Primitive Cusp Forms

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Let $S_k(N)$ denote the vector space of weight k cusp forms on $\Gamma_0(N)$ with trivial character; see [1] for background. There are two circumstances under which $f \in S_k(N)$ might fail to be **primitive** [2]:

- $f \in S_k(N/d)$ for some divisor $d > 1$ of N
- $f(z) = g(d\,z)$ and $g \in S_k(N/d)$ for some divisor $d > 1$ of N.

For example, let f_{11A} denote the (unique) level 11 weight 2 cusp form, then both $f_{11A}(z)$ and $f_{11A}(2z)$ are level 22 cusp forms. Similarly, both $f_{14A}(z)$ and $f_{14A}(2z)$ are level 28 cusp forms, and both $f_{15A}(z)$ and $f_{15A}(2z)$ are level 30 cusp forms. None of these are "new" at $N = 22$, 28 or 30 since they arise from lower levels.

Define $S_k^{\#}(N)$ to be the vector space of weight k primitive cusp forms (or **Hecke newforms**) on $\Gamma_0(N)$ with trivial character. We restrict attention to the case $k = 2$ henceforth. The dimension $\delta_0^{\#}(N)$ of $S_2^{\#}(N)$ over $\mathbb{C}$ possesses the following formula [3, 4, 5]:

$$\delta_0^{\#}(N) = \mu(N) + \frac{\lambda(N)}{12} - \frac{\omega_2(N)}{4} - \frac{\omega_3(N)}{3} - \frac{\kappa(N)}{2}$$

where λ, κ, ω_2, ω_3 are multiplicative functions with

$$\lambda(p^e) = \begin{cases} p-1 & \text{if } e = 1, \\ p^2 - p - 1 & \text{if } e = 2, \\ p^{e-3}(p+1)(p-1)^2 & \text{if } e \geq 3, \end{cases}$$

$$\kappa(p^e) = \begin{cases} 0 & \text{if } e \equiv 1 \bmod 2, \\ p-2 & \text{if } e = 2, \\ p^{e/2-2}(p-1)^2 & \text{if } 4 \leq e \equiv 0 \bmod 2, \end{cases}$$

$$\omega_2(p^e) = \begin{cases} -1 & \text{if } p = 2 \text{ and } e \leq 2, \\ 1 & \text{if } p = 2 \text{ and } e = 3, \\ 0 & \text{if } p = 2 \text{ and } e \geq 4, \\ \left(\frac{-4}{p}\right) - 1 & \text{if } p \neq 2 \text{ and } e = 1, \\ -\left(\frac{-4}{p}\right) & \text{if } p \neq 2 \text{ and } e = 2, \\ 0 & \text{if } p \neq 2 \text{ and } e \geq 3, \end{cases}$$

$$\omega_3\left(p^e\right) = \begin{cases} -1 & \text{if } p = 3 \text{ and } e \le 2, \\ 1 & \text{if } p = 3 \text{ and } e = 3, \\ 0 & \text{if } p = 3 \text{ and } e \ge 4, \\ \left(\frac{-3}{p}\right) - 1 & \text{if } p \ne 3 \text{ and } e = 1, \\ -\left(\frac{-3}{p}\right) & \text{if } p \ne 3 \text{ and } e = 2, \\ 0 & \text{if } p \ne 3 \text{ and } e \ge 3, \end{cases}$$

$\mu(N)$ is the Möbius mu function [6], and $(-4/p)$, $(-3/p)$ are Kronecker-Jacobi-Legendre symbols [7]. We have asymptotic extreme results [4]

$$\tfrac{1}{12}(0.3739558136...) = \tfrac{1}{12}\prod_p \left(1 - \tfrac{1}{p(p-1)}\right) = \liminf_{N\to\infty} \frac{\delta_0^{\#}(N)}{\varphi(N)} < \limsup_{N\to\infty} \frac{\delta_0^{\#}(N)}{\varphi(N)} = \tfrac{1}{12}$$

and average behavior

$$\sum_{N \le x} \delta_0^{\#}(N) = \frac{45}{2\pi^6}x^2 + o\left(x^2\right)$$

as $x \to \infty$, where $\varphi(N)$ is the Euler totient function [8] and the infinite product is Artin's constant [9].

