ON SOME NEW TYPES OF PARTITIONS ASSOCIATED WITH GENERALIZED FERRERS GRAPHS

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1. In this paper we find generating functions and asymptotic expressions for the number of partitions of a positive integer n into two sets of positive integers satisfying the conditions

$$n = \sum_{k=1}^{r} a_k + \sum_{j=1}^{s} b_j, \tag{1}$$

$$\left.\begin{matrix} a_1 \leqslant a_2 \leqslant a_3 \leqslant \ldots \leqslant a_r, \quad b_1 \leqslant b_2 \leqslant b_3 \leqslant \ldots \leqslant b_s, \\ b_s < a_r. \end{matrix}\right\} \tag{2}$$

The set 'b' can be empty. Such partitions are considered by Temperley (1) in a forthcoming paper on the roughness of crystal surfaces. We shall consider them in more detail and under different sets of conditions on the a's and b's.

Type A. In this type we take $b_s = a_r - 1$. Every integer up to a_r is taken at least once in the set 'a' and every integer up to b_s is taken at least once in the set 'b'. A typical partition of 13 may be written as 1123321. If $P(n)$ denotes the total number of such partitions of n for all possible values of r, it can be easily verified that $P(6) = 5$, $P(10) = 19$.

Type B. There are no restrictions on the a's and b's except those stated in (2). If $Q(n)$ denotes the total number of partitions of n under these conditions, then it is easily found that $Q(6) = 27$, $Q(10) = 209$.

Type C. In this type every integer up to a_r is taken at least once in the set 'a', there being no such restriction on the set 'b'. Denoting by $R(n)$ the total number of such partitions of n we see that $R(6) = 8$, $R(10) = 38$.

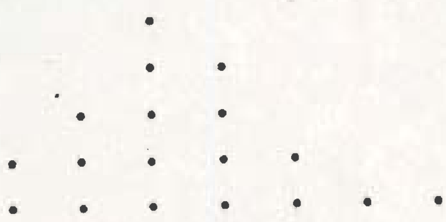

Fig. 1

We can depict these partitions by means of Ferrers graphs in which the number of dots in the columns, as reckoned from the left, first increase steadily, reach a maximum and then decrease steadily. For example, the type B partition 2354211 of the number 18 can be depicted as in Fig. 1. In particular, the partitions of the type C can be denoted by such graphs and a second type of graphical representation which will be described in § 4 and the equivalence of the two methods proved. Partitions of type B also have a second type of graphical representation, as Temperley (1) has shown. Asymptotic expressions enumerating the three types of partitions for large numbers, together with recurrence formulae, will be obtained in the following sections.

2. In this section we consider partitions of the type A. If m is the maximum value of the a's in any partition, then the number of such partitions will be the coefficient of x^n in the expression

$$\frac{x^{\frac{1}{2}m(m+1)}}{(1-x)(1-x^2)(1-x^3)\dots(1-x^m)}\,\frac{x^{\frac{1}{2}m(m-1)}}{(1-x)(1-x^2)\dots(1-x^{m-1})}.$$

Hence the generating function enumerating partitions of the type A is

$$f(x) = \sum_{m=1}^{\infty} \frac{x^{m^2}}{(1-x)^2(1-x^2)^2\dots(1-x^{m-1})^2(1-x^m)}. \tag{3}$$

Following MacMahon (2) we consider the product

$$\{(1+zx)(1+zx^3)(1+zx^5)\dots\}\left\{\left(1+\frac{x}{z}\right)\left(1+\frac{x^3}{z}\right)\left(1+\frac{x^5}{z}\right)\dots\right\}$$
$$\equiv L + M\left(z+\frac{1}{z}\right) + N\left(z^2+\frac{1}{z^2}\right) + \dots, \tag{4}$$

where $L, M, N, \dots$ are functions of x only. By changing z into zx^2 we get

$$L + M\left(zx^2 + \frac{1}{zx^2}\right) + N\left(z^2x^4 + \frac{1}{z^2x^4}\right) + \dots = zx\left\{L + M\left(z+\frac{1}{z}\right) + N\left(z^2+\frac{1}{z^2}\right) + \dots\right\}. \tag{5}$$

