Identities in Jordan algebras

C. M. GLENNIE

THE main part of this paper is the calculation of the dimensions of certain subspaces of Jordan algebras. From a knowledge of these dimensions we deduce a theorem on identities in Jordan algebras. This is given in the third and final section. In the first section we set up some notation and give some preliminary results. The results are not new but it is convenient to gather them together here. The second section gives the statement and proof of the main theorem. The reader should consult the preceding paper by L. J. Paige in this volume for background material.

1. We shall work throughout over a fixed but arbitrary field of characteristic zero and shall not refer to the ground field again. The restriction on the characteristic can almost certainly be relaxed but this would require further investigation which we have not carried out. We shall be working in certain free Jordan and free associative algebras and shall use $a, b, c, \dots$ to denote the free generators. In particular places we shall write $p, q, r, \dots$ instead of $a, b, c, \dots$ when the result we are stating remains true if the variables are permuted or if we wish to indicate a typical monomial. The element $pqrs+srqp$ in an associative algebra will be denoted by $\overline{pqrs}$, and called a tetrad. Similarly $pqrst+tsrqp$ is $\overline{pqrst}$ and so on. Tetrads such as $\overline{abcd}, \overline{dcba}, \overline{acef}, \overline{feca}$ in which the letters appear in alphabetical or reversed alphabetical order will be called ordered tetrads. As associative products occur only under bars we shall also use juxtaposition to denote the Jordan product $\frac{1}{2}\overline{pq}$. Products in the Jordan algebras will be left normed, i.e. xyz means $(xy)z$ and so on. We use the following notation.

$L(n)$	subspace of the free Jordan algebra on n generators spanned by monomials linear in each generator,
$M(n)$	subspace of the free special Jordan algebra on n generators spanned by monomials linear in each generator,
$N(n)$	subspace of the free associative algebra on n generators spanned by the $\frac{1}{2}n!$ elements $\overline{w}$ arising from the $n!$ monomials w linear in each generator,
$S(n)$	$(n \geq 2)$ subspace of $L(n)$ spanned by monomials pw where w is a monomial linear in each of the generators other than p,

$T(n)$ ($n \geq 3$) subspace of $L(n)$ spanned by monomials pqw where w is a monomial linear in each of the generators other than p and q,

$U(n)$ ($n \geq 2$) subspace of $S(n)$ spanned by monomials pw with $p \neq a$,

$V(n)$ ($n \geq 3$) subspace of $T(n)$ spanned by monomials pqw with $p \neq a$ and $q \neq a$,

$[W]$ subspace spanned by the subset W of a vector space,

$R(x)$ the mapping $y \to yx$, x and y elements in the Jordan algebra under consideration,

$P(x, y, z)$ $R(x)R(yz)+R(y)R(zx)+R(z)R(xy)$,

$Q(x, y, z)$ $R(yz)R(x)+R(zx)R(y)+R(xy)R(z)$,

$S(x, y, z)$ $R(x)R(z)R(y)+R(y)R(z)R(x)$.

With the above notation the linearized form of the Jordan identity $xyy^2 = xy^2y$ is

$$xP(y, z, t) = xQ(y, z, t) \tag{1}$$

or

$$xR(yzt) = xP(y, z, t) - xS(y, z, t). \tag{2}$$

From (1) and (2) we have at once

$$xR(yzt) = xQ(y, z, t) - xS(y, z, t). \tag{3}$$

It is clear that $M(n) \subseteq N(n)$. We have also

Lemma 1. *For* $n \geq 3$, $U(n)+V(n) = L(n)$.

Proof. Let $w \in L(n)$. Then w is a sum of elements aR where R is a monomial in operators $R(x)$ and each x is a monomial in some of the generators $b, c, \ldots$. If x contains more than two generators then by (2) $R(x)$ can be expanded as a sum of words $R(y)$ where each y contains fewer generators than x. Repeating such expansions as often as necessary gives the result.

Corollary. $S(n)+T(n) = L(n)$ *and* $S(n)+V(n) = L(n)$.