For concreteness' sake, here is a list of basis elements of $S_2^{\#}(N)$ for $1 \le N \le 32$ [10, 11, 12, 13]:

$$f_{11A}(z) = \eta(z)^2\eta(11z)^2,$$

$$f_{14A}(z) = \eta(z)\eta(2z)\eta(7z)\eta(14z),$$

$$f_{15A}(z) = \eta(z)\eta(3z)\eta(5z)\eta(15z),$$

$$f_{17A}(z) = \frac{\eta(z)\eta(4z)^2\eta(34z)^5}{\eta(2z)\eta(17z)\eta(68z)^2} - \frac{\eta(2z)^5\eta(17z)\eta(68z)^2}{\eta(z)\eta(4z)^2\eta(34z)},$$

$$f_{19A}(z) = \left(\frac{\eta(8z)^2\eta(76z)^5}{\eta(4z)\eta(38z)^2\eta(152z)^2} - \frac{\eta(2z)^2\eta(38z)^2}{\eta(z)\eta(19z)} + \frac{\eta(4z)^5\eta(152z)^2}{\eta(2z)^2\eta(8z)^2\eta(76z)}\right)^2,$$

$$f_{20A}(z) = \eta(2z)^2\eta(10z)^2,$$

$$\begin{aligned} f_{21A}(z) \quad = \quad & \frac{\eta(7z)\left[3\eta(z)^2\eta(7z)^2\eta(9z)^3 - \eta(3z)^5\eta(7z)\eta(21z) + 7\eta(z)\eta(3z)^2\eta(21z)^4\right]}{2\eta(z)^2\eta(3z)\eta(21z)} + \\ & \frac{3\eta(7z)\eta(63z)\left[\eta(z)^2\eta(7z)\eta(9z)^3 - \eta(3z)^5\eta(21z)\right]}{2\eta(z)\eta(3z)\eta(9z)\eta(21z)} + \frac{3\eta(z)^2\eta(7z)\eta(9z)\eta(63z)^2}{2\eta(3z)\eta(21z)}, \end{aligned}$$

$$f_{23A}(z) = q - \tfrac{1-\sqrt{5}}{2}q^2 - \sqrt{5}q^3 - \tfrac{1+\sqrt{5}}{2}q^4 - (1-\sqrt{5})q^5 - \tfrac{5-\sqrt{5}}{2}q^6 + \cdots,$$

$$f_{23B}(z) = q - \tfrac{1+\sqrt{5}}{2}q^2 + \sqrt{5}q^3 - \tfrac{1-\sqrt{5}}{2}q^4 - (1+\sqrt{5})q^5 - \tfrac{5+\sqrt{5}}{2}q^6 + \cdots,$$

$$f_{24A}(z) = \eta(2z)\eta(4z)\eta(6z)\eta(12z),$$
$$f_{26A}(z) = q - q^2 + q^3 + q^4 - 3q^5 - q^6 - q^7 - q^8 - 2q^9 + 3q^{10} + 6q^{11} + q^{12} + \cdots,$$
$$f_{26B}(z) = q + q^2 - 3q^3 + q^4 - q^5 - 3q^6 + q^7 + q^8 + 6q^9 - q^{10} - 2q^{11} - 3q^{12} + \cdots,$$
$$f_{27A}(z) = \eta(3z)^2\eta(9z)^2,$$
$$f_{29A}(z) = q - (1-\sqrt{2})q^2 + (1-\sqrt{2})q^3 + (1-2\sqrt{2})q^4 - q^5 - (3-2\sqrt{2})q^6 + \cdots,$$
$$f_{29B}(z) = q - (1+\sqrt{2})q^2 + (1+\sqrt{2})q^3 + (1+2\sqrt{2})q^4 - q^5 - (3+2\sqrt{2})q^6 + \cdots,$$
$$f_{30A}(z) = \eta(3z)\eta(5z)\eta(6z)\eta(10z) - \eta(z)\eta(2z)\eta(15z)\eta(30z),$$
$$f_{31A}(z) = q + \tfrac{1-\sqrt{5}}{2}q^2 - (1-\sqrt{5})q^3 - \tfrac{1+\sqrt{5}}{2}q^4 + q^5 - (3-\sqrt{5})q^6 + \cdots,$$
$$f_{31B}(z) = q + \tfrac{1+\sqrt{5}}{2}q^2 - (1+\sqrt{5})q^3 - \tfrac{1-\sqrt{5}}{2}q^4 + q^5 - (3+\sqrt{5})q^6 + \cdots,$$
$$f_{32A}(z) = \eta(4z)^2\eta(8z)^2$$