Comparing the coefficients of powers of z we obtain

$$M = Lx, \quad N = Mx^3 = Lx^4, \quad \text{etc.}$$

Thus the product in (4) becomes equal to

$$L\left\{1 + x\left(z+\frac{1}{z}\right) + x^4\left(z^2+\frac{1}{z^2}\right) + x^9\left(z^3+\frac{1}{z^3}\right) + \dots\right\}.$$

Putting $z = -1$, we get an expression for L:

$$L = \frac{\{(1-x)(1-x^3)(1-x^5)\dots\}^2}{1-2x+2x^4-2x^9\dots}$$
$$= \frac{\{(1-x)(1-x^3)(1-x^5)\dots\}^2}{\vartheta_4(0,x)}$$
$$= \frac{\{(1-x)(1-x^3)(1-x^5)\dots\}^2}{\prod_{n=1}^{\infty}(1-x^{2n-1})(1-x^n)}$$
$$= \prod_{n=1}^{\infty}(1-x^{2n})^{-1}. \tag{6}$$

Thus by first changing z into zx and then x^2 into x in equation (4) we finally obtain

$$\left(1+\frac{1}{z}\right)\{(1+zx)(1+zx^2)(1+zx^3)\dots\}\left\{\left(1+\frac{x}{z}\right)\left(1+\frac{x^2}{z}\right)\left(1+\frac{x^3}{z}\right)\dots\right\}$$
$$\equiv \frac{1 + x^{\frac{1}{2}}\left(zx^{\frac{1}{2}} + \dfrac{1}{zx^{\frac{1}{2}}}\right) + x^2\left(z^2x + \dfrac{1}{z^2x}\right) + \dots}{(1-x)(1-x^2)(1-x^3)\dots}. \tag{7}$$

Using the well-known result

$$(1+zx)(1+zx^2)(1+zx^3)\ldots = 1+\frac{zx}{1-x}+\frac{z^2x^3}{(1-x)(1-x^2)}+\frac{z^3x^6}{(1-x)(1-x^2)(1-x^3)}+\ldots, \tag{8}$$

we get from (7), by equating the terms independent of z,

$$\begin{aligned}&\left\{1+\frac{x^2}{(1-x)^2}+\frac{x^6}{(1-x)^2(1-x^2)^2}+\frac{x^{12}}{(1-x)^2(1-x^2)^2(1-x^3)^2}\ldots\right\}\\ &\quad+\left\{\frac{x}{1-x}+\frac{x^4}{(1-x)^2(1-x^2)}+\frac{x^9}{(1-x)^2(1-x^2)^2(1-x^3)}\ldots\right\}\\ &\quad=\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots},\end{aligned} \tag{9}$$

the indices of the numerators in the first series being twice the triangular numbers. By putting $z=e^{i\theta}$ and $e^{-i\theta}$ in (8), multiplying the two expressions and then integrating with respect to θ between 0 and 2π we obtain

$$\begin{aligned}1+\frac{x^2}{(1-x)^2}+\frac{x^6}{(1-x)^2(1-x^2)^2}+\ldots &= \frac{1}{2\pi}\int_0^{2\pi}\prod_{m=1}^{\infty}(1+2x^m\cos\theta+x^{2m})\,d\theta\\ &=\frac{1}{4\pi x^{\frac18}\prod\limits_1^{\infty}(1-x^n)}\int_0^{2\pi}\frac{\vartheta_2(\frac12\theta,x^{\frac12})}{\cos\frac12\theta}\,d\theta\\ &=\frac{2}{\pi\prod\limits_1^{\infty}(1-x^n)}\int_0^{\frac12\pi}\left(1+x\frac{\cos3\theta}{\cos\theta}+x^3\frac{\cos5\theta}{\cos\theta}+x^6\frac{\cos7\theta}{\cos\theta}\ldots\right)d\theta\\ &=\frac{1-x+x^3-x^6+x^{10}\ldots}{(1-x)(1-x^2)(1-x^3)\ldots},\end{aligned} \tag{10}$$

where the integers 1, 3, 6, 10, ... are the triangular numbers. Hence the generating function for $P(n)$ is

$$f(x)=\frac{x-x^3+x^6-x^{10}\ldots}{(1-x)(1-x^2)(1-x^3)\ldots}. \tag{11}$$

It is evident that if $p(n)$ is the number of unrestricted partitions of n, then

$$\begin{aligned}P(n)&=p(n-1)-p(n-3)+p(n-6)-p(n-10)\ldots\\ &=\sum_{r(r+1)\leqslant 2n}(-)^{r-1}p(n-\tfrac12 r(r+1)),\end{aligned} \tag{12}$$

which gives a formula for calculating $P(n)$ for small values of n. In the appendix is given a table for values of $P(n)$ up to $n=20$.