Lemma 2. $\dim S(n) \leq n \dim L(n-1)$, $\dim T(n) \leq \frac{1}{2}n(n-1) \dim L(n-2)$, $\dim U(n) \leq (n-1) \dim L(n-1)$, $\dim V(n) \leq \frac{1}{2}(n-1)(n-2) \dim L(n-2)$.

Proof. The proofs of these inequalities follow at once from the definitions of $S(n)$, etc.

The following relations, in which $p, q, r, \ldots$ denote distinct elements from $b, c, d, \ldots$ and x is a monomial in the remaining generators, are either clear from the definitions of the operators or follow easily from (1), (2), (3) and previous relations in the set.

$$xQ(p, q, r) \in U(n) \tag{4}$$

$$xP(p, q, r) \in U(n) \tag{5}$$

$$xS(p, q, r) \in U(n) \tag{6}$$

$$xR(pqr) \in U(n) \tag{7}$$

$$x(qr(st))-xQ(q, r, st) \in U(n) \tag{8}$$

$$x(qr)(st)-xQ(q, r, st) \in U(n) \tag{9}$$

$$xp(qr(st))-xP(p, qr, st) \in U(n) \tag{10}$$

$$x(qrp)(st)+x(stp)(qr)-xQ(p, qr, st) \in U(n). \tag{11}$$

The following lemmas are due to Cohn. Proofs will be found in [1].

LEMMA 3. $\overline{abcd}-(\operatorname{sgn}\pi)\overline{pqrs} \in M(4)$ *where* p, q, r, s *is the permutation* π *of* a, b, c, d.

LEMMA 4. $M(n)+[W(n)] = N(n)$ *for* $n = 1, \ldots, 7$ *where* $W(n) = \phi$ *(the empty set) for* $n = 1, 2, 3$, $W(4) = \{\overline{abcd}\}$, $W(5) = \{\overline{pqrs}t\}$, $W(6) = \{\overline{pqrs}tu+\overline{pqrs}ut, \overline{pqrs}(tu), \overline{pqrs}tu-\overline{pqrs}ut\}$, $W(7) = \{\overline{pqrs}tuv+\overline{pqrs}utv, \overline{pqrs}(tu)v, \overline{pqrs}tuv-\overline{pqrs}utv\}$. *In the cases* $n = 5, 6, 7$ *the set is to include all elements obtained by replacing* $p, q, r, \ldots$ *by any permutation of* $a, b, c, \ldots$ *such that* $\overline{pqrs}$ *is an ordered tetrad.*

Let U be a subspace of the vector space V and $W = \{w_1, \ldots, w_n\}$ be a subset of V. If r_i $(i = 1, \ldots, m)$ denotes the relation

$$\sum_{j=1}^{n} \lambda_{ij} w_j \in U$$

amongst the elements of W and $R = \{r_1, \ldots, r_m\}$ we shall call the $m \times n$ matrix $\Lambda = (\lambda_{ij})$ the word-relation matrix for W and R. We have

LEMMA 5. $\dim (U+[W]) \leqslant \dim U+(n-\operatorname{rank} \Lambda)$.

Proof. Let $r = \operatorname{rank} \Lambda$. We can find r elements from W each expressible as a linear combination of some element in U and the remaining $n-r$ elements of W. So $U+[W]$ is spanned by any basis of U together with $n-r$ elements from W, and the result follows.

2. THEOREM 1. *For* $n = 1, \ldots, 7$, $\dim L(n) = \dim M(n)$. *The dimensions are respectively* 1, 1, 3, 11, 55, 330, 2345.

Proof. The mapping $a \to \overline{a}$, $b \to \overline{b}$, etc., can be extended to a linear transformation of $L(n)$ onto $M(n)$. So $\dim M(n) \leqslant \dim L(n)$. For each n we now find a number $d(n)$ such that $\dim L(n) \leqslant d(n)$ and $\dim M(n) \geqslant d(n)$. It follows at once that $\dim L(n) = \dim M(n) = d(n)$. We shall use $w(n)$ to denote the number of elements in $W(n)$. For simplicity we write L for $L(m)$ and so on when dealing with the case $n = m$.

$n = 1$. Take $d = 1$. L is spanned by a single monomial. So $\dim L \leqslant d$. By Lemma 4, $M = N$. So $\dim M = \dim N = 1 \geqslant d$.

