SOME LINEAR DIVISIBILITY SEQUENCES OF ORDER 6

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ABSTRACT. An integer sequence $(a_n)_{n\geq 1}$ is a *linear divisibility sequence of order* k if the sequence satisfies a linear recurrence of order k and if a_n divides a_m whenever n divides m and $a_n \neq 0$. Examples include the 2-parameter family of Lucas sequences of the first kind of order 2, the 2-parameter family of Lehmer sequences of order 4 and a 3-parameter family of fourth order linear divisibility sequences due to Williams and Guy [1]. We add to this list by constructing two families of linear divisibility sequences of order 6, both families depending on two integer parameters P and Q.

Our purpose in these notes is to prove the following result.

Theorem. *Let* $f(x) = \pm 1 + Px + Qx^2 + x^3$ *be a monic cubic polynomial with integer coefficients.* Let $\widetilde{f}(x) = 1 + Qx + Px^2 \pm x^3$ denote the reciprocal polynomial of $f(x)$. *Then the rational function* $x\frac{d}{dx}\left(log\left(\frac{f(x)}{\widetilde{f}(x)}\right)\right)$ *generates a linear divisibility sequence of order 6.*

Remark. There are corresponding results for monic quartic polynomials - see my notes uploaded to A327541.

We outline the proof of the theorem when the constant term of the polynomial f is $+1$ - the proof when constant term of the polynomial f is -1 being exactly similar. Suppose then

$$f(x) = 1 + Px + Qx^2 + x^3 \in \mathbb{Z}[x]$$

and suppose f factorises over $\mathbb{C}$ as

$$f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3). \tag{1}$$

Then the reciprocal polynomial $\widetilde{f}(x) = 1 + Qx + Px^2 + x^3$ factorises over $\mathbb{C}$ as

$$\widetilde{f}(x) = (1 - x\alpha_1)(1 - x\alpha_2)(1 - x\alpha_3). \tag{2}$$

We define a 2-parameter family of sequences $U_n \equiv U_n(P, Q)$ by means of the rational function expansion

$$x\frac{d}{dx}\left(\log\left(\frac{f(x)}{\widetilde{f}(x)}\right)\right) \quad = \quad \sum_{n\geq 1} U_n x^n. \tag{3}$$

Calculation gives

$$\sum_{n\geq 1} U_n x^n \quad = \quad (P-Q)x\frac{\left(x^4-2x^3-(P+Q)x^2-2x+1\right)}{(1+Px+Qx^2+x^3)\,(1+Qx+Px^2+x^3)}, \qquad (4)$$

showing U_n is an integer sequence satisfying a linear recurrence of order 6. Our aim is to show that $U_n(P,Q)$ is a divisibility sequence: that is, U_n divides U_m whenever n divides m and $U_n \neq 0$.

Suppose $P = Q$. In this case $U_n = 0$ for all n, a trivial divisibility sequence. However, we note that there is a non-trivial divisibility sequence hidden in (4). If we remove the factor of $P - Q$ from the right side of (4) before setting $Q = P$, the result is the rational function

$$\begin{aligned} F(x) &= \frac{x\left(x^4-2x^3-2Px-2x+1\right)}{\left(1+Px+Px^2+x^3\right)^2} \\ &= x-2(P+1)x^2+3P^2x^3-4(P+1)(P-1)^2x^4 \\ &\quad +5(P^2-P-1)^2x^5-6P^2(P+1)(P-2)^2x^6+\cdots. \end{aligned}$$

We claim the coefficients of $F(x)$ form a divisibility sequence. It is easy to verify that

$$F(x) = x\frac{dG(x)}{dx}, \qquad (5)$$

where

$$G(x) = \frac{x\left(1-x^2\right)}{1+(P+1)x+2Px^2+(P+1)x^3+x^4}.$$

Williams and Guy [1] found a family of linear divisibility sequences $U_n(P1,P2,Q)$ of order four, depending on three integer parameters $P1, P2$ and Q, and with the rational generating function

$$\sum_{n\geq 1} U_n(P1,P2,Q)x^n = \frac{x(1-Qx^2)}{1-P1x+(P2+2Q)x^2-P1Qx+Q^2x^4}.$$

We see that the rational function $G(x)$ is of this type with $P1 = -P-1$, $P2 = 2P-2$, and $Q = 1$. It follows from (5) that $F(x)$ is the generating function for the divisibility sequence $nU_n(-P-1, 2P-2, 1)$, proving the claim.

Suppose now $P \neq Q$. We will show that $U_n(P,Q)$ is a divisibility sequence. The normalised sequence $U_n(P,Q)/(P-Q)$ will then be a divisibility sequence with initial term equal to 1.