To obtain an asymptotic expression for $P(n)$ we again use the formula (12) for $P(n)$. Now by the Hardy-Ramanujan-Rademacher asymptotic formula for $p(n)$ we have

$$p(n)=\frac{1}{4n\sqrt3}e^{\pi\sqrt{(\frac23 n)}}\{1+O(n^{-\frac12})\}. \tag{13}$$

Substituting it in (12) we get

$$P(n)=\Sigma(-)^{r-1}\frac{1}{4\{n-\frac12 r(r+1)\}\sqrt3}e^{\pi\sqrt{(\frac23 n)}(n-\frac12 r(r+1))^{\frac12}}\left\{1+O\left(\frac{1}{\{n-\frac12 r(r+1)\}^{\frac12}}\right)\right\}.$$

We split the sum into two parts (i) $r < n^{\frac{3}{8}}$, (ii) $r \geqslant n^{\frac{3}{8}}$. We compare the expression for $P(n)$ with the expression

$$\sum_{r=1}^{\infty} (-)^{r-1} \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{(\frac{2}{3}n)}} e^{-\pi\sqrt{(\frac{1}{6})}n^{-\frac{1}{2}}\frac{1}{2}r(r+1)}.$$

For $r < n^{\frac{3}{8}}$ the corresponding terms differ by $O(n^{-\frac{7}{8}} e^{\pi\sqrt{(\frac{2}{3}n)}})$. The remaining terms of both series contribute

$$O(e^{\pi\sqrt{(\frac{2}{3}n)} - An^{\frac{1}{4}}}),$$

where A is a constant independent for n. Hence

$$P(n) = \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{(\frac{2}{3}n)}} \sum_{r=1}^{\infty} (-)^{r-1} e^{-\frac{1}{2}\pi\sqrt{\frac{1}{6}}n^{-\frac{1}{2}} r(r+1)} + O(n^{-\frac{7}{8}} e^{\pi\sqrt{(\frac{2}{3}n)}}). \tag{14}$$

We now evaluate the series $\sum_{m=0}^{\infty} (-)^m e^{-m(m+1)x}$ for small x:

$$\begin{aligned}
\sum_{m=0}^{\infty} (-)^m e^{-m(m+1)x} &= \sum_{m=0}^{\infty} \{e^{-2m(2m+1)x} - e^{-(2m+1)(2m+2)x}\} \\
&= \sum_{m=0}^{\infty} e^{-(2m+1)^2 x}\{e^{(2m+1)x} - e^{-(2m+1)x}\} \\
&= 2x \sum_{m=0}^{\infty} (2m+1)\, e^{-(2m+1)^2 x} + O(x^2) \sum_{m=0}^{\infty} m^2 e^{-m^2 x} \\
&= x \int_0^{\infty} u\, e^{-u^2 x}\, dx + O(x^{\frac{1}{2}}) \\
&= \tfrac{1}{2} + O(x^{\frac{1}{2}}).
\end{aligned}$$

Hence

$$\lim_{x \to 0} \sum_{m=0}^{\infty} (-)^m e^{-m(m+1)x} = \frac{1}{2}.$$

Using this result in (14) we obtain the asymptotic expression

$$P(n) \sim \frac{1}{8n\sqrt{3}} e^{\pi\sqrt{(\frac{2}{3}n)}}. \tag{15}$$

3. We now consider partitions of the type B. The generating function of these partitions is

$$g(x) = \sum_{m=0}^{\infty} \frac{x^m}{(1-x)^2 (1-x^2)^2 \ldots (1-x^{m-1})^2 (1-x^m)}. \tag{16}$$