$n = 2$. Take $d = 1$. L is spanned by the single monomial ab. So $\dim L \leqslant d$. By Lemma 4, $M = N$. So $\dim M = \dim N = 1 \geqslant d$.

$n = 3$. Take $d = 3$. L is spanned by abc, bca, cab. So $\dim L \leqslant d$. By Lemma 4, $M = N$. So $\dim M = \dim N = 3 \geqslant d$.

$n = 4$. Take $d = 11$. L is spanned by the twelve monomials $apqr$, $ap(qr)$, $a(pq)r$ (see proof of Lemma 1). These are subject to the relation

$$aP(b, c, d) = aQ(b, c, d)$$

and the word-relation matrix has rank 1. So by Lemma 5 (with $U = \{0\}$) $\dim L \leqslant 11 = d$. From Lemma 4 we have that $M + [W] = N$ where $w = 1$. So $\dim M \geqslant \dim N - \dim [W] \geqslant \dim N - w \geqslant 12 - 1 = 11 = d$.

$n = 5$. Take $d = 55$. Since $pqr(st) = stR(pqr) = stQ(p, q, r) - stS(p, q, r)$ we have that $V \subseteq U$ and $U = L$. Then $\dim L = \dim U \leqslant 5 \dim L(4) = 5 \times 11 = 55 = d$. From Lemma 4, $M + [W] = N$ with $w = 5$. So $\dim M \geqslant \dim N - w = 60 - 5 = 55 = d$.

$n = 6$. Take $d = 330$. From Lemma 2, $\dim U \leqslant 5 \dim L(5) = 275$. V is spanned by (i) 60 elements $apqr(st)$, (ii) 30 elements $a(pq)r(st)$, (iii) 30 elements $ap(qr)(st)$. From (1), (8), (9), (10), (11) we have

$$ap(qr)(st) - a(qrp)(st) - a(stp)(qr) \in U.$$

Defining $T(p, q, r, s, t)$ as

$$[Q(q, r, p) - S(q, r, p)]R(st) + [Q(s, t, p) - S(s, t, p)]R(qr)$$

we have

$$ap(qr)(st) - aT(p, q, r, s, t) \in U. \tag{12}$$

Also, from (5):

$$apqP(r, s, t) \in U \tag{13}$$

$$a(pq)P(r, s, t) \in U. \tag{14}$$

and from (1):

$$aP(p, q, r)R(st) - aQ(p, q, r)R(st) \in U. \tag{15}$$

(12) to (15) give respectively 30, 20, 10, 10 relations. Setting up the word-relation matrix for the 120 spanning elements of V and these 70 relations we get a 70×120 matrix of which the rank is 65. Then by Lemma 5, $\dim (U+V) \leqslant \dim U + (120 - 65)$. So

$$\dim L \leqslant \dim (U+V) \leqslant 275 + 55 = 330 = d.$$

From Lemma 4, $M + [W] = N$ with $w = 45$. Now let W' be the subset of W consisting of the 30 elements $\overline{pqrs}tu + \overline{pqrs}ut$, $\overline{pqrs}(tu)$, and let $N' = M + [W']$. We have 45 relations amongst elements of $W - W'$ obtained from

$$\overline{abcd}ef - \overline{abcd}fe + \overline{bcde}fa - \overline{bcde}af + \overline{cdef}ab - \overline{cdef}ba + \overline{acdf}eb - \overline{acdf}be \in N' \tag{16}$$

by permuting a, b, c, d, e, f and using Lemma 3. We have a further 6 relations obtained from

$$\overline{cdef}ab - \overline{cdef}ba + \overline{defb}ac - \overline{defb}ca + \overline{efbc}ad - \overline{efbc}da$$
$$+ \overline{fbcd}ae - \overline{fbcd}ea + \overline{bcde}af - \overline{bcde}fa \in N' \tag{17}$$

by permuting a, b, c, d, e, f and using Lemma 3. (16) is the linearized form of

$$\overline{abcd}ab-\overline{abcd}ba \in N'$$

which comes from

$$\overline{acdb^2}a-\overline{cdb^2a}a+\overline{cdb^2}a^2-\overline{bdca^2}b+\overline{dca^2b}b-\overline{dca^2}b^2 = 0$$

using Lemma 3 and

$$\overline{pqrst} = \overline{qrstp}-\overline{rstpq}+\overline{stpqr}-\overline{tpqrs}+\overline{pqrst}. \tag{18}$$

