Proof. Since

$$\xi = \lim_{\nu=\infty} C_\nu / D_\nu,$$

we may write

$$\xi = \frac{C_\nu}{D_\nu} + \frac{C_{\nu+1}}{D_{\nu+1}} - \frac{C_\nu}{D_\nu} + \frac{C_{\nu+2}}{D_{\nu+2}} - \frac{C_{\nu+1}}{D_{\nu+1}} + \cdots.$$

Using the fundamental relation

$$C_\mu D_{\mu-1} - C_{\mu-1} D_\mu = (-1)^{\mu-1},$$

we obtain

$$D_\nu^3 \left| \xi - \frac{C_\nu}{D_\nu} \right| = \frac{D_\nu^2}{D_{\nu+1}} - \frac{D_\nu^3}{D_{\nu+1} D_{\nu+2}} + \frac{D_\nu^3}{D_{\nu+2} D_{\nu+3}} - \cdots. \tag{42}$$

The first term on the right may be shown to tend to R_0 or R_1 as follows:

$$\frac{D_{\nu+1}}{D_\nu^2} = \frac{D_\nu a_\nu + D_{\nu-1}}{D_\nu^2} = \frac{a_\nu}{D_\nu} + \frac{D_{\nu-1}}{D_\nu^2}.$$

As ν tends to infinity through integers of the same parity, $D_{\nu-1}/D_\nu^2$ tends rapidly to zero, while a_ν / D_ν tends to R_0^{-1} or R_1^{-1} according as the value of ν is even or odd by Theorem 17.

Each of the other terms of (42) tends to zero as $\nu \to \infty$ since for $\lambda > 0$

$$\frac{D_\nu^3}{D_{\nu+\lambda} D_{\nu+\lambda+1}} \leq \frac{D_\nu^3}{D_{\nu+1} D_{\nu+2}} < \frac{D_\nu^3}{D_{\nu+1}^2} < \frac{D_\nu^3}{(D_\nu a_\nu)^2} < \frac{D_\nu}{a_\nu^2} < \frac{1}{a_\nu}$$

by Theorem 14. Hence the theorem is proved.

Theorem 16 now follows from Theorem 18 and from the fact that the convergents C_ν / D_ν are the fractions of best approximation to ξ.

We conclude with a theorem concerning the above-mentioned limits R_0 and R_1.

THEOREM 19.

$$R_0 R_1 = \frac{1}{1 + \xi^2}.$$

For the proof of this theorem we need the following result of interest in itself.

THEOREM 20. *If* C_ν / D_ν *is the* ν*-th convergent of* (36), *then*

$$C_\nu C_{\nu+1} + D_\nu D_{\nu+1} = n_{\nu+1}.$$

Proof. Let $A_{\nu+1}$ and $B_{\nu+1}$ be the numerator and denominator of the $(\nu + 1)$-st convergent $\sigma_{\nu+1}$ of the continued cotangent (8) defining ξ, or, what is the same, the numerator and denominator of the ν-th convergent of the continued fraction (31). From the theory of irregular continued fractions we have in view of (32) the following recursion formulas for the A's and the B's:

$$A_{\nu+1} = (n_{\nu-1}^2 + 1)(A_\nu + A_{\nu-1}), \tag{43}$$

$$B_{\nu+1} = (n_{\nu-1}^2 + 1)(B_\nu + B_{\nu-1}). \tag{44}$$

We now prove that

$$A_{\nu+1} = C_\nu a_\nu a_{\nu-1} a_{\nu-2} \cdots a_0 , \tag{45}$$

$$B_{\nu+1} = D_\nu a_\nu a_{\nu-1} a_{\nu-2} \cdots a_0 . \tag{46}$$

In fact, (45) is true for $\nu = 0$, since $A_1 = n_0 = 0$ and $C_0 = 0$. If (45) is true for $\nu < \mu$ we may prove it for $\nu = \mu$ as follows. By (43) and the hypothesis of induction

$$\begin{aligned} A_{\mu+1} &= (n_{\mu-1}^2 + 1)(C_{\mu-1} a_{\mu-1} a_{\mu-2} \cdots a_0 + C_{\mu-2} a_{\mu-2} \cdots a_0) \\ &= (n_{\mu-1}^2 + 1) a_{\mu-2} a_{\mu-3} \cdots a_0 (C_{\mu-1} a_{\mu-1} + C_{\mu-2}) = a_\mu a_{\mu-1} a_{\mu-2} \cdots a_0 C_\mu \end{aligned}$$

by (35). (46) is established in the same way. By Theorem 11

$$A_{\nu+1}A_{\nu+2} + B_{\nu+1}B_{\nu+2} = n_{\nu+1}(n_\nu^2 + 1)(n_{\nu-1}^2 + 1) \cdots (n_0^2 + 1), \tag{47}$$

while by (45) and (46)

$$\begin{aligned} A_{\nu+1}A_{\nu+2} + B_{\nu+1}B_{\nu+2} &= (C_\nu C_{\nu+1} + D_\nu D_{\nu+1})(a_{\nu+1}a_\nu)(a_\nu a_{\nu-1}) \cdots (a_1 a_0) a_0 \\ &= (C_\nu C_{\nu+1} + D_\nu D_{\nu+1})(n_\nu^2 + 1)(n_{\nu-1}^2 + 1) \cdots (n_0^2 + 1) a_0 . \end{aligned}$$