The $(m+1)$th term of this series enumerates partitions into unrestricted parts not greater than m, but at least one summand being equal to m, followed by unrestricted parts less than or equal to $(m-1)$. We have identically

$$1 + \frac{2x - x^2}{(1-x)^2} + \frac{2x^2 - x^4}{(1-x)^2 (1-x^2)^2} + \ldots \equiv \frac{1}{(1-x)^2 (1-x^2)^2 (1-x^3)^2 \ldots}, \tag{17}$$

which gives

$$g(x) = \frac{1}{2} \frac{1}{(1-x)^2 (1-x^2)^2 (1-x^3)^2 \ldots} - \frac{1}{2}\left\{1 + \frac{x^2}{(1-x)^2} + \frac{x^4}{(1-x)^2 (1-x^2)^2} + \ldots\right\}.$$

Using the relation

$$\frac{1}{(1-zx)(1-zx^2)(1-zx^3)\ldots} = 1 + \frac{zx}{1-x} + \frac{z^2x^2}{(1-x)(1-x^2)} + \ldots, \tag{18}$$

we find, by the same process that was used in deriving equation (10) from equation (8), that

$$1+\frac{x^2}{(1-x)^2}+\frac{x^4}{(1-x)^2(1-x^2)^2}+\ldots = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{\prod_{r=1}^{\infty}(1-2x^r\cos\theta+x^{2r})}$$

$$= \frac{\vartheta_1'(0,x^{\frac{1}{2}})}{2\pi G^2}\int_0^{2\pi}\frac{\sin\frac{1}{2}\theta\, d\theta}{\vartheta_1(\frac{1}{2}\theta,x^{\frac{1}{2}})},$$

where $G = \prod_{n=1}^{\infty}(1-x^n)$. Using the known result (3)

$$\frac{\vartheta_1'(0)}{\vartheta_1(z)} = \frac{1}{\sin z}+4\sum_{m=1,3,5\ldots}\alpha_m \sin mz, \tag{19}$$

where

$$\alpha_m = \sum_{r=1}^{\infty}(-)^r q^{r(r+m)},$$

the above integral reduces to

$$\int_0^{2\pi}\frac{\vartheta_1'(0)}{\vartheta_1(\frac{1}{2}\theta)}\sin\tfrac{1}{2}\theta\, d\theta = 2\pi+4\alpha_1\int_0^{2\pi}\sin^2\tfrac{1}{2}\theta\, d\theta$$

$$= 2\pi(1+2\alpha_1)$$

$$= 2\pi\left\{1+2\sum_{m=1}^{\infty}(-)^m x^{\frac{1}{2}m(m+1)}\right\}.$$

Thus we obtain the expression for $g(x)$,

$$g(x) = \frac{x-x^3+x^6-x^{10}+\ldots}{[(1-x)(1-x^2)(1-x^3)\ldots]^2} = \left\{\sum_{n=0}^{\infty}P(n)x^n\right\}\prod_{m=1}^{\infty}(1-x^m)^{-1}, \tag{20}$$

which gives

$$\sum_{n=0}^{\infty}P(n)x^n = (1-x-x^2+x^5+x^7\ldots)\sum_{m=0}^{\infty}Q(m)x^m,$$

and thus we obtain

$$P(n) = Q(n)-Q(n-1)-Q(n-2)+Q(n-5)+Q(n-7)\ldots$$

$$= Q(n)+\sum_{k=1}^{\infty}(-)^k\{Q(n-\tfrac{1}{2}k(3k-1))+Q(n-\tfrac{1}{2}k(3k+1))\}, \tag{21}$$

and

$$Q(n) = \sum_{m=1}^{n}P(m)p(n-m). \tag{22}$$

In the appendix we give a table giving values of $Q(n)$ up to $n = 20$.

To obtain the asymptotic formula for $Q(n)$ we substitute the asymptotic formulae for $P(n)$ and $p(n)$ in (22) and deduce

$$Q(n)\sim\sum\frac{1}{96(n-m)m}e^{\pi\sqrt{\frac{2}{3}}\{(n-m)^{\frac{1}{2}}+m^{\frac{1}{2}}\}}, \tag{23}$$

since all the terms are positive. Putting $m = \frac{1}{2}n+\xi$ and expanding the indices in powers of ξ, we finally obtain

$$Q(n)\sim\frac{1}{96}\frac{4}{n^2}e^{\sqrt{(\frac{4}{3}n)}}\int_{-\infty}^{\infty}e^{-\frac{\pi}{3^{\frac{1}{2}}n^{\frac{3}{2}}}\xi^2}d\xi = \frac{1}{8\,.\,3^{\frac{3}{4}}n^{\frac{5}{4}}}e^{2\pi\sqrt{(\frac{1}{3}n)}}. \tag{24}$$

4. We proceed now to the consideration of the partitions of the type C. There is an interesting way of looking at these partitions. Suppose that we want to place n coins in a plane in continuous rows touching each other such that every coin (except those in the first row) lies in the groove formed by two other coins in the row below it. For example,

thirteen coins can be arranged in a pattern shown in Fig. 2. If there are r coins in the first row, then there may be $r-1, r-2, r-3, \ldots, 1, 0$ coins in the second row. If the number in the second row is $(r-k)$, the two rows can be arranged in k distinct ways.

