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A PROBLEM OF ZARANKIEWICZ

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1. Introduction.

K. Zarankiewicz [22] and others generalizing the problem, asked for the least positive integer $k_{a,b}(m,n)$ such that every subset of k points of an m by n rectangle of the unit lattice should contain ab points situated simultaneously in a rows and b columns. If $a = b$ or $m = n$, we omit one suffix or argument; both suffixes are omitted if their values are implied by the context.

Sierpiński [18] solved the original problem, showing that $k_3(4) = 14$, $k_3(5) = 21$ and $k_3(6) = 27$. Then $k_3(n)$ was unknown for $n \geq 7$, but later [19] J. Brzeziński is credited with $k_3(7) = 34$, and Čulík [2] showed $k_3(8) = 43$. Additional values are calculated and listed in tables 2 ($a = b = 2$), 3 (a, $b = 2,3$) and 4 ($a = b = 3$).

Hartmann, Mycielski and Ryll-Nardzewski [12] showed that

(1) $$c_1 n^{4/3} < k_2(n) < c_2 n^{3/2},$$

where $c_1 = \frac{3}{4} \cdot 2^{1/3} - \varepsilon$, $c_2 = 2 + \varepsilon$; Kövari, Sös and Turán [14, see 21 for graph-theoretic connexions] showed

(2) $$k_a(n) < an + [(a-1)^{1/a} n^{2-1/a}],$$

where, here and elsewhere, square brackets denote 'integer part',

(3) $$\lim_{n\to\infty} n^{-3/2} k_2(n) = 1,$$

(4) $$k_2(p^2+p,\ p^2) = p^2(p+1) + 1, \qquad p \text{ prime},$$

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and Hyltén-Cavallius [13] observed that the same method yields

(5) $$k_{a,b}(m,n) < (a-1)n + (b-1)^{1/a} n^{1-1/a} m\,,$$

and for $a = 2$ improved this to

(6) $$k_{2,b}(m,n) < \tfrac{1}{2}n + \{(b-1)nm(m-1) + n^2/4\}^{1/2}\,.$$

He also obtained

(7) $$\lim_{n\to\infty} n^{-3/2} k_{2,3}(n) = 2^{1/2}\,,$$

(8) $$\underline{\lim}_{n\to\infty} n^{-3/2} k_{2,2h}(n) \geq h^{1/2}\,,$$

(9) $$\lim_{\substack{n\to\infty \\ m/n\to t}} n^{-3/2} k_{2,s^2+1}(m,n) = ts, \qquad 0 \leq t \leq 1/s \leq 1.$$

Čulík [3] showed that if $1 \leq a \leq m$ and $n \geq (b-1)\binom{m}{a}$, then

(10) $$k_{a,b}(m,n) = (a-1)n + (b-1)\binom{m}{a} + 1.$$

The methods given below demonstrate the 'neighboring' theorem

(11) $$k_{a,b}(m,n) = [\{(a^2-1)n + (b-1)\binom{m}{a}\}/a] + 1\,,$$

provided $(b-1)\binom{m}{a} + 1 \geq n \geq l(m,a,b)$, where the lower terminal for n has the approximate value $(b-1)\binom{m}{a}/(a+1)$. Its value is made precise for small a.

Reiman [16] showed that

(12) $$k_2(m,n) \leq \tfrac{1}{2}\{m + \sqrt{m^2 + 4mn(n-1)}\} + 1\,,$$

with equality in infinitely many cases, thus improving (2) when $a = 2$, and noted the connexion between this and (4) with finite projective and affine planes. Znám [23] improved (5) in the case $a = b$, $m = n$, to

$$k_a(n) < [\tfrac{1}{2}n(a-1) + (a-1)^{1/a}n^{2-1/a}] + 1, \tag{13}$$

noted that this was not as good as (12) in case $a = 2$, and later [24] made the further improvements

$$k_a(n) < [\tfrac{1}{2}n(a-1) + (a-1)^{1/a}n\{n - \tfrac{3}{8}(a-1)\}^{1-1/a}], \tag{14}$$

$$k_a(n) < [n(a-1)/\varepsilon + (a-1)^{1/a}n^{2-1/a}], \tag{15}$$

where $\varepsilon = \{2(n/(a-1))^{1/a} - 1\}/\{(n/(a-1))^{1/a} - 1\}$, in which (15) appeared better than (14), though this is true only for $a = 2$ and 3, and for sufficiently small n for $a \geq 4$, since the improvement of (14) over (13) is

$$(a-1)^{1/a}n^{2-1/a}\{1 - (1 - 3(a-1)/8n)^{1-1/a}\} \tag{16}$$

$$= (a-1)^{1/a}n^{2-1/a}\left\{\frac{a-1}{a}\cdot\frac{3(a-1)}{8n} + \frac{1}{2!}\,\frac{a-1}{a^2}\cdot\frac{3^2(a-1)^2}{8^2n^2} + \frac{1}{3!}\,\frac{(a-1)(a+1)}{a^3}\cdot\frac{3^3(a-1)^3}{8^3n^3} + \ldots\right\}$$

$$= \frac{3(a-1)^{2+1/a}n^{1-1/a}}{8a}\left\{1 + \frac{3(a-1)}{2!\;8an} + \frac{a+1}{3!}\cdot\frac{3^2(a-1)^2}{8^2a^2n^2} + \frac{(a+1)(a+2)}{4!}\cdot\frac{3^3(a-1)^3}{8^3a^3n^3} + \ldots\right\},$$