If we compare this with (47), the theorem is seen to follow from $a_0 = 1$. In the same way Theorem 10 yields

THEOREM 21.

$$C_\nu^2 + D_\nu^2 = a_{\nu+1} .$$

Theorem 19 is now a simple consequence of Theorem 20. In fact, we have by definition of R_0 and R_1

$$\begin{aligned} R_0 R_1(\xi^2 + 1) &= \lim_{\nu\to\infty} \left(\frac{D_\nu}{a_\nu} \frac{D_{\nu+1}}{a_{\nu+1}} \left\{ \frac{C_\nu}{D_\nu} \frac{C_{\nu+1}}{D_{\nu+1}} + 1 \right\} \right). \\ &= \lim_{\nu\to\infty} \left(\frac{C_\nu C_{\nu+1} + D_\nu D_{\nu+1}}{n_{\nu+1} - n_\nu} \right) = \lim_{\nu\to\infty} \frac{n_{\nu+1}}{n_{\nu+1} - n_\nu} = 1. \end{aligned}$$

Regular continued cotangents of familiar constants. The converse problem of discovering a law enjoyed by the partial cotangents of the regular continued cotangent expansion of a familiar constant appears to be even more difficult than in continued fractions. There are no periodic regular continued cotangents in view of (4). In fact, a periodic continued cotangent would not converge. Hence equation (22) cannot be used as in continued fractions to study the roots of a quadratic equation with rational coefficients. Furthermore, it is practically impossible to find more than 6 or 8 partial cotangents of a given irrational number. By Theorem 12, ten terms of the continued cotangent expansion of a number x between 10 and 11 would give x correctly to more than 1000 decimal places, 20 terms would give more than a million digits. This dependence of the continued cotangent expansion upon the "size" of x is brought out more sharply by the fact that two numbers x_1 and x_2 which merely differ by an integer may have widely different continued cotangent expansions while their continued fraction expansions are essentially the same. Thus, for example, $13/25 = \cot(\operatorname{arc\,cot} 0 - \operatorname{arc\,cot} 1 + \operatorname{arc\,cot} 3 - \operatorname{arc\,cot} 44)$, while $5 + (13/25) = \cot(\operatorname{arc\,cot} 5 - \operatorname{arc\,cot} 55)$.

The writer has been unable to discover any combination of familiar constants whose regular continued cotangent expansion is in any way predictable; that is, we have found nothing comparable with

$$\frac{3-e}{e-1} = \frac{1|}{|6} + \frac{1|}{|10} + \frac{1|}{|14} + \cdots + \frac{1|}{|4\nu+2} + \cdots$$

or with the irregular continued cotangent

$$2+\sqrt{2} = \cot(\operatorname{arc\,cot} 3 + \operatorname{arc\,cot} 17 + \operatorname{arc\,cot} 99 + \operatorname{arc\,cot} 577 + \cdots)$$

whose partial cotangents satisfy the difference equation $n_{\nu+1} = 6n_\nu - n_{\nu-1}$. The continued cotangents for $\sqrt{2}$, π and e begin as follows:

$$\sqrt{2} = \cot(\operatorname{arc\,cot} 1 - \operatorname{arc\,cot} 5 + \operatorname{arc\,cot} 36 - \operatorname{arc\,cot} 3406 + \operatorname{arc\,cot} 14694817 - \operatorname{arc\,cot} 727050997716715 + \cdots),$$

$$\pi = \cot(\operatorname{arc\,cot} 3 - \operatorname{arc\,cot} 73 + \operatorname{arc\,cot} 8599 - \operatorname{arc\,cot} 400091364 + \cdots),$$

$$e = \cot(\operatorname{arc\,cot} 2 - \operatorname{arc\,cot} 8 + \operatorname{arc\,cot} 75 - \operatorname{arc\,cot} 8949 + \operatorname{arc\,cot} 11964723 \cdots).$$

Although this paper is concerned with developing the general properties of regular continued cotangents, the reader cannot have failed to notice that many of the theorems have number-theoretic implications. The applications of the above theory to Diophantine analysis will be given in another paper.

An interesting generalization of the regular continued cotangent is an expansion of the form

$$\cot \sum_{\nu=0} c_\nu \operatorname{arc\,cot} n_\nu, \tag{48}$$

in which c_ν are ± 1 and the n_ν satisfy certain inequalities. This is called a "semi-regular" continued cotangent and has many properties in common with the semi-regular continued fraction

$$q_0 \pm \frac{1|}{|q_1} \pm \frac{1|}{|q_2} \pm \cdots.$$

A discussion of semi-regular continued cotangents will appear later. However, if the coefficients c_ν of (48) are unrestricted, the analogy with continued fractions breaks down.

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