From (3), the generating function for the sequence U_n is given by

$$\begin{aligned}\sum_{n\geq 1} U_n x^n &= x\left(\frac{f'(x)}{f(x)} - \frac{\widetilde{f}\,'(x)}{\widetilde{f}(x)}\right)\\ &= x\sum_{i=1}^{3}\left\{-\frac{1}{\alpha_i\left(1-\dfrac{x}{\alpha_i}\right)} + \frac{\alpha_i}{1-\alpha_i x}\right\}\end{aligned} \tag{6}$$

using (1) and (2). Here the prime $'$ indicates differentiation with respect to x.

Expanding the right side of (6) into geometric series yields

$$U_n = \alpha_1^n + \alpha_2^n + \alpha_3^n - \frac{1}{\alpha_1^n} - \frac{1}{\alpha_2^n} - \frac{1}{\alpha_3^n}. \tag{7}$$

We shall recast (7) into a form more suitable for proving divisibility properties of the numbers U_n. It is straightforward to verify that if A, B, C are complex numbers such that $ABC = -1$ then

$$A + B + C - \frac{1}{A} - \frac{1}{B} - \frac{1}{C} = (A+1)(B+1)(C+1), \tag{8}$$

while if $ABC = 1$ then

$$A + B + C - \frac{1}{A} - \frac{1}{B} - \frac{1}{C} = (A-1)(B-1)(C-1). \tag{9}$$

Now α_1, α_2 and α_3 were defined in (1) as the zeros of the polynomial $f(x) = 1 + Px + Qx^2 + x^3$. Hence $\alpha_1\alpha_2\alpha_3 = -1$. Therefore $\alpha_1^n\alpha_2^n\alpha_3^n = -1$ for odd n and $\alpha_1^n\alpha_2^n\alpha_3^n = 1$ for even n. Thus setting $A = \alpha_1^n$, $B = \alpha_2^n$, $C = \alpha_3^n$ in (8) and (9) puts (7) into the form

$$U_n = \begin{cases} (\alpha_1^n + 1)(\alpha_2^n + 1)(\alpha_3^n + 1) & n \text{ odd} \\ (\alpha_1^n - 1)(\alpha_2^n - 1)(\alpha_3^n - 1) & n \text{ even.}\end{cases} \tag{10}$$

The proof that U_n is a divisibility sequence is an easy consequence of (10) and proceeds on a case-by-case basis depending on the parities of n and m. For example, let us prove that U_n divides U_{mn} when n is odd and m is even. In this case we define

$$P(x) = \frac{x^{nm} - 1}{x^n + 1} = x^{n(m-1)} - x^{n(m-2)} + \cdots + x^n - 1$$

and put $S(x,y,z) = P(x)P(y)P(z)$, a symmetric polynomial function in x, y and z. Then the quotient

$$\frac{U_{nm}}{U_n} = S\left(\alpha_1, \alpha_2, \alpha_3\right)$$

is a symmetric polynomial function of the roots α_i of the cubic equation $f(x) = 0$ and so is an integer by the fundamental theorem of symmetric polynomials. Thus we have shown that U_n divides U_{mn} when n is odd and m is even. The remaining cases can be proven in a similar manner. This concludes the proof of the theorem for the case when the polynomial $f(x)$ has constant term $+1$.

In the other case, where the polynomial

$$f(x) = -1 + Px + Qx^2 + x^3 \in \mathbb{Z}[x]$$

has constant term -1, the three roots α_1, α_2 and α_3 of $f(x) = 0$ satisfy $\alpha_1\alpha_2\alpha_3 = 1$ and so by (7) and (9)

$$U_n = \left(\alpha_1^n - 1\right)\left(\alpha_2^n - 1\right)\left(\alpha_3^n - 1\right)$$

for all n. The divisibility property of U_n follows easily from this representation. In fact, up to signs, U_n is the Lehmer-Pierce sequence associated to the cubic polynomial $f(x)$. The generating function is

$$\sum_{n\geq 1} U_n x^n = -(P+Q)x\frac{\left(x^4 + 2x^3 + (Q-P)x^2 + 2x + 1\right)}{\left(1 - Px - Qx^2 - x^3\right)\left(1 + Qx + Px^2 - x^3\right)}. \tag{11}$$

□

Example 1. Let $f(x) = x^3 - x^2 - 1$. Then

$$x\frac{d}{dx}\left(\log\left(\frac{f(x)}{\widetilde{f}(x)}\right)\right) = x + 3x^2 + x^3 + 3x^4 + 11x^5 + 9x^6 + \cdots$$

is the generating function for the sequence of associated Mersenne numbers A001351, which was shown to be a divisibility sequence by Haselgrove - see the link in A001351.

Example 2. Let $f(x) = x^3 - x - 1$. Then

$$x\frac{d}{dx}\left(\log\left(\frac{f(x)}{\widetilde{f}(x)}\right)\right) = x + x^2 + x^3 + 5x^4 + x^5 + 7x^6 + \cdots$$

is the generating function for A001945.

REFERENCES

[1] Hugh C. Williams & Richard K. Guy, Some fourth order linear divisibility sequences, Internat. J. Number Theory, 7, No. 5 (2011) 1255–1277.