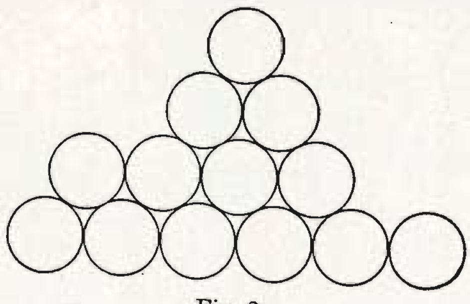

Fig. 2

Thus if $R(n,r)$ denotes the number of ways in which r coins can be arranged in the above type of pattern with r coins in the first row, then we have

$$R(n,r) = R(n-r,r-1)+2R(n-r,r-2)+3R(n-r,r-3)+\ldots + (r-1)R(n-r,1)+\delta_n^r. \quad (25)$$

If $R(n,r)$ is the coefficient of x^n in $h_r(x)$ we obtain

$$h_r(x) = x^r\{h_{r-1}(x)+2h_{r-2}(x)+\ldots+(r-1)h_1(x)+1\}.$$

Thus

$$h_1(x) = x,$$
$$h_2(x) = x^2\{h_1(x)+1\},$$
$$h_3(x) = x^3\{h_2(x)+2h_1(x)+1\},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Denoting $\sum_{r=1}^{\infty} h_r(x)$ by $h(x)$, we get by adding the above equations

$$h(x) = \frac{x}{1-x}+\frac{x^2}{(1-x)^2}\{h_1(x)+xh_2(x)+x^2h_3(x)+\ldots\}.$$

But

$$h_1(x)+xh_2(x)+x^2h_3(x)+\ldots = \frac{x}{1-x^2}+\frac{x^3}{(1-x^2)^2}\{h_1(x)+x^2h_2(x)+x^4h_3(x)+\ldots\}.$$

We obtain, by continuing this process,

$$h(x) = \frac{x}{1-x}+\frac{x^3}{(1-x)^2(1-x^2)}+\frac{x^6}{(1-x)^2(1-x^2)^2(1-x^3)}+\ldots. \quad (26)$$

This proves the equivalence of the two definitions because each term in (26) can be interpreted in the way of the definition of partitions of the type C given in § 1. Temperley (1) has demonstrated the same type of correspondence between partitions of the type B and arrangements of coins in rows forming a square lattice.

From the equations (8) and (18) we obtain the following results:

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{(1+e^{-i\theta})(1+xe^{-i\theta})(1+x^2e^{-i\theta})\ldots}{(1-xe^{i\theta})(1-x^2e^{i\theta})(1-x^3e^{i\theta})\ldots}\,d\theta$$

$$= \frac{x}{(1-x)^2}+\frac{x^3}{(1-x)^2(1-x^2)^2}+\frac{x^6}{(1-x)^2(1-x^2)^2(1-x^3)^2}+\ldots,$$