where $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta function and $q = e^{2\pi i z}$ [14]. It is natural to ask whether basis elements possessing integer coefficients necessarily have an eta expression. Counterexamples might include $f_{26A}(z)$ and $f_{26B}(z)$. Another counterexample might be $f_{49A}(z)$, which evidently can be represented via Ramanujan's two-variable theta function [15].

What can be said about the relative number of newforms to cusp forms in $\Gamma_0(N)$? Martin [4] proved that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{N\le n}\frac{\delta_0^{\#}(N)}{\delta_0(N)} = \prod_p\left(1+\frac{1}{p}\right)^{-1}\left(1-\frac{1}{p}\right)\left(1+\frac{2}{p}-\frac{1}{p^4}-\frac{1}{p^5}\right) = 0.444301....$$

A parallel theory can be developed for weight 2 primitive cusp forms on $\Gamma_1(N)$ with trivial character [5]. The answer to the same question over $\Gamma_1(N)$ is [4]

$$\lim_{n\to\infty}\frac{1}{n}\sum_{N\le n}\frac{\delta_1^{\#}(N)}{\delta_1(N)} = \prod_p\left(1+\frac{1}{p}\right)^{-1}\left(1+\frac{1}{p}-\frac{2}{p^3}-\frac{2}{p^4}-\frac{2}{p^5}+\frac{1}{p^6}+\frac{1}{p^7}+\frac{1}{p^8}\right) = 0.652036....$$

Given a weight k primitive cusp form $f(z) = \sum_{m=1}^{\infty} a_m q^m$ on $\Gamma_0(N)$, define

$$L_f(z) = \sum_{m=1}^{\infty} a_m m^{-z}, \qquad \mathrm{Re}(z) > (k+1)/2.$$

This admits analytic continuation to all of $\mathbb{C}$. What can be said about L-series moments over all such f at $z = 1/2$? Conrey [16, 17] proved that, for $k = 2$,

$$\tfrac{1}{\delta_0^{\#}(N)}\sum_{f\in S_2^{\#}(N)} L_f(1/2) \sim \zeta(2),$$

$$\frac{1}{\delta_0^{\#}(N)} \sum_{f\in S_2^{\#}(N)} L_f^2(1/2) \sim 2\zeta(2)^2 \prod_p \left(1+\tfrac{1}{p^2}\right)\cdot \ln\left(\sqrt{N}\right),$$

$$\frac{1}{\delta_0^{\#}(N)} \sum_{f\in S_2^{\#}(N)} L_f^3(1/2) \sim 8\zeta(2)^3 \prod_p \left(1-\tfrac{1}{p}\right)\left(1+\tfrac{1}{p}+\tfrac{4}{p^2}+\tfrac{1}{p^3}+\tfrac{1}{p^4}\right)\cdot \frac{\ln\left(\sqrt{N}\right)^3}{3!},$$

$$\frac{1}{\delta_0^{\#}(N)} \sum_{f\in S_2^{\#}(N)} L_f^4(1/2) \sim 128\zeta(2)^5 \prod_p \left(1-\tfrac{1}{p}\right)^3\left(1+\tfrac{3}{p}+\tfrac{11}{p^2}+\tfrac{10}{p^3}+\tfrac{11}{p^4}+\tfrac{3}{p^5}+\tfrac{1}{p^6}\right)\cdot \frac{\ln\left(\sqrt{N}\right)^6}{6!}$$

as $N \to \infty$ passes through the prime numbers. Is it necessary that N be prime? Compare and contrast with [18, 19]. What is the $\Gamma_1(N)$-analog of the first four moments? It would be illustrative to perform the same calculations for weight 4 newforms as well.