while the improvement of (15) over (13) is

$$n(a-1)\left\{\tfrac{1}{2} - \frac{(n/(a-1))^{1/a} - 1}{2(n/(a-1))^{1/a} - 1}\right\} = \tfrac{1}{4}(a-1)^{1+1/a}n^{1-1/a}\left\{1 - \frac{(a-1)^{1/a}}{2n^{1/a}}\right\}^{-1} \tag{17}$$

$$= \tfrac{1}{4}(a-1)^{1+1/a}n^{1-1/a}\left\{1 + \frac{(a-1)^{1/a}}{2n^{1/a}} + \frac{(a-1)^{2/a}}{2^2n^{2/a}} + \ldots + \frac{a-1}{2^a n} + \frac{(a-1)^{1+1/a}}{2^{a+1}n^{1+1/a}} + \ldots\right\},$$

which is greater than (16) for $a = 2$ or 3, since then $\frac{1}{4} \geq \frac{3(a-1)}{8a}$.
However, for $a \geq 4$, (17) is greater than (16) only if

$$\tfrac{1}{4}\left[1 - \frac{(a-1)^{1/a}}{2n^{1/a}}\right] < \frac{3(a-1)}{8a},$$

that is, only for n in the range

$$a \leq n \leq \frac{3^a(a-1)^{a+1}}{2^a(a-3)^a}. \qquad (18)$$

2. Graph-theoretic and matrix considerations.

To visualize the problem it is convenient to define the *complete a-graph*, $K_m^a = K_m^a(V,E)$ on m vertices, consisting of a set, V, of m objects, usually labelled $1, 2, \ldots, m$, and the set E of *all* a-edges, where an *a-edge* is a subset of V of cardinal a. A graph in the usual sense is a 2-graph; for brevity we use 'graph' more generally. The problem is then equivalent to finding maximal packings into such a complete graph of complete graphs on p vertices, $a \leq p \leq m$. Refinements are obtained from mixed packings of complete graphs on $p, p-1, \ldots, a$ vertices.

The theorem of Čulík, (10), corresponds to the case $p = a$. We shall be concerned with the cases $p = a + 1$, $a + 2$. In the case $b = 2$, the packings are all *edge-disjoint* in the sense of a-edge as defined, two graphs on p vertices having no a-edge in common, though edges of lower cardinal may occur repeatedly. In the general problem, a maximum of $b-1$ repetitions of the a-edges is allowed.

A neighboring problem [1,7,8] is that of determining the *coarseness* of the complete graph, in which an optimal mixed packing of K_5-graphs and complete bipartite graphs, $K_{3,3}$, is sought. Others, including the present one, are discussed by Erdös [4,5].

It is also useful to consider the problem in terms of an m by n matrix, whose elements are 0's or 1's. Define the *total* of a matrix as the sum of its elements, and call an a by b submatrix of total ab an a,b-*grid* (a-*grid* if $a = b$, or *grid* if the context is clear). Then $k = k_{a,b}(m,n)$ is such that every m by n matrix of total k contains an a,b-grid, while there is a matrix of total k-1 which contains no such grid. A matrix of total k-1, containing no a,b-grid is said [16] to be *saturated*. A 1 by n submatrix of total r is called an r-*row*, and an m by 1 of total c, a c-*col*(umn). When $a = 2$, 2-cols, 3-cols, 4-cols and 5-cols will also be called *pairs*, *triangles*, (complete) *quadrangles*, and *pentangles* respectively. When $a = 3$, 3-cols and 4-cols are referred to as *faces* and *tetrahedra*. Since the interchange of any two rows or columns does not affect the definitions or arguments, we will normally assume that the rows are arranged with totals $r_1 \geq r_2 \geq \ldots \geq r_m$, reading from top to bottom, and the columns with totals $c_1 \geq c_2 \geq \ldots \geq c_n$, from left to right. By interchanging rows and columns,

(19) $$k_{a,b}(m,n) = k_{b,a}(n,m),$$

but no other symmetry is to be expected; for example $k_{2,3}(5,3) = 12$ while $k_{3,2}(5,3) = 11$.

In the particular calculations of sections 5, 6 and 7, the method is always to establish a lower bound for $k_{a,b}(m,n)$ as one more than the total of an exhibited saturated matrix, and an (equal) upper bound, by assuming that it is possible to have an m by n matrix of total k without a grid, and arriving at a contradiction by various arguments. To avoid repetition, some of these are stated here and labelled with a capital; a prime denotes the transposed argument, e.g., .' refers to rows in place of columns.

A. There are $\binom{c_i}{a}$ a-edges in column i, and a total of $\sum\binom{c_i}{a}$. If this exceeds $(b-1)\binom{m}{a}$, then, by the pigeon-hole principle, there are b *coincident* (i.e. occupying the same a rows) edges, forming a grid.

B. If $\sum\binom{c_i}{a} = (b-1)\binom{m}{a}$, the matrix is said to be *colmax*, i.e. every a-edge of K_m^a is occupied exactly $b-1$ times. Since $c_i > c_j$ implies

$$\binom{c_i}{a} + \binom{c_j}{a} \geq \binom{c_i+1}{a} + \binom{c_j+1}{a}, \tag{20}$$

with equality only if $c_i = c_j + 1$, it follows that for a given total, $\sum c_i$, we minimize $\sum\binom{c_i}{a}$ by taking the *most level partition* of the total, i.e.

$$\sum c_i = np + q, \qquad 0 \leq q < n, \tag{21}$$

with $c_1 = \ldots = c_q = p + 1$, $c_{q+1} = \ldots = c_n = p$. Note that a colmax matrix is not necessarily saturated (e.g. $a = b = 3$, $c_1 = c_2 = m$, $c_3 = \ldots = c_n = 2$ is colmax, but saturated only if $n = 3$), but is so if its columns form the most level partition. Formula (10) follows from this argument.

