Bull Calcutta Math Soc
33 119-127 1941
→ 186
1044
1623
-1627

THE ENUMERATION OF THE LATIN RECTANGLE OF DEPTH THREE BY MEANS OF A DIFFERENCE EQUATION

By

S. M. Kerawala

(Communicated by the Secretary)

1. *Introduction*: MacMohon[1] first gave an operational formula for the number of permutations of any number of non-clashing rows, each row consisting of the same totality of n letters, each letter differing from all the rest. Jacob[2] has supplied reduction formulae for three non-clashing rows, but he remarks that he could not obtain one single recurrence relation for the enumeration. Besides, Jacob's fundamental recurrence formula contains an error, which has vitiated the tables given at the end of his work and deflected him from probably a correct conjecture as to the limit of the ratio of the enumeration to $n!^2$. In this paper I follow closely the method of Jacob, derive a difference equation for the enumeration which Jacob did not, compile fresh tables for values of n up to 15 and justify the conjecture which Jacob abandoned.

2. μ_n will denote the enumeration of the Latin rectangle of depth three. The nature of letters in μ_n is as shown below:

$$\begin{matrix} A_1, & A_2, & A_3,\ldots\ldots\ldots A_n \\ B_1, & B_2, & B_3,\ldots\ldots\ldots B_n \\ C_1, & C_2, & C_3,\ldots\ldots\ldots C_n. \end{matrix}$$

For a permutation to be non-clashing, any one letter of A can appear with any one of $(n-1)$ letters of B and any one of $(n-2)$ letters of C. We have, therefore, that

$$\mu_n = (n-1)(n-2)\alpha_{(n-1)}, \qquad \ldots \quad (1)$$

where α_n is the enumeration of the non-clashing permutations with the letters

$$\begin{matrix} A_1, & A_2, & A_3,\ldots\ldots\ldots A_n \\ B_0, & B_2, & B_3,\ldots\ldots\ldots B_n \\ C_0, & C_1, & C_3,\ldots\ldots\ldots C_n \end{matrix}$$

the letters in different columns being identical only if their numerical suffixes are identical. Further β_n will denote the non-clashing enumerations with the distribution

$$\begin{matrix} A_1, & A_2, & A_3, & A_4, \ldots\ldots\ldots A_n \\ B_0, & B_2, & B_3, & B_4, \ldots\ldots\ldots B_n \\ C_1, & C_{(n+1)}, & C_3, & C_4, \ldots\ldots\ldots C_n \end{matrix}$$

γ_n the non-clashing enumerations with the distribution

$$\begin{matrix} A_1, & A_2, & A_3, & A_4, \ldots\ldots\ldots A_n \\ B_1, & B_2, & B_3, & B_4, \ldots\ldots\ldots B_n \\ C_0, & C_{(n+1)}, & C_3, & C_4, \ldots\ldots\ldots C_n \end{matrix}$$

δ_n the non-clashing enumerations with the distribution

$$\begin{matrix} A_1, & A_2, & A_3, & A_4, \ldots\ldots\ldots A_n \\ B_1, & B_2, & B_3, & B_4, \ldots\ldots\ldots B_n \\ C_0, & C_2, & C_3, & C_4, \ldots\ldots\ldots C_n \end{matrix}$$

and finally ϵ_n the non-clashing enumerations with the distribution

$$\begin{matrix} A_1, & A_2, & A_3, & A_4, \ldots\ldots\ldots A_n \\ B_0, & B_2, & B_3, & B_4, \ldots\ldots\ldots B_n \\ C_{(n+1)}, & C_2, & C_3, & C_4, \ldots\ldots\ldots C_n. \end{matrix}$$

Considering now the value of α_n, obviously $\alpha_n = \beta_n -$ (all the cases in which B_0 and $C_{(n+1)}$ appear together in β_n).

If B_0, $C_{(n+1)}$ and A_1 turn up together, $\delta_{(n-1)}$ cases are possible; if B_0, $C_{(n+1)}$ and A_2 turn up together, $\delta_{(n-1)}$ more cases result; and if B_0, $C_{(n+1)}$ and any of the remaining $(n-2)$ letters of A appear together, $(n-2)\alpha_{(n-1)}$ further cases are obtained. Thus

$$\alpha_n = \beta_n - 2\delta_{(n-1)} - (n-2)\alpha_{(n-1)}. \qquad \ldots \quad (2)$$

Considering the value of δ_n, it is evident that $\delta_n = \mu_n +$ the number of cases in which A_1 appears with $C_0 +$ the number of cases in which B_1 appears with C_0. These last are $2(n-1)\delta_{(n-1)}$ in number. Hence

$$\delta_n = \mu_n + 2(n-1)\delta_{(n-1)}. \qquad \ldots \quad (3)$$

The value of ϵ_n is made up of the following different cases:

when

(a) A_1, B_0, $C_{(n+1)}$ go together, yielding $\mu_{(n-1)}$ cases.