(17) comes from $\sum (\overline{cdef}ab-\overline{cdef}(ab)) = 0$ where the sum is taken over the cyclic permutations of b, c, d, e, f and Lemma 3 is used where necessary. The rank of the word-relation matrix for the 15 elements in $W-W'$ and the 51 relations above is 15. So $\dim N = \dim (N'+[W-W']) \leqslant \dim N'+15-15 = \dim N'$. Whence $N = N'$. So $\dim M \geqslant \dim N-30 = 360-30 = 330 = d$. .

$n = 7$. Take $d = 2345$. From Lemma 2, $\dim S \leqslant 7 \dim L(6) = 7\times 330 = 2310$. V is spanned by elements of types (i) $apqrs(tu)$, (ii) $a(pq)rs(tu)$, (iii) $ap(qr)s(tu)$, (iv) $apq(rs)(tu)$, (v) $a(pq)(rs)(tu)$. Now $tuR(apqrs)$, $tuR(a(pq)rs)$, $tuR(apq(rs))$, and $tuR(a(pq)(rs))$ are in S. This follows at once on expanding the operator R using (3) and then using (3) again where necessary. So $L = S+V$ is spanned by S and the set of 180 elements $ap(qr)s(tu)$. Now let X be the set of the 48 elements of type (iii) in which $q = b$ and $t = c$ or $q = c$ and $t = b$. Consider the following table, in which each element is to represent the set of elements obtained from it by replacing p, q, r, s by all permutations of d, e, f, g:

	$ap(bq)r(cs)$	$ap(cq)r(bs)$	
$ap(qr)b(cs)$	$ap(bq)c(rs)$	$ap(cq)b(rs)$	$ap(qr)c(bs)$
$ab(pq)r(cs)$	$ap(qr)s(bc)$	$ap(bc)q(rs)$	$ac(pq)r(bs)$
$ab(cp)q(rs)$	$ab(pq)c(rs)$	$ac(pq)b(rs)$	$ac(bp)q(rs)$

Each element in the table can be expressed modulo S as a linear combination of elements in higher rows. Thus, for example:

$$ap(qr)b(cs) = -ap(bq)r(cs)-ap(br)q(cs) \pmod S$$

since $apQ(q, r, b)R(cs) = apP(q, r, b)R(cs)$ and the elements in this last expression are all of type (iv) and so in S. The expression for $ab(cp)q(rs)$ arises from

$$cpQ(a, b, q)R(rs)+rsQ(a, c, p)R(bq)+bqQ(a, r, s)R(cp)-aQ(bq, cp, rs) \in S.$$

So we now have that $S+[X] = L$. But there are further relations modulo S amongst the elements of X. These are:

$$\sum ap(bq)r(cs) \in S \tag{19}$$

$$\sum ap(cq)r(bs) \in S, \tag{20}$$

where in each case s is fixed as one of d, e, f, g, and the sum is taken as p, q, r run over all the permutations of the remaining variables, and

$$ap(bq)r(cs)+ap(cq)r(bs)-ar(bp)s(cq)-ar(cp)s(bq) \in S, \qquad (21)$$

where the sum is taken as p, q run over the permutations of two of the variables and r, s over the permutations of the remaining two. For (19) it is sufficient to show that $ap(bp)p(cs)$ is in S for (19) can then be obtained by linearization. But $2ap(bp)p(cs)+abpp(cs) \in S$ and $abp^2p(cs) = abpp^2(cs) \in S$. (20) is obtained similarly. (21) is the linearized form of $ap(bp)r(br) - ar(bp)r(bp) \in S$. Now

$$\begin{aligned} 8[ap(bp)r(br)-ar(bp)r(bp)] &\equiv 8[ap(bp)r(br)+ap(br)r(bp)+ar(br)p(bp)] \\ \text{(by (19) and (20))} \quad &\equiv 2(abp^2br^2+apr^2pb^2+arb^2rp^2) \\ &\equiv -a[R(b^2p^2)R(r^2)+R(p^2r^2)R(b^2)] \\ &\quad +R(r^2b^2)R(p^2) \\ &\equiv aP(b^2, p^2, r^2) \equiv 0 \text{ (all congruences mod } S). \end{aligned}$$