C. If, for some c, $k < n(c+1)$ and $k_{a,b}(m, n-1) \leq k - c$, then $k_{a,b}(m,n) \leq k$, since, by the pigeon-hole principle, some column totals $\leq c$, and the remaining columns contain a grid.

D. If an r-row meets columns $c_1, \ldots, c_r$ and $\sum_{i=1}^{r} \binom{c_i-1}{a-1} > (b-1)\binom{m-1}{a-1}$ then the pigeon-hole principle ensures at least b $(a-1)$-tuples coincide and form a grid with the r-row.

E. If the meet of r rows and c columns totals at least $k_{a,b}(r,c)$, there is a grid.

F. A matrix which does not contain a grid can be augmented by any number of $(a-1)$-cols or $(b-1)$-rows, without producing a grid, so

(22) $$k_{a,b}(m+m', n+n') \geq k_{a,b}(m,n) + m'(a-1) + n'(b-1).$$

Note also that if $(b-1)\binom{m}{a} - \sum\binom{c_i}{a}$ is positive, then that number of a-cols can also be added.

G. Every a by b submatrix of a saturated m by n matrix contains a zero. There are $\binom{m}{a}\binom{n}{b}$ such, and each zero belongs to $\binom{m-1}{a-1}\binom{n-1}{b-1}$ of them, so the matrix contains at least mn/ab zeros, so

(23) $$k - 1 \leq mn(ab-1)/ab.$$

More generally, every r by c submatrix contains at least $rc - (k-1)$ zeros, where $k = k_{a,b}(r,c)$, and

(24) $$k_{a,b}(m,n) - 1 \leq mn\{k_{a,b}(r,c) - 1\}/rc.$$

3. Maximal packings of triangles.

To obtain results in case $a = 2$, we require maximal packings of triangles (and complete graphs of higher orders) in K_m^2. These can be obtained inductively, but the process is complicated, so instead we make suitable modifications of the work of Fort and Hedlund [6], who solved the corresponding covering problem. Define an *m-pipt* (packing, in pairs, of triangles) as a set S of triples chosen from $\{1,2, \ldots, m\}$ so that no pair occurs in more than one triple, and call it *maximal* if $|S| \geq |S'|$ for all m-pipts S'. For $m \equiv 1$ or 3, mod 6, m-pipts are *exact* and are well known [11,15,17,20] as Steiner triple systems.

Since a triangle has two edges at a vertex, the number of triangles at a vertex of K_m is at most $[\frac{1}{2}(m-1)]$, and a maximal packing therefore contains at most $[\frac{1}{3}m[\frac{1}{2}(m-1)]]$ triangles. Denote this by t_m, unless $m \equiv 5$, mod 6, when t_m is diminished by one. When $m = 6s + 5$, $\binom{m}{2} = 18s^2 + 27s + 10$ is not a multiple of 3, so not all edges occur in an m-pipt. However, if one is omitted at a vertex, at least 2 must be, since K_{6s+5} has even valence $6s + 4$, so the least number of omitted edges is four, forming the sides of a quadrilateral, and $t_m = \frac{1}{3}(18s^2 + 27s + 10 - 4) = 6s^2 + 9s + 2$.

Theorem 1. *For all positive integers m, maximal m-pipts contain t_m triangles.*

We have seen that the number is at most t_m. The proof that t_m can be achieved follows that of Fort and Hedlund, and is not given in detail. The value of t_m for $m = 6s + r$, $0 \leqslant r \leqslant 5$ is shown in table 1. When $r = 5$, and the unused pairs are AC, CB, BD, DA, say, the appropriate analog of an *admissible n-copt* [6] is an *admissible m-pipt*, i.e. a maximal m-pipt which contains triangles ABB', $B'CD$. In the remaining odd cases the m-pipts are Steiner systems, but for $r = 3$, Fort and Hedlund define as *admissible* those which contain $2s + 1$ triples which together occupy all the $6s + 3$ vertices. When r is even, the valence of K_m is odd, and an edge is unused at each vertex. In fact there are just $\frac{1}{2}m$ unused edges, which are vertex-disjoint (form a 1-factor), except when $r = 4$, and $\binom{m}{2} - \frac{1}{2}m = 18s^2 + 18s + 4$ is again not a multiple of three, and there are $3s + 3$ unused edges, $3s$ of which are vertex-disjoint, the remaining 3 forming a *trefoil* (3-claw). If this is AB', AC, AD, the concept of admissibility requires $B'CD$ to be a triangle of the system, although there are maximal m-pipts for which this is not true. In the present application, however, the former type gives an economical mixed packing of a complete quadrangle $ABCD$ and $6s(s+1)$ triangles. For $r = 5$, the corresponding result is a mixed packing of a pentangle, $ABB'CD$, and $3s(2s+3)$ triangles.