and
$$\frac{1}{2\pi}\int_0^{2\pi}\frac{(1+e^{-i\theta})(1+xe^{-i\theta})(1+x^2e^{-i\theta})\ldots}{(1-x^2e^{i\theta})(1-x^3e^{i\theta})(1-x^4e^{i\theta})\ldots}\,d\theta$$
$$=\frac{x^2}{(1-x)^2}+\frac{x^5}{(1-x)^2(1-x^2)^2}+\frac{x^9}{(1-x)^2(1-x^2)^2(1-x^3)^2}+\ldots,$$
the indices of the numerator in the first series being the triangular numbers and the indices of the numerator in the second series numbers of the form $\frac{1}{2}n(n+3)$. The difference of the two integrals gives
$$h(x)=\frac{x}{2\pi}\int_0^{2\pi}e^{i\theta}\prod_{m=0}^{\infty}\left(\frac{1+x^me^{-i\theta}}{1-x^{m+1}e^{i\theta}}\right)d\theta. \tag{27}$$
Since
$$\vartheta_2(\tfrac{1}{2}\theta,x^{\frac{1}{2}})=2Gx^{\frac{1}{4}}\cos\tfrac{1}{2}\theta\prod_{n=1}^{\infty}(1+2x^n\cos\theta+x^{2n}),$$
where G has been defined earlier. We can put $h(x)$ in the form
$$h(x)=\frac{x^{\frac{3}{4}}}{2\pi G}\int_0^{2\pi}\frac{e^{\frac{1}{2}i\theta}\vartheta_2(\frac{1}{2}\theta,x^{\frac{1}{2}})}{(1-x^2e^{2i\theta})(1-x^4e^{2i\theta})\ldots}\,d\theta$$
$$=\frac{x^{\frac{3}{4}}}{\pi G}\int_0^{2\pi}\frac{e^{\frac{1}{2}i\theta}\sum\limits_{n=0}^{\infty}x^{\frac{1}{2}(n+\frac{1}{2})^2}\cos(n+\frac{1}{2})\theta}{\prod\limits_{m=1}^{\infty}(1-x^{2m}e^{2i\theta})}\,d\theta$$
$$=\frac{x}{2\pi G}\int_0^{2\pi}\sum_{n=0}^{\infty}x^{\frac{1}{2}(n^2+n)}\left(e^{(n+1)i\theta}+e^{-ni\theta}\right)\left\{1+\frac{x^2}{1-x^2}e^{2i\theta}+\frac{x^4}{(1-x^2)(1-x^4)}e^{4i\theta}+\ldots\right\}d\theta$$
$$=\frac{x}{2\pi G}\int_0^{2\pi}\sum_{n=0}^{\infty}x^{n(2n+1)}e^{-2ni\theta}\left\{1+\frac{x^2}{1-x^2}e^{2i\theta}+\frac{x^4}{(1-x^2)(1-x^4)}e^{4i\theta}+\ldots\right\}d\theta$$
$$=\frac{x}{G}\sum_{n=0}^{\infty}\frac{x^{n(2n+3)}}{(1-x^2)(1-x^4)\ldots(1-x^{2n})}$$
$$=\frac{1}{\prod\limits_{n=1}^{\infty}(1-x^n)}\sum_{n=0}^{\infty}\frac{x^{(n+1)(2n+1)}}{(1-x^2)(1-x^4)\ldots(1-x^{2n})}. \tag{28}$$
From this expression for $h(x)$ we obtain a formula for $R(n)$:
$$R(n)=\sum_{m,k}p(n-2k^2-k-2m-1)\,p_k(m), \tag{29}$$
where k and m take all positive integral values consistent with the relation
$$2k^2+k+2m\leqslant n-1,$$
and $p_k(m)$ denotes the number of ways in which m can be represented as the sum of exactly k positive integers. We can also put $R(n)$ in the alternative form
$$R(n)=\sum_{m,k}p(n-k^2-2k-2m-1)\,q_k(m), \tag{30}$$

where $q_k(m)$ denotes the number of ways in which m can be represented as the sum of exactly k *different* positive integers. From (29) and (30) it follows that

$$\sum_m p(n-2m^2-3m-1)\,p(m) < R(n) < \sum_m p(n-2m-4)\,q(m).$$

This is a very weak inequality and leads to the result that for $n \to \infty$

$$\pi\sqrt{\tfrac{2}{3}} \leqslant n^{-\frac{1}{2}} \log R(n) \leqslant \pi\sqrt{\tfrac{5}{6}}. \tag{31}$$

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APPENDIX

1522 1523 1524

n	$P(n)$	$Q(n)$	$R(n)$
1	~~1~~	1	~~1~~
2	~~1~~	2	1
3	1	4	2
4	2	8	3
5	3	15	5
6	5	27	8
7	7	47	12
8	10	79	18
9	14	130	26
10	19	209	38
11	26	330	53
12	35	512	75
13	47	784	103
14	62	1,183	142
15	82	1,765	192
16	107	2,604	260
17	139	3,804	346
18	179	5,504	461
19	230	7,898	~~605~~ 607
20	293	11,240	~~706~~ 697

type 1 type 1 type 1

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Plane partitions