(b) No two of A_1, B_0, $C_{(n+1)}$ occur together, giving μ_n cases.

(c) Only two of A_1, B_0, $C_{(n+1)}$ appear together, giving $3(n-1)\delta_{(n-1)}$ cases.

Hence $$\epsilon_n = \mu_n + \mu_{(n-1)} + 3(n-1)\delta_{(n-1)}. \qquad \ldots \quad (4)$$

Again, β_n is composed entirely of the following different cases:

(a) A_1, B_0, $C_{(n+1)}$ turn up together, giving $\delta_{(n-1)}$ cases.

(b) A_1, B_0 and any one of the $(n-2)$ C's other than C_1 and $C_{(n+1)}$ occur together, resulting in $(n-2)\gamma_{(n-1)}$ cases.

(c) C_1, B_0 and any one of the $(n-2)$ A's other than A_1 and A_2 come together, yielding $(n-2)\beta_{(n-1)}$ cases.

(d) C_1, B_0 and A_2 occur together, giving rise to $\epsilon_{(n-1)}$ cases.

(e) No two of A_1, B_0, C_1 occur together, giving δ_n cases.

Thus $$\beta_n = \delta_n + \delta_{(n-1)} + \epsilon_{(n-1)} + (n-2)\beta_{(n-1)} + (n-2)\gamma_{(n-1)}. \qquad \ldots \quad (5)$$

Finally, γ_n is composed of all the non-clashing permutations possible when

(a) C_0, B_1, A_2 come together, giving $\epsilon_{(n-1)}$ cases.

(b) C_0, B_1 and any one of the $(n-2)$ A's other than A_1 and A_2 appear together, giving $(n-2)\beta_{(n-1)}$ cases.

(c) C_0, A_1 and B_2 appear together, giving $\epsilon_{(n-1)}$ cases.

(d) C_0, A_1 and any one of the $(n-2)$ A's other than B_1 and B_2 appear together, giving rise to $(n-2)\beta_{(n-1)}$ cases.

(e) No two of A_1, B_1, C_0 come together, giving δ_n cases.

$$\therefore \quad \gamma_n = \delta_n + 2\epsilon_{(n-1)} + 2(n-2)\beta_{(n-1)}. \qquad \ldots \quad (6)$$

Multiplying (2) by $(n-1)$ and using (1),

$$(n-1)(\beta_n - \alpha_n) = 2(n-1)\delta_{(n-1)} + \mu_n$$

is obtained. With the help of (3), this becomes

$$\delta_n = (n-1)(\beta_n - \alpha_n). \qquad \ldots \quad (7)$$

Substituting for β_n from (2) in (5), we get

$$\alpha_n = \delta_n - \delta_{(n-1)} + \epsilon_{(n-1)} + (n-2)(\beta_{(n-1)} + \gamma_{(n-1)} - \alpha_{(n-1)}).$$

Substituting for $\delta_{(n-1)}$ from (7), we get

$$\alpha_n = \delta_n + \epsilon_{(n-1)} + (n-2)\gamma_{(n-1)}. \qquad \ldots \quad (8)$$

I shall consider now only the equations (1), (8), (3), (4), (6) and (7). These are six simultaneous difference equations for the six unknowns α_n, β_n, γ_n, δ_n, ϵ_n and μ_n. Five of the equations are first order equations and one is a zero order equation. If all the unknowns except μ_n be eliminated we should expect to get a fifth order difference equation.

From (1)

$$\mu_{(n+5)}=(n+4)(n+3)\alpha_{(n+4)}.$$

With the help of (8), this becomes

$$\mu_{(n+5)}=(n+4)(n+3)[\delta_{(n+4)}+\epsilon_{(n+3)}+(n+2)\gamma_{(n+3)}].$$

Using (6), this becomes

$$\mu_{(n+5)}=(n+4)(n+3)[\delta_{(n+4)}+\epsilon_{(n+3)}+(n+2)\delta_{(n+3)}+2(n+2)\epsilon_{(n+2)}$$
$$+2(n+2)(n+1)\beta_{(n+2)}].$$

Using (7), this assumes the form

$$\mu_{(n+5)}=(n+4)(n+3)[\delta_{(n+4)}+\epsilon_{(n+3)}+(n+2)\delta_{(n+3)}+2(n+2)\epsilon_{(n+2)}$$
$$+2(n+2)(n+1)\alpha_{(n+2)}+2(n+2)\delta_{(n+2)}].$$