1839 1840 (divide by 2) 1841 1842 1843

0

m	t_m	$t_{2,m}$	$t_{3,m}$	u_m	$u_{2,m}$	T_m	$T_{2,m}$
3	1	2	3	0	0	0	0
4	1	4	5	1	2	1	2
5	2	6	10	1	2	1	5
6	4	10	14	1	4	3	9
7	7	14	21	2	7	7	15
8	8	18	26	2	8	14	28
9	12	24	36	3	9?	18	
10	13	30	43	5	15	30	60
11	17	36	55	6	15?	34	
12	20	44	64	9		51	108
13	26	52	78	13	26	65	
14	28	60	88	14		91	182
15	35	70	105	15		105	
16	37	80	117	20	40	140	280
$6s$	$6s^2-2s$	$12s^2-2s$	$18s^2-4s$			$\frac{3}{2}s(6s^2-3s-1)$?	$9s^2(2s-1)$?
$6s+1$	$6s^2+s$	$12s^2+2s$	$18s^2+3s$			$\frac{1}{2}s(3s-1)(6s+1)$	
$6s+2$	$6s^2+2s$	$12s^2+6s$	$18s^2+8s$			$\frac{1}{2}s(3s+1)(6s+1)$	$s(3s+1)(6s+1)$
$6s+3$	$6s^2+5s+1$	$12s^2+10s+2$	$18s^2+15s+3$			$\frac{3}{2}s(3s+1)(2s+1)$	
$6s+4$	$6s^2+6s+1$	$12s^2+14s+4$	$18s^2+20s+5$			$\frac{1}{2}s(2s+1)(3s+1)(3s+2)$	$(2s+1)(3s+1)(3s+2)$
$6s+5$	$6s^2+9s+2$	$12s^2+18s+6$	$18s^2+27s+10$			$\frac{1}{2}(3s+1)(6s^2+9s+2)$?	

TABLE 1

type 1 1 1 4 4

Packing triangles

Related to Z's problem

If $m = 6s + 1$ is prime, a direct construction is the following. Use least non-negative residues, mod m, and let x be a primitive root of m, so that $x, x^2, \ldots, x^{3s}$ are distinct and $x^{3s} \equiv -1$, $(x^s + 1)(x^{2s} - x^s + 1) \equiv 0$, $x^{2s} - x^s + 1 \equiv 0$, $x^{2s+i} - x^{s+i} + x^i \equiv 0$ $(i = 0,1, \ldots, s-1)$ and $x^{2s+i} + j$, $x^{s+i} + j$, $x^i + j$ $(i = 0,1, \ldots, s-1;$ $j = 0,1, \ldots, 6s)$ form edge disjoint triangles, $s(6s + 1) = t_m$ in number.

Another construction forms an exact pq-pipt from exact p- and q-pipts. In K_p replace each vertex by K_q, and each triangle by q^2 triangles. If A has been replaced by $A_1, \ldots, A_q$, etc., and ABC was a triangle originally, the q^2 new triangles are $A_q B_i C_i$ $(i = 1, \ldots, q)$, and $A_d B_i C_j$, $A_d B_j C_i$, where i plays j on the dth day of a round robin tournament. The construction extends to the cases $q \equiv 0$ or 2, mod 6, since there is just a 1-factor of unused edges. It can also be used in case $p = 2$, q odd, provided the q-pipt is admissible since if $q \equiv 1$ or 3, mod 6, the duplication process merely produces a 1-factor of unused edges, while if $q \equiv 5$, mod 6, in addition to the quadrilateral $ACBD$, there are triangles ABB', $B'CD$. These form a K_5 graph which on duplication becomes a K_{10}, which has an admissible 10-pipt.

Theorem 2. *If* $\binom{m}{2} - 2t_m - 1 < n \leqslant \binom{m}{2} + 1$, *then* $k_2(m,n) = 1 + 2n + [\tfrac{1}{2}\binom{m}{2} - \tfrac{1}{2}n]$, *with equality at the lower terminal for* n, *unless* $m \equiv 1$ *or* 3, *mod* 6.

Proof. For $n = \binom{m}{2}$ or $\binom{m}{2} + 1$ this is a special case of (10). Consider a matrix with $t(\leqslant n)$ 3-cols and $n-t$ 2-cols. They contain $3t + n - t$ pairs. Choose t so that these pairs are distinct, in order to avoid a grid, $2t + n \leqslant \binom{m}{2}$, and as large as possible, $t = \min(n, [\tfrac{1}{2}\binom{m}{2} - \tfrac{1}{2}n])$. By theorem 1, this choice is possible if and only if $0 \leqslant t \leqslant t_m$, so

$\binom{m}{2} \geq n \geq \binom{m}{2} - 2t_m - 1$, provided $3t_m < \binom{m}{2}$, which is true unless the m-pipt is exact. If $m \equiv 1$ or 3, mod 6, the second inequality for n must be strict. The matrix has total $3t + 2(n-t)$, so if it is saturated, $k - 1 = 2n + [\frac{1}{2}\binom{m}{2} - \frac{1}{2}n]$. Note that a total of $1 + 2n + t$ implies, by (20) and argument A, that there is a grid, since

$$\sum\binom{c_i}{2} \geq (t+1)\binom{3}{2} + (n-t-1)\binom{2}{2} > \binom{m}{2}.$$

To obtain the corresponding results for larger b, we need the maximal packing of $t_{b-1,m}$ triangles in K_m, where each edge may be used up to $b-1$ times. For $m \equiv 1$ or 3, mod 6, this is $t_{b-1,m} = (b-1)t_m$, since the m-pipt is exact. For $m \equiv 5$, mod 6, a maximal m-pipt has a quadrilateral of unused edges *ACBD*. If the edges may be used twice, reproduce the m-pipt with *BCD* permuted cyclically. The edges *ACBD*, *ADCB* form 2 additional triangles, *ABD*, *ACD*, leaving *BC* unused (twice). A third use allows an exact packing by another permutation of *BCD*, giving *ABDC*, which with *BC* twice forms triangles *ABC*, *BCD*. For $m \equiv 5$, mod 6, $t_{b-1,m} = [\frac{b-1}{3}\binom{m}{2}]$, except it is 1 less if $b \equiv 2$, mod 3.

