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Matthew Coster
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From gauss!coster Tue Nov 20 18:42:33 EST 1990

He[illegible]y I'll send you some interesting sequences, some of them are accompanied b[illegible]me references. For other sequences I don't have references at the moment. I[illegible]ou are writing your book please send me (at cwi.nl!coster) or Frits Beukers (in Utrecht e-mail address unknown) a mail.

Matthew. Coster

All the sequences are based on recurrences of the form:

(1) u(n+1) = p1(n)*u(n) + p2(n)*u(n-1).

The degree of the recurrence is max degree(denominator(p1),numerator(p1), denominator(p2),numerator(p2)).

Degree 1:

(2a) (n+1)*u(n+1) = a*(2*n+1)*u(n) - b*n*u(n-1)

(2b) u(0) = 1, u(1) = a.

These sequences can be derived from the the Legendre polynomials and they appear as the coefficients of special hypergeometric functions.

Let f(x) = \Sum(n=0,\Inf) u(n)*x^n. The differential equation which satisfies f(x) is

([illegible]) (b*x^2-2*a*x+1)*y' + (b*x-a)*y = 0

Solving this equation gives:

(4) y = (1-2*a*x+b*x^2)^(-1/2).

The definition of the Legendre polynomials can be given in several ways (see [5], pp. 90-91). One definition is

(5) (1-2*t*x+x^2)^(-1/2) = \Sum(n=0,\Inf) P_n(t)*x^n.

One important sequence is the sequence which we get with a = 3 and b = 1. In this case

(6) u(n) = \Sum(k=0,n) \Choose(n,k)*\Choose(n+k,k).

with u(0) = 1, u(1) = 3.

Start of sequence: 1, 3, 13, 63, 321, 1683, 8989, 48639, 265729, 1462563, 8097453, 45046719, 251595969, 1409933619, 7923848253, ... 1850

Let v(n) satisfy (2a) with a, b as above and v(0) = 0, v(1) = 1. However the numbers v(n) are not integers, the quotients v(n)/u(n) are important to find a measure of irrationality of log(2).

De[illegible]ee 2.

[illegible] now consider sequences which satisfy the recurrence-equation

(7a) (n+1)^2*u(n+1) = (a*n^2+a*n+b)*u(n) - c*n^2*u(n-1), (7b) u(0) = 1, u(1) = b.

There are only a few sequences with integer coefficients which satisfy (7) are se[quen]ces which are related to power series which appear as fibers in some s[urfa]ces which were described by Beauville [3].
Un[fo]rtunaly I forgot some of those sequences. But those I remember are:

a=7, b=2, c=8: u(n) = \Sum(k=0,n) \Choose(n,k)^3.
Start of sequence: 1, 2, 10, 56, 346, 2252, 15184, 104960, 739162, 5280932, 38165260, 278415920, 2046924400, 15148345760, ...

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a=11, b=3, c=-1: u(n) = \Sum(k=0,n) \Choose(n,k)^2*\Choose(n+k,k).
Start of sequence: 1, 3, 19, 147, 1251, 11253, 104959, 1004307, 9793891, 96918753, 970336269, 9807518757, 99912156111, ...

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These numbers play a main role in finding a measure of irrationality of \pi^2.
See [2] and [4].

a=10, b=3, c=9: u(n) = \Sum(k=0,n) \Choose(n,k)^2*\Choose(2*k,k).
Start of sequence: 1, 3, 15, 93, 639, 4653, 35169, 272835, 2157759, 17319837, 14066065, 1153462995, 9533639025, ...

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See [4].

a=9, b=3, c=27: Start of sequence: 1, 3, 9, 21, 9, -297, -2421, -12933, -52407, -145293, -35091, 2954097, 25228971, 142080669, ...

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Degree 3

S[equ]ences which satisfy the recurrence-equation

(8a) (n+1)^3*u(n+1) = (a*n^3+(3/2)*a*n^2+bn+c)*u(n) - (d*n^3+e*n)*u(n-1),

(8b) u(0) = 1, u(1) = c.

a=34, b=27, c=5, d=1, e=0: u(n) = \Sum(k=0,n) \Choose(n,k)^2*\Choose(n+k,k)^2.
Start of sequence: 1, 5, 73, ... 5259

These numbers were used by Ape'ry proving the irrationality of \zeta(3). See [2] and [4].

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a=12, b=10, c=2, d=-64, e=4: u(n) = \Sum(k=0,n) \Choose(n,k)^4.
Start of sequence: 1, 2, 18, 164, ...

References.

[1] K. Alladi and M.L. Robinson, On certain values of the logarithm, Lecture Notes 751, 1-9.

[2] R. Ape'ry, Irrationalite' de \zeta(2) et \zeta(3), Aste'risque 61 (1979), 11-13.

[3] A. Beauville, Les familles stables de courbes elliptiques sur P^2 admettant quatre fibres singulie'res, C.R. Acad. Sci. Paris 294 (1982), 657.

[4] F. Beukers, Another Congruence for the Ape'ry Numbers, J. Num. Th. 25 (1987). p 201
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[5] M.J. Coster, Supercongruences, Thesis, Leiden 1988.