Using (1) and (4), this reduces to

$$\mu_{(n+5)}=(n+4)(n+3)[\delta_{(n+4)}+(n+2)\delta_{(n+3)}+5(n+2)\delta_{(n+2)}$$
$$+6(n+1)(n+2)\delta_{(n+1)}+3\mu_{(n+3)}+(2n+5)\mu_{(n+2)}+2(n+2)\mu_{(n+1)}. \quad \ldots \quad (9)$$

If now in (9) the μ's be replaced by δ's with the help of (3), we have on simplifying

$$\delta_{(n+5)}=(n+5)(n+4)\delta_{(n+4)}+(n+5)(n+4)(n+3)\delta_{(n+3)}+(n+4)(n+3)^2\delta_{(n+2)}$$
$$+2(n+4)(n+3)(n^2+3n+3)\delta_{(n+1)}-4(n+4)(n+3)(n+2)n\delta_n. \quad \ldots \quad (10)$$

Equation (10) corresponds to equation (1.12) in Jacob's work. There is an error in this result of Jacob. For the last term of Jacob's equation translated into the notation of this paper is

$$-4(n+4)(n+3)(n+2)(n+1)n\delta_n,$$

whereas the last term of equation (10) is

$$-4(n+4)(n+3)(n+2)n\delta_n.$$

It is this error which has vitiated Jacob's results on pages 336-337.

With the help of (7), (9) becomes

$$\mu_{(n+5)} = (n+3)(n+4)\left[\mu_{(n+4)} - \mu_{(n+3)} - (n+1)\mu_{(n+2)} + 2(n+2)\mu_{(n+1)}\right] + 3(n+4)^2(n+3)\delta_{(n+3)}.$$

We have thus the value of δ in terms of μ. Substituting these in (3), we have immediately on simplification

$$(n+3)\mu_{(n+5)} = (n+4)(n^2+8n+17)\mu_{(n+4)} + (n+3)(n+4)(n^2+8n+17)\mu_{(n+3)} + (n+3)(n+4)(n^2+8n+13)\mu_{(n+2)} + 2(n+2)(n+3)(n+4) \times (n^2+5n+3)\mu_{(n+1)} - 4(n+1)(n+2)(n+3)(n+4)^2\mu_n. \quad \ldots \quad (11)$$

This is, therefore, the fifth order difference equation looked for, which, though quite simple to derive, Jacob could not achieve.

The problem thus reduces to solving the difference equation (11) being given that $\mu_1=0$, $\mu_2=0$, $\mu_3=2$, $\mu_4=24$ and $\mu_5=552$, these latter values being derived from elementary considerations of non-clashing permutations possible when $n=1, 2, 3, 4, 5$. I could not bring the equation (11) to any of the standard types considered by Milne-Thompson in his *Calculus of Finite Differences*. However, I have calculated the values of μ_n for values of n up to 15, and of the allied enumerations for values of n up to 10, and the results are given in Tables I and II.

To find values of $\frac{\mu_n}{n!^2}$ for $n>15$, we substitute $v_n = \frac{\mu_n}{n!^2}$ in (11) and obtain

$$v_{(n+5)} = \frac{(n+4)(n^2+8n+17)}{(n+3)(n+5)^2}v_{(n+4)} + \frac{(n^2+8n+17)}{(n+4)(n+5)^2}v_{(n+3)} + \frac{(n^2+8n+13)}{(n+3)^2(n+4)(n+5)^2}v_{(n+2)} + \frac{2(n^2+5n+3)}{(n+2)(n+3)^2(n+4)(n+5)^2}v_{(n+1)} - \frac{4}{(n+1)(n+2)(n+3)^2(n+5)^2}v_n. \quad \ldots \quad (12)$$

The volumes of v calculated from (12) are given in Table III.

(186) Crave. (1044) Crave.

TABLE I

n	μ_n	$(n!)^2$	$\mu_n/(n!)^2$
1	0	1	0
2	0	4	0
3	2	36	·055 555 556
4	24	576	·041 666 667
5	552	14, 400	·038 333 333
6	21, 280	518, 400	·041 049 383
7	1, 073, 760	25, 401, 600	·042 271 353
8	70, 299, 264	1, 625, 702, 400	·043 242 394
9	5, 792, 853, 248	131, 681, 894, 400	·043 991 266
10	587, 159, 944, 704	13, 168, 189, 440, 000	·044 589 269
11	71, 822, 743, 499, 520	1, 593, 350, 922, 240, 000	·045 076 538
12	10, 435, 278, 503, 677, 440	229, 442, 532, 802, 560, 000	·045 480 990
13	1, 776, 780, 700, 509, 416, 448	38, 775, 788, 043, 632, 640, 000	·045 821 911
14	350, 461, 958, 856, 515, 690, 496	7, 600, 054, 456, 551, 997, 440, 000	·046 113 085
15	79, 284, 041, 282, 622, 163, 140, 608	1, 710, 012, 252, 724, 199, 424, 000, 000	·046 364 604

Related to 3 line Latin Rect.