If m is even, the unused edges in a maximal m-pipt form a 1-factor, except this is modified to include a trefoil when $m \equiv 4$, mod 6. If the edges may be used twice, we occupy those in the 1-factor in sets of 3: select a triangle *ABC* of the m-pipt; this defines unused edges *AY*, *BZ*, *CX*. In duplicating the m-pipt, interchange A, B, C with X, Y, Z respectively, giving a triangle *XYZ* and unused edges *BX*, *CY*, *AZ*. Replace *XYZ* and the unused edges by triangles *AYZ*, *BZX*, *CXY*. If $m \equiv 0$, mod 6, and b is odd, the packing is exact, so $t_{b-1,m} = \frac{b-1}{3}\binom{m}{2}$ in this case, and $\frac{b-2}{3}\binom{m}{2} + t_m$, with an unused 1-factor, if b is even. If $m \equiv 4$, mod 6, there is a trefoil *AB'*, *AC*, *AD*, and, if the m-pipt is admissible, a triangle *B'CD*. These form a complete quadrangle which, on duplication, yields 4 triangles $B'CD$, ACD, $AB'D$, $AB'C$, so for b odd we again have an exact packing. If $m \equiv 2$, mod 6, an edge remains unused (twice) when $b - 1 = 2$, so $t_{2,m} = [\tfrac{2}{3}\binom{m}{2}]$. For $b - 1 = 3$, we must waste a 1-factor, and another edge (twice), so $t_{3,m} = [\tfrac{2}{3}\binom{m}{2}] + t_m$. For $b - 1 = 4$, one edge, and hence 4, must be wasted and $t_{4,m} = 2[\tfrac{2}{3}\binom{m}{2}]$. For $b - 1 = 5$ we need waste only a 1-factor with trefoil, and for $b - 1 = 6$ the packing can be exact. Thereafter the pattern is repeated, mod 6. The results are summarized as

Theorem 3. A maximal packing of triangles in a complete graph on m vertices, where each edge may be used up to b-1 times, contains $t_{b-1,m}$ triangles, where $t_{b-1,m} = \frac{b-2}{3}\binom{m}{2} + t_m$ if m and b are both even, and the integer part is to be taken if $m \equiv 2$, mod 6, and where $t_{b-1,m} = \frac{b-1}{3}\binom{m}{2}$ otherwise, and the packing is exact, except that the integer part has to be taken if $m \equiv 5$, mod 6 and $b \equiv 0$, mod 3, or if $m \equiv 2$, and $b \equiv 3$, mod 6, and the result is one less if $m \equiv 5$, mod 6 and $b \equiv 2$, mod 3, or if $m \equiv 2$ and $b \equiv 5$, mod 6.

Theorem 2 now generalizes to

Theorem 4. *If* $(b-1)\binom{m}{2} - 2t_{b-1,m} - 1 < n \leq (b-1)\binom{m}{2} + 1$, *then* $k_{2,b}(m,n) = 1 + 2n + [\frac{b-1}{2}\binom{m}{2} - \tfrac{1}{2}n]$, *with equality in the lower terminal for* n *unless the corresponding packing of triangles is exact.*

If $b = 3$, this simplifies to

Theorem 5. *If* $\tfrac{2}{3}\binom{m}{2} \leq n \leq 2\binom{m}{2} + 1$, *then* $k_{2,3}(m,n) - 1 = \binom{m}{2} + [\frac{3n}{2}]$.

In order to obtain theorems for $n < \frac{b-1}{3}\binom{m}{2}$, optimal packings of complete quadrangles in K_m are needed. The largest number of quadrangles,

u_m, is given for small m in table 1, as are some values of $u_{2,m}$, where edges may be used twice. In order to give an exact result, mixed packings of quadrangles and triangles must be sought. We give a few further results in the neighbourhood of theorem 2.

Theorem 6. (a) $k_2(6s,\ 6s^2 + s - 3 + \varepsilon) = 18s^2 - 5 + 2\varepsilon$, $\varepsilon = 0$ or 1,

(b) $k_2(6s + 1,\ 6s^2 + s - 2 + \varepsilon) = 18s^2 + 3s - 4 + 2\varepsilon$ $\varepsilon = 0, \pm 1$,

(c) $k_2(6s + 2,\ 6s^2 + 5s - 2 + \varepsilon) = 18s^2 + 12s - 3 + 2\varepsilon$, $\varepsilon = 0, \pm 1$,

(d) $k_2(6s + 3,\ 6s^2 + 5s - 1 + \varepsilon) = 18s^2 + 15s - 1 + 2\varepsilon$, $\varepsilon = 0, \pm 1$,

(e) $k_2(6s + 4,\ 6s^2 + 5s + 1) = 18s^2 + 17s + 5$,

(f) $k_2(6s + 4,\ 6s^2 + 9s + \varepsilon) = 18s^2 + 24s + 3 + 2\varepsilon$, $\varepsilon = 0, \pm 1$,

(g) $k_2(6s + 4,\ 6s^2 + 9s + 2) = 18s^2 + 24s + 6$,

(h) $k_2(6s + 5,\ 6s^2 + 9s + 3 + \varepsilon) = 18s^2 + 27s + 9 + 2\varepsilon$, $\varepsilon = 0, \pm 1$.

Proof. (a) $m = 6s$, $t_m = 6s^2 - 2s$, so $3^{6s^2-2s+1}2^{3s-4+\varepsilon}$ are not possible, while $4 \cdot 3^{6s^2-2s-1}2^{3s-3}$ is colmax, though the quadrangle and $3s-3$ pairs occupy at most $6s-2$ vertices, leaving at least one edge unoccupied. On the other hand, $3^{6s^2-2s}2^{3s-3+\varepsilon}$ are possible columns.