0.1. Half-Integer Weights. Let $k \geq 1$ be an odd integer and $N \geq 4$ be a multiple of 4. A **modular form of weight $k/2$ and level N** is an analytic function f defined on the complex upper half plane that transforms under the action of $\Gamma_0(N)$ according to [2, 20, 21]

$$f\left(\frac{az+b}{cz+d}\right) = \left(\frac{c}{d}\right)^k \varepsilon_d^{-k}(cz+d)^{k/2} f(z) \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$$

and whose Fourier series $f(z) = \sum_{n=-\infty}^{\infty} \gamma_n e^{2\pi i n z}$ satisfies $\gamma_n = 0$ for all $n < 0$. For the preceding relation, define

$$\varepsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \bmod 4, \\ i & \text{if } d \equiv 3 \bmod 4. \end{cases}$$

Note that d must be odd since otherwise $ad-bc$ would be divisible by 2, contradicting $ad-bc=1$. For negative odd d or zero c, let

$$\left(\frac{c}{d}\right) = \begin{cases} \left(\dfrac{c}{|d|}\right) & \text{if } d<0 \text{ and } c>0, \\ -\left(\dfrac{c}{|d|}\right) & \text{if } d<0 \text{ and } c<0, \\ 1 & \text{if } d=\pm 1 \text{ and } c=0. \end{cases}$$

If, additionally, we have $\gamma_0 = 0$, then f is a **cusp form** of weight $k/2$ and level N. The space $M_{k/2}(N)$ of modular forms and the space $S_{k/2}(N)$ of cusp forms satisfy

$$\dim(M_{k/2}(4)) = \left\lfloor \frac{k}{4} \right\rfloor + 1$$

and $\dim(S_{k/2}(4)) = \dim(M_{k/2}(4)) - 2$ if $k \geq 9$. Straightforward formulas for $\dim(S_{1/2}(N))$ and $\dim(S_{3/2}(N))$ have not yet been found, but we know that [22, 23, 24, 25]

$$\dim(S_{5/2}(N)) = \frac{1}{8}\psi(N) - \frac{1}{2\,\alpha(N)}\beta(N)\chi(N)$$

where

$$\psi(N) = N\prod_{p|N}\left(1+\frac{1}{p}\right), \qquad \chi(N) = \sum_{d|N}\varphi\left(\gcd\left(d, \frac{N}{d}\right)\right),$$

$$\alpha(N) = \begin{cases} 3 \cdot 2^{r_2(N)/2-1} & \text{if } r_2(N) \text{ is even,} \\ 2^{(r_2(N)+1)/2} & \text{if } r_2(N) \text{ is odd,} \end{cases}$$

$r_p(N)$ is the largest exponent e such that p^e divides N for prime p, and

$$\beta(N) = \begin{cases} \alpha(N) & \text{if } r_2(N) \geq 4, \\ 3 & \text{if } r_2(N) = 3, \\ 2 & \begin{array}{l}\text{if } r_2(N) = 2 \text{ and there exists } p \equiv 3 \bmod 4 \\ \text{such that } p|N \text{ and } r_p(N) \text{ is odd,}\end{array} \\ 3/2 & \text{otherwise.} \end{cases}$$

There are slightly different formulas for $\dim(S_{k/2}(N))$ for larger k as well. The proof, due to Cohen & Oesterlé [22], has never been published.