TABLE II

n	α_n (1623)	β_n (1624)	γ_n (1625)	δ_n (1626)	ϵ_n (1627)
1	—	—	—	0	1
2	1	1	2	0	0
3	4	5	4	2	2
4	46	58	60	36	44
5	1, 064	1, 274	1, 276	840	1, 008
6	35, 792	41, 728	41, 888	29, 680	34, 432
7	1, 673, 792	1, 912, 112	1, 916, 064	1, 429, 920	1, 629, 280
8	103, 443, 808	116, 346, 400	116, 522, 048	90, 318, 144	101, 401, 344
9	8, 154, 999, 232	9, 059, 742, 176	9, 069, 595, 840	7, 287, 943, 552	8, 030, 787, 968
10	798, 030, 483, 328	877, 746, 364, 288	878, 460, 379, 392	717, 442, 928, 640	788, 377, 273, 856
n	α_n	β_n	γ_n	δ_n	ϵ_n

Ann, these 5 only.

TABLE III

v_{16}	·046 584 014
v_{17}	·046 777 073
v_{18}	·046 948 245
v_{19}	·047 101 050
v_{20}	·047 238 276
v_{21}	·047 362 192
v_{22}	·047 474 638
v_{23}	·047 577 122
v_{24}	·047 670 925
v_{25}	·047 757 096

With the help of equation (12) and the tables given above, it can be readily shown that for $n \geq 7$,

$$v_n < v_{(n-1)} \left[1 + \frac{1}{(n-1)(n-2)} \right].$$

It follows that for $s \geq 7$

$$v_n < v_s \prod_{r=s}^{(n-1)} \left\{ 1 + \frac{1}{(r-1)r} \right\}. \qquad \dots \quad (13)$$

Again it is possible to show that for $n \geq 19$,

$$v_n > v_{(n-1)} \left[1 + \frac{1}{(n-1)(n-2)} - \frac{2}{(n+1)^4} \right]$$

from which it follows that for every $s \geq 19$

$$v_n > v_s \prod_{r=s}^{(n-1)} \left[1 + \frac{1}{(r-1)r} - \frac{2}{(r+2)^4} \right]. \qquad \dots \quad (14)$$

Thus the sequence v_n is monotonic and bounded and must therefore tend to some limit l such that for every $s \geq 19$

$$v_s \prod_{r=s}^{\infty} \left[1 + \frac{1}{(r-1)r} - \frac{2}{(r+2)^4} \right] < l < v_s \prod_{r=s}^{\infty} \left[1 + \frac{1}{(r-1)r} \right].$$

Taking $s=25$, we have from (13)

$$\log l < \log v_{25} + \sum_{25}^{\infty} \frac{1}{(r-1)r} - \frac{1}{2}\sum_{25}^{\infty} \frac{1}{(r-1)^2 r^2} + \frac{1}{3}\sum_{25}^{\infty} \frac{1}{(r-1)^3 r^3}$$

$$= \log v_{25} + \frac{6805}{41472} - \frac{\pi^2}{2} + \sum_{r=1}^{24} \frac{3}{r^4}$$

$$= \log v_{25} + \frac{6805}{41472} - \cdot 122431993$$

$$\therefore \quad l < \cdot 0497884.$$

Again, from (14), we have that

$$\log l > \log v_{25} + \sum_{r=25}^{\infty} \frac{1}{r(r-1)} - \sum_{r=25}^{\infty} \frac{2}{(r+2)^4} - \frac{1}{2}\sum_{r=25}^{\infty} \frac{1}{r^2(r-1)^2}$$

$$+ \sum_{r=25}^{\infty} \frac{2}{r(r-1)(r+2)^4} > \log v_{25} + \frac{95}{1152} - \sum_{r=25}^{\infty} \frac{1}{r^2} - \int_{24}^{\infty} \frac{2dx}{(x+2)^4}$$

$$+ \int_{25}^{\infty} \frac{2dx}{(x+2)^6} = \log v_{25} + \frac{95}{1152} - \cdot 040810664 - \frac{1}{26364} + \frac{2}{71744535}$$

from which $l > \cdot 0497865$.

Now the value of $\frac{1}{e^3} = \cdot 0497871$:

Thus the values of both l and $\frac{1}{e^3}$ correct to five places of decimals are the same, *viz.*, ·04979. It is, therefore, highly probable that the conjecture $l = \frac{1}{e^3}$ which Jacob discarded is correct.

Department of Mathematics,
Muslim University,
Aligarh.

References

[1] Trans. Camb. Phil. Soc., 16 (1898), pp. 262-290.

[2] Jacob, S. M., The enumeration of the Latin rectangle etc., Proc. Lond. Math. Soc. (2), Vol. 31 (1930), pp. 329-336.