(b) $m = 6s + 1$, $t_m = 6s^2 + s$. $4^{2-\varepsilon}3^{6s^2+s-4+2\varepsilon}$ is colmax, while the presence of quadrangles requires some edges are unoccupied. However 3^{6s^2+s-1} and $4^2 3^{6s^2+s-5}(2)$ are possible column arrangements. To see the latter, add a $(3s-1)$-row to $t_{6s} = 6s^2 - 2s$, and the accompanying $3s$ unoccupied edges, to form 2 4-cols with 2 of the triangles, $3s-3$ triangles from $3s-3$ of the unoccupied edges, leaving 3 unoccupied edges. There is a total of $6s^2 - 2s - 2 + (3s-3) = 6s^2 + s - 5$ triangles.

(c) and (d) are proved as in (a) and (b).

(e) since $t_{6s+3} = 6s^2 + 5s + 1$, and an *admissible* $(6s+3)$-pipt contains $2s+1$ vertex-disjoint triangles, we may add a $(2s+1)$-row to $6s^2 + 5s + 1$ 3-cols, to form $4^{2s+1}3^{6s^2+3s}$ which is saturated and colmax.

(f) $m = 6s + 4$, $t_m = 6s^2 + 6s + 1$, so $3^{6s^2+6s+3}2^{3s-3+\varepsilon}$ is impossible, and so is $4\cdot 3^{6s^2+6s+1}2^{3s-2+\varepsilon}$, while $4^2 3^{6s^2+6s-1}2^{3s-1+\varepsilon}$ fails by argument A. On the other hand, in an admissible $(6s+4)$-pipt, the tips of the trefoil form a triangle of the system and they may be replaced by a 4-col, so that $4\cdot 3^{6s^2+6s}2^{3s}$ is possible.

(g) $m = 6s + 4$. $3^{6s^2+6s+1}2^{3s+1}$ is possible, but $3^{6s^2+6s+2}2^{3s}$ is not, since it is colmax and the number of triangles exceeds t_m.

(h) $m = 6s + 5$, $t_m = 6s^2 + 9s + 2$ and an admissible m-pipt contains a quadrilateral so that $3^{6s^2+9s+2}2^{1+\varepsilon}$ is possible, while $4\cdot 3^{6s^2+9s+1}$ is impossible, since the quadrangle implies that at least 2 edges are unoccupied. So is $4^2 3^{6s^2+9s-1}2$, which is colmax. $3^{6s^2+9s+3}(2)$ and $4\cdot 3^{6s^2+9s+1}2$ are also impossible.

4. Maximal packing of tetrahedra.

We use the work of Hanani [9,10] who has shown that for $m \equiv 2$ or 4, mod 6, there is an exact packing of K_4^3 in K_m^3. In these cases the maximum number of tetrahedra is $T_m = \frac{1}{4}\binom{m}{3}$. If $m \equiv 1$ or 3, mod 6, the number of faces at an edge is odd, so at least one does not belong to any tetrahedron and there are at least $\frac{1}{2}(m-1)$ such at a vertex. So the maximum number of faces at a vertex which belong to tetrahedra is $\binom{m-1}{2} - \frac{1}{2}(m-1) = \frac{1}{2}(m-1)(m-3)$; there are at most $\frac{1}{6}(m-1)(m-3)$ tetrahedra at a vertex, so $T_m \leq \frac{1}{24}m(m-1)(m-3)$. On the other hand, delete a vertex from the cases $m + 1 \equiv 2$ or 4, mod 6, and $\frac{1}{4}\binom{m+1}{3} - \frac{1}{3}\binom{m}{2} = \frac{1}{24}m(m-1)(m-3)$ tetrahedra remain.

If $m = 6s$, the projection from any vertex on to the remainder of the graph, of a packing of tetrahedra, induces a packing of triangles in K^2_{6s-1}, and since this contains at least a quadrilateral of unoccupied edges, there are at least 4 unused faces at each vertex, and at most $\binom{m-1}{2} - 4 = 18s^2 - 9s - 3$ faces there belong to tetrahedra. At most $6s^2 - 3s - 1$ tetrahedra occur at a vertex and there are at most $\frac{3}{2}s(6s^2 - 3s - 1)$ altogether. It is hoped to show elsewhere that this number can be achieved, so that $T_m = \frac{3}{2}s(6s^2 - 3s - 1)$ in this case. The unused faces may be visualized as the (8 external) faces of each of s vertex-disjoint octahedra. There is a duplication construction, analogous to that for $2q$-pipts given in section 3, which derives an optimal packing for $m = 12s$ if there is one of this form for $m = 6s$.

If we delete a vertex from the previous case, there remain $\frac{1}{2}(3s - 2)(6s^2 - 3s - 1)$ tetrahedra, so that for $m = 6s + 5$, at least $\frac{1}{2}(3s + 1)(6s^2 + 9s + 2)$ tetrahedra may be packed. It is hoped to show that this is also an optimal packing. Packings using faces more than once have yet to be investigated in detail, though close bounds can be obtained by present methods. The following theorems are proved in the same way as theorem 2.

Theorem 7. *If* $\binom{m}{3} - 3T_m \leq n \leq \binom{m}{3} + 1$, *then* $k_{3,2}(m,n) = 1 + 3n + [\frac{1}{3}\binom{m}{3} - \frac{1}{3}n]$, *where the range for* n *may be extended downwards by* 2 *unless* $m \equiv 2$ *or* 4, *mod* 6.

Theorem 8. *If* $2\binom{m}{3} - 3T_{2,m} \leq n \leq 2\binom{m}{3} + 1$, *then* $k_3(m,n) = 1 + 3n + [\frac{2}{3}\binom{m}{3} - \frac{1}{3}n]$, *where the lower terminal for* n *may be reduced by* 2 *if the corresponding packing of tetrahedra is not exact.*

5. Special values for $a = b = 2$.

In this section, $k(m,n)$ means $k_{2,2}(m,n)$ and grid means 2,2-grid. Capital letters refer to the arguments at the end of section 2, where the method of proof is outlined. For small m,n, the values of $k(m,n)$ are shown in table 2; those above the indicated line are given by (10) or by theorem 2.