In the following, we will need one of the two basis elements of $M_2(4)$:

$$F(z) = \sum_{n=0}^{\infty}\sigma(2n+1)q^{2n+1}$$

where $\sigma(m)$ is the sum of all divisors of m. It can be shown that [2, 23]

$$F(z) = \frac{\eta(4z)^8}{\eta(2z)^4}.$$

The simplest half-integer weight modular form has weight 1/2 and level 4:

$$\theta(z) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{\eta(2z)^5}{\eta(z)^2\eta(4z)^2}.$$

(It turns out that $\theta(z)^4$ is the other basis element of $M_2(4)$.) Let us focus on cusp forms henceforth [26]. The first nonzero cusp form of weight 1/2 occurs at level 1728:

$$\frac{1}{2}\sum_{n=-\infty}^{\infty}\left(\frac{12}{n}\right)q^{3n^2} = \eta(72z)$$

and the first nonzero cusp form of weight 3/2 occurs at level 28:

$$\frac{\eta(z)\eta(4z)\eta(14z)^4}{\eta(2z)\eta(7z)\eta(28z)}.$$

The first nonzero cusp form of level 4 has weight 9/2:

$$\theta(z)F(z)\left(\theta(z)^4 - 16F(z)\right) = \frac{\eta(2z)^{12}}{\theta(z)^3};$$

the first nonzero cusp form of level 8 has weight 7/2:

$$\frac{\eta(z)^2\eta(4z)^6}{\eta(2z)};$$

the first nonzero cusp form of level 12 has weight 5/2:

$$\frac{\eta(2z)^3\eta(6z)^3}{\theta(3z)}.$$

A prominent example is one of the two basis elements of $S_{13/2}(4)$:

$$\theta(z)F(z)\left(\theta(z)^4 - 16F(z)\right)\left(\theta(z)^4 - 2F(z)\right)$$

which is the image of $\Delta(z) \in S_{12}(1)$ under what is called the *Shimura correspondence* [2, 27]. Further discussion of this topic, with application to Tunnell's solution of the congruent number problem, is beyond our scope. We have not mentioned newforms of half-integer weight thus far – in fact, two distinct definitions are commonly used, one due to Serre & Stark [28] and the other due to Kohnen [29] – more details and examples are forthcoming.

0.2. Complex Multiplication. A cusp form $f(z) = \sum_{n=1}^{\infty} \gamma_n q^n \in S_k(N)$ has **complex multiplication (CM)** by a nontrivial Dirichlet character ξ if [30]

$$f(z) = \sum_{n=1}^{\infty} \xi(n)\gamma_n q^n;$$

equivalently, $\xi(p) = 1$ or $\gamma_p = 0$ for each prime p. It can be shown that ξ is necessarily a quadratic character, thus we often refer to CM by the corresponding quadratic field. There is a one-to-one correspondence between imaginary quadratic fields of class number one: [31]

$$\mathbb{Q}(\sqrt{-1}),\ \mathbb{Q}(\sqrt{-2}),\ \mathbb{Q}(\sqrt{-3}),\ \mathbb{Q}(\sqrt{-7}),\ \mathbb{Q}(\sqrt{-11}),$$
$$\mathbb{Q}(\sqrt{-19}),\ \mathbb{Q}(\sqrt{-43}),\ \mathbb{Q}(\sqrt{-67}),\ \mathbb{Q}(\sqrt{-163})$$

and CM-newforms of weight 2 (elliptic curves with CM) up to twisting: [32]

$$64A4,\ 256A1,\ 27A3,\ 49A1,\ 121B1,\ 361A1,\ 1849A1,\ 4489A1,\ 26569A1$$

with rational coefficients. Schütt [33] classified similarly CM-newforms of weight 3 and 4.

0.3. Singular K3 Surfaces. We merely mention a class of projective varieties, called **K3 surfaces**, that are two-dimensional analogs of elliptic curves [34]. The name K3 is given in honor of Kummer, Kahler & Kodaira and also refers to the mountain K2 [35]. Existence of rational points is one theme; canonical heights of such points can be computed [36, 37] as with elliptic curves.

A K3 surface over $\mathbb{Q}$ is *not* modular, in general [38]. If we restrict attention to what are called **singular** (or **extremal**) K3 surfaces, however, then modularity holds with associated newform of weight 3 and possibly nontrivial Nebentypus character [39, 40, 41]. Further, the newform is CM.