We now have 14 relations (4 each of (19) and (20) and 6 of (21)) amongst the 48 elements of X, and the word-relation matrix has rank 13. So

$$\dim L = \dim (S+U) \leqslant \dim S+(48-13) \leqslant 2310+35 = 2345 = d.$$

Now $M+[W] = N$ from Lemma 4. If W' consists of the 210 elements $\overline{pqrs}tuv+\overline{pqrs}utv$, $\overline{pqrs}(tu)v$ it follows from work done in the $n = 6$ case that $M+[W'] = N$. Also we have

$$\overline{pqrs}qsp+\overline{pqrs}sqp+\overline{qprs}psq+\overline{qprs}spq \in M. \qquad (22)$$

To establish (22) we use the following (congruences are modulo M):

$$\begin{aligned} 8p^2q^2rs^2 &= \overline{p^2q^2rs^2}+\overline{q^2p^2rs^2}+\overline{rq^2p^2s^2}+\overline{rp^2q^2s^2} \\ &\equiv 8\overline{pq^2rs^2}p+8\overline{qp^2rs^2}q \\ \overline{pq^2rs^2}p &\equiv 2\overline{rs^2pq}qp \equiv 4\overline{pqrs}qsp \\ \overline{pq^2rs^2}p &\equiv 2\overline{pq^2rs}sp \equiv 4\overline{rspq}sqp \equiv 4\overline{pqrs}sqp \end{aligned}$$

and the relations obtained by interchanging p and q. If we linearize (22) and substitute all permutations of a, b, c, d, e, f, g we obtain 315 relations corresponding to the 315 words $pq(rs)t(uv)$. But we know that $\dim (S+U) - \dim S \leqslant 35$. So at most 35 of these relations are linearly independent. If we choose 35 relations corresponding to 35 words in U which are linearly independent mod S we can set up the word-relation matrix for these and the 105 words of W' involved in them. The rank of this matrix is 35 (see comment at end of proof of theorem). So $\dim M \geqslant \dim N-(210-35) = 2345 = d$. This completes the proof of the theorem.

Comment. The proof requires at several stages the calculation of the rank of a matrix. In all cases but the last this calculation was carried out by hand. The work involved is not as bad as might be feared because of the large number of zero entries and the pattern of blocks within the matrix. For the last matrix (which has 35 rows and 105 columns) use was made of the KDF9 computer at Edinburgh University. The program was designed to print out a basis for the space of vectors x such that $xA = 0$ for a given matrix A. In the present case the matrix was augmented by five rows known to be linearly dependent on the chosen 35. The print-out showed correctly the known linear dependences and this was regarded as being a check on the accuracy of the program.

3. In [2], the cases $n \leq 5$ of Theorem 1 were proved although no explicit values for the dimensions were established. An example of an identity in three variables valid in all special Jordan algebras but not valid in all Jordan algebras was given. This identity is of total degree 8, so in a linearized form shows that Theorem 1 is not valid for $n > 7$. The following theorem, which is a corollary of Theorem 1, bridges the gap left in [2] for $n = 6, 7$.

THEOREM 2. *A multilinear identity of total degree 6 or 7 which is valid in all special Jordan algebras is valid in all Jordan algebras.*

It should be possible using the methods of Part 2 to find dim $L(8)$, dim $M(8)$ and the degree 8 multilinear identities holding in special Jordan algebras but not in all Jordan algebras. These correspond to the elements in the kernel of the canonical linear transformation of $L(8)$ onto $M(8)$.

I should like to record my gratitude to Mr. J. K. S. McKay for his encouragement in general and his help with the programming and computer work in particular.

REFERENCES

1. P. M. COHN: On homomorphic images of special Jordan algebras. *Canadian J. Maths.* 6 (1954), 253–264.
2. C. M. GLENNIE: Some identities valid in special Jordan algebras but not valid in all Jordan algebras. *Pacific J. Maths.* 16 (1966), 47–59.