$k_2(m,n)$

m \ n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
2	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
3		7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
4			10	11	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
5				13	15	16	18	19	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
6					17	19	20	22	23	25	26	28	29	31	32	33	34	35	36	37	38	39	40	41	42	43	44
7						22	23	25	26	28	29	31	32	34	35	37	38	40	41	43	44	45	46	47	48	49	50
8							25	27	29	31	33	34	36	37	39	40	42	43	45	46	48	49	51	52	54	55	57
9								30	32	34	37	38	40	41	43	44	46	47	49	50	52	53	55	56	58	59	61
10									35	37	40	41	43	45	47	48	50	52	53	55	56	58	59	61	62	64	65
11										40	43	45	46	48	51	52	54	56	58	60	61	63	64	66	67	69	70
12											46	49	50	52	54	56	58	61	62	64	66	67	69	71	73	74	76
13												53	54	56	58	60	62	65	67	68	70	72	74	76	79	80	82
14													57	59	61	64	66	69	71	73	74	76	78	80	82	84	86
15														61	64	67	70	73	76	78							
16																			81								

TABLE 2

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Figure 1 will show $k(8) = 25$, $k(8,9) = 27$ and $k(8,10) = 29$ if we use C' in the last case, with $c = 3$, and show that $4 \cdot 3^7$ and 3^9 (i.e. a 4-col with 7 3-cols, and 9 3-cols) are impossible. The presence of a (trivalent) quadrangle at four vertices means that at least two edges are not occupied by triangles and $\binom{4}{2} + 7\binom{3}{2} > \binom{8}{2} - 2$, while $t_8 < 9$.

$$\begin{array}{cccccccc|c|c|c|c}
1&1&1&0&0&0&0&0&1&0&0&0\\
1&0&0&1&0&1&0&0&0&1&0&0\\
1&0&0&0&1&0&1&0&0&0&1&0\\
0&1&0&0&1&1&0&0&0&0&0&1\\
0&0&1&1&0&0&1&0&0&0&0&1\\
0&1&0&1&0&0&0&1&0&0&1&0\\
0&0&1&0&1&0&0&1&0&1&0&0\\
0&0&0&0&0&1&1&1&1&0&0&0
\end{array}$$

FIGURE 1

$$\begin{array}{ccccccccc|c|c|c}
0&0&1&1&1&0&0&0&1&0&0&0\\
0&0&0&0&0&1&1&1&1&0&0&0\\
1&0&1&0&0&0&0&1&0&1&0&0\\
1&0&0&0&1&0&1&0&0&0&1&0\\
1&0&0&1&0&1&0&0&0&0&0&1\\
0&1&0&0&1&1&0&0&0&1&0&0\\
0&1&0&1&0&0&0&1&0&0&1&0\\
0&1&1&0&0&0&1&0&0&0&0&1\\
1&1&0&0&0&0&0&0&1&0&0&0
\end{array}$$

FIGURE 2

$$\begin{array}{cccccccccc|c}
1&0&0&0&0&0&0&1&1&1&0\\
0&1&0&0&0&1&1&0&0&1&0\\
0&0&1&0&1&0&1&0&1&0&0\\
0&0&0&1&1&1&0&1&0&0&0\\
0&0&1&1&0&0&0&0&0&1&1\\
0&1&0&1&0&0&0&0&1&0&0\\
0&1&1&0&0&0&0&1&0&0&0\\
1&0&0&1&0&0&1&0&0&0&0\\
1&0&1&0&0&1&0&0&0&0&0\\
1&1&0&0&1&0&0&0&0&0&1
\end{array}$$

FIGURE 3

Figure 2 and C' with $c = 3$ show $k(9) = 30$, $k(9,10) = 32$ and $k(9,11) = 34$.

Figure 3 and C' with $c = 3$ show $k(10) = 35$ and $k(10,11) = 37$.

$$\begin{array}{cccccccccccc}
1&0&0&1&1&1&0&0&0&0&0&0\\
1&0&0&0&0&0&1&1&1&0&0&0\\
1&0&0&0&0&0&0&0&0&1&1&1\\
0&1&0&1&0&0&1&0&0&1&0&0\\
0&1&0&0&1&0&0&1&0&0&1&0\\
0&1&0&0&0&1&0&0&1&0&0&1\\
0&0&1&1&0&0&0&1&0&0&0&1\\
0&0&1&0&1&0&0&0&1&1&0&0\\
0&0&1&0&0&1&1&0&0&0&1&0\\
1&1&1&0&0&0&0&0&0&0&0&0
\end{array}$$

FIGURE 4

$$\begin{array}{ccccccccccccc|c|c|c}
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0\\
0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0\\
0&1&0&0&0&1&0&0&1&0&0&1&0&0&1&0\\
0&0&1&0&1&0&0&0&1&0&0&0&1&0&0&1\\
0&0&1&0&0&0&1&0&0&1&0&1&0&1&0&0\\
0&0&0&1&0&0&1&1&0&0&0&0&1&0&1&0\\
0&0&0&1&0&1&0&0&0&1&1&0&0&0&0&1
\end{array}$$

FIGURE 5

Figure 4 and B show $k(10,12) = 40$. F with $a = 2$ shows that $k(10,13) = 41$ since $4^2 3^{11}$ is colmax, while the 2 quadrangles occupy at most 8 of the

10 vertices, so an unoccupied edge would occur. From figure 5, $k(10,14) = 43$ since $4\cdot3^{13}$ is colmax while unoccupied edges occur. From the same figure, $k(10,15) = 45$ and $k(10,16) = 47$ by C' with $c = 4$. F with $a = 2$ and C' with $c = 4$ show $k(10,17) = 48$. See also theorem 6, (f) and (g).

To show $k(11) = 40$, add a row 00000100101 to figure 3 and note that $4^7 3^4$ and $5\cdot4^5 3^5$ are impossible by E, since $k(11,7) = 28$ and $k(11,6) = 25$, and the latter would be colmax.

$$\begin{array}{cccccccccccc|c}
1&0&0&1&0&0&1&1&0&0&0&0&0\\
1&0&0&0&1&0&0&0&1&1&0&0&0\\
1&0&0&0&0&1&0&0&0&0&1&1&0\\
0&1&0&1&0&0&0&0&1&0&1&0&0\\
0&1&0&0&1&0&1&0&0&0&0&1&0\\
0&1&0&0&0&1&0&1&0&1&0&0&0\\
0&0&1&1&0&0&0&0&0&1&0&1&0\\
0&0&1&0&1&0&0&1&0&0&1&0&0\\
0&0&1&0&0&1&1&0&1&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&0&1\\
0&0&0&1&1&1&0&0&0&0&0&0&1
\end{array}$$

FIGURE 6

$$\begin{array}{cccccccccccccc|c}
1&0&0&0&0&1&1&0&0&0&0&0&1&1&1\\
0&1&0&0&0&0&0&1&1&1&0&0&1&0&0\\
0&0&1&0&0&0&0&1&0&0&1&1&0&1&0\\
0&0&0&1&1&0&0&1&0&0&0&0&0&0&1\\
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&1&0&0&0&0&1&0&1&0&0&0&0\\
1&0&0&0&1&0&0&0&0&1&0&1&0&0&0\\
0&1&0&1&0&0&1&0&0&0&0&1&0&0&0\\
0&1&0&0&1&1&0&0&0&0&1&0&0&0&0\\
0&0&1&1&0&1&0&0&0&1&0&0&0&0&0\\
0&0&1&0&1&0&1&0&1&0&0&0&0&0&0
\end{array}$$

FIGURE 7

Figure 6 and A show that $k(11,12) = 43$ and $k(11,13) = 45$. Figure 7 shows $k(11,14) = 46$ since $4^4 3^{10}$ contains at least 4 vertices where just one quadrangle occurs, implying 2 or more unoccupied edges, while $4\binom{4}{2} + 10\binom{3}{2} > \binom{11}{2} - 2$, and $5\cdot4^2 3^{11}$ is colmax but contains unoccupied edges, while $4^5 3^8 2$ is also colmax, so the 5 quadrangles occur exactly 2 at each of 10 vertices, and all triangles contain the eleventh vertex, so they cannot number more than 5, by D. Similarly $k(11,15) = 48$, since if $4^3 3^{12}$, three quadrangles leave at least 3 edges unoccupied while $3\binom{4}{2} + 12\binom{3}{2} > \binom{11}{2} - 3$; $5\cdot4\cdot3^{13}$ is colmax while there are unoccupied edges, and $4^4 3^{10} 2$ contains a grid by E, since $k(11,14) = 46$.

Figure 8 and A show $k(11,16) = 51$, and on replacing column 1 by 2 3-cols and up to 4 2-cols, that $k(11,17) = 52$ and $k(11,18) = 54$ by C' with $c = 4$;

$$\begin{array}{cccccccccccccccc}
0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&1&1&1&1&0&0&0&0&0&0\\
0&0&1&0&0&0&1&0&0&0&1&1&1&0&0&0\\
0&0&0&1&0&0&0&1&0&0&1&0&0&1&1&0\\
0&0&0&0&1&0&0&0&1&0&0&1&0&1&0&1\\
0&0&0&0&0&1&0&0&0&1&0&0&1&0&1&1\\
1&1&0&0&0&0&0&0&0&0&1&0&0&0&0&1\\
1&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0\\
1&0&0&1&0&0&0&0&0&1&0&1&0&0&0&0\\
1&0&0&0&1&0&0&1&0&0&0&0&1&0&0&0\\
1&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0
\end{array}$$

FIGURE 8

$$\begin{array}{cccccccccccc|c}
1&1&1&0&0&0&0&0&0&1&0&0&0\\
1&0&0&1&1&0&0&0&0&0&1&0&0\\
1&0&0&0&0&1&1&0&0&0&0&1&0\\
0&1&0&1&0&0&0&1&0&0&0&1&0\\
0&1&0&0&0&1&0&0&1&0&1&0&0\\
0&0&1&0&1&0&0&0&1&0&0&1&0\\
0&0&1&0&0&0&1&1&0&0&1&0&0\\
0&0&0&1&0&0&1&0&1&1&0&0&0\\
0&0&0&0&1&1&0&1&0&1&0&0&0\\
1&0&0&0&0&0&0&1&1&0&0&0&1\\
0&1&0&0&1&0&1&0&0&0&0&0&1\\
0&0&1&1&0&1&0&0&0&0&0&0&1
\end{array}$$

FIGURE 9

that $k(11,19) = 56$ by C with $c = 3$; and that $k(11,20) = 58$ and $k(11,21) = 60$ by C' with $c = 5$. See also theorem 6(h) with $s = 1$.

Figure 9 shows $k(12) = 46$ since $4^{10}3^2$ is colmax and $u_{12} < 10$, and $k(12,13) = 49$ by C with $c = 3$. By F with $a = 2$, $k(12,14) = 50$, since $4^8 3