For example, the Fermat quartic surface in $\widetilde{\mathbb{C}^3}$:

$$Z_0^4 + Z_1^4 + Z_2^4 + Z_3^4 = 0$$

has corresponding unique CM-newform of weight 3 and level 16: [34]

$$\eta(4z)^6$$

which has character $(-4/\cdot)$. There are unique CM-newforms of weight 3 and levels 7, 8, 11 and 15: [33, 42, 43]

$$\eta(z)^3\eta(7z)^3,$$
$$\eta(z)^2\eta(2z)\eta(4z)\eta(8z)^2,$$
$$\left(G(z)^2 + 4G(2z)^2 + 8G(4z)^2\right) G(z)^2/G(2z),$$
$$\eta(3z)^3\eta(5z)^3 - \eta(z)^3\eta(15z)^3$$

where

$$G(z) = \eta(z)\eta(11z)$$

and we wonder if algebraic expressions for geometric realizations of these (for example, as intersections of varieties) can be found.

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References

[1] S. R. Finch, Modular forms on $\mathrm{SL}_2(\mathbb{Z})$, unpublished note (2005).

[2] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Springer-Verlag, 1984, pp. 145–146, 173–174, 176–184, 213, 216, 238; MR0766911 (86c:11040).

[3] E. Halberstadt and A. Kraus, Courbes de Fermat: résultats et problèmes, *J. Reine Angew. Math.* 548 (2002) 167–234; MR1915212 (2003h:11068).

[4] G. Martin, Dimensions of the spaces of cusp forms and newforms on $\Gamma_0(N)$ and $\Gamma_1(N)$, *J. Number Theory* 112 (2005) 298–331; math.NT/0306128; MR2141534 (2005m:11069).

[5] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A127788 and A159046.

[6] S. R. Finch, Meissel-Mertens constants, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 94–98.

[7] S. R. Finch, Quadratic Dirichlet L-series, unpublished note (2005).

[8] S. R. Finch, Euler totient constants, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 115–118.

[9] S. R. Finch, Artin's constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 104–109.

[10] Y. Martin and K. Ono, Eta-quotients and elliptic curves, *Proc. Amer. Math. Soc.* 125 (1997) 3169–3176; MR1401749 (97m:11057).

[11] W. Stein, Table of q-expansions of normalized new eigenforms, http://wstein.org/Tables/an.html.

[12] M. Somos, Eta expressions for newforms $17A$, $19A$, $21A$ and $30A$, unpublished note (2011).

[13] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A006571, A030187, A030184, A128263, A187096, A030205, A187846, A030188, A247198, A251913, A030206, A122777, A002171.

[14] S. R. Finch, Dedekind eta products, unpublished note (2007).

[15] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A140686.

[16] J. B. Conrey, L-functions and random matrices, *Mathematics Unlimited - 2001 and Beyond*, ed. B. Engquist and W. Schmid, Springer-Verlag, 2001, pp. 331–352; math.NT/0005300; MR1852163 (2002g:11134).

[17] S. R. Finch, Confirmation of Conrey's formulas, http://www.people.fas.harvard.edu/~sfinch/csolve/lconfirm.pdf.

[18] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein and N. C. Snaith, Integral moments of L-functions, *Proc. London Math. Soc.* 91 (2005) 33–104; math.NT/0206018; MR2149530 (2006j:11120).

[19] H. M. Bui and J. P. Keating, On the mean values of L-functions in orthogonal and symplectic families, *Proc. Lond. Math. Soc.* 96 (2008) 335–366; arXiv:math/0612348; MR2396123 (2009a:11172).

[20] G. Shimura, Modular forms of half integral weight, *Modular Functions of One Variable. I*, Proc. 1972 Antwerp conf., Lect. Notes in Math. 320, ed. W. Kuyk, Springer-Verlag, 1973, pp. 57–74; correction, *Modular Functions of One Variable. IV*, Lect. Notes in Math. 476, p. 145; MR0480346 (58 #519).

[21] G. Shimura, On modular forms of half integral weight, *Annals of Math.* 97 (1973) 440–481; MR0332663 (48 #10989).

[22] H. Cohen and J. Oesterlé, Dimensions des espaces de formes modulaires, *Modular Functions of One Variable. VI*, Proc. 1976 Bonn conf., ed. J.-P. Serre and D. B. Zagier, Lect. Notes in Math. 627, Springer-Verlag, 1977, pp. 69–78; MR0472703 (57 #12396).

[23] K. Ono, *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-series*, Amer. Math. Soc., 2004, pp. 10–16, 19; MR2020489 (2005c:11053).

[24] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A159630–A159636.

[25] P. Humphries, A conjecture related to the Cohen-Oesterlé dimension formula of spaces of modular forms for half-integer weights, http://mathoverflow.net/questions/161948.

[26] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A000122, A159813, A029769, A159814, A159584 and A054891.

[27] W. Kohnen and D. Zagier, Values of L-series of modular forms at the center of the critical strip, *Invent. Math.* 64 (1981) 175–198; MR0629468 (83b:10029).

[28] J.-P. Serre and H. M. Stark, Modular forms of weight 1/2, *Modular Functions of One Variable. VI*, Proc. 1976 Bonn conf., ed. J.-P. Serre and D. B. Zagier, Lect. Notes in Math. 627, Springer-Verlag, 1977, pp. 27–67; MR0472707 (57 #12400).

[29] W. Kohnen, Newforms of half-integral weight, *J. Reine Angew. Math.* 333 (1982) 32–72; MR0660784 (84b:10038).

[30] K. A. Ribet, Galois representations attached to eigenforms with Nebentypus, *Modular Functions of One Variable. V*, Proc. 1976 Bonn conf., ed. J.-P. Serre and D. B. Zagier, Lect. Notes in Math. 601, Springer-Verlag, 1977, pp. 17–51; MR0453647 (56 #11907).

[31] S. R. Finch, Class number theory, unpublished note (2005).

[32] J. H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Springer-Verlag, 1994, pp. 101–102, 109–111, 140–143, 483; MR1312368 (96b:11074).

[33] M. Schütt, CM newforms with rational coefficients, *Ramanujan J.* 19 (2009) 187–205; arXiv:math/0511228; MR2511671 (2010c:11052).

[34] M. Schütt, Arithmetic of K3 surfaces, *Jahresbericht. Deutsch. Math.-Verein.* 111 (2009) 23–41; arXiv:0808.1061; MR2508556.

[35] W. Barth, C. Peters and A. Van de Ven, *Compact Complex Surfaces*, Springer-Verlag, 1984, pp. 238–288; MR0749574 (86c:32026).

[36] J. H. Silverman, Rational points on $K3$ surfaces: a new canonical height, *Invent. Math.* 105 (1991) 347–373; MR1115546 (92k:14025).

[37] G. S. Call and J. H. Silverman, Computing the canonical height on $K3$ surfaces, *Math. Comp.* 65 (1996) 259–290; MR1322885 (96g:11067).

[38] N. D. Elkies and M. Schütt, Modular forms and $K3$ surfaces, *Adv. Math.* 240 (2013) 106–131; arXiv:0809.0830; MR3046305.

[39] T. Shioda and H. Inose, On singular $K3$ surfaces, *Complex Analysis and Algebraic Geometry*, ed. W. L. Baily and T. Shioda, Cambridge Univ. Press, 1977, pp. 119–136; MR0441982 (56 #371).

[40] R. Livné, Motivic orthogonal two-dimensional representations of Gal($\overline{Q}/Q$), *Israel J. Math.* 92 (1995) 149–156; MR1357749 (96m:11050).

[41] N. Yui, Arithmetic of Calabi-Yau varieties, *Mathematisches Institut, Georg-August-Universität Göttingen: Seminars Summer Term 2004*, ed. Y. Tschinkel, Universitätsdrucke Göttingen, 2004, pp. 9–29; http://www.uni-math.gwdg.de/tschinkel/SS04/yui.pdf; MR2183123 (2007h:11050).

[42] M. Somos, Eta expression for CM-newforms of weight 3 and levels 11 & 15, unpublished note (2007).

[43] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A002656, A030207, A129522, A136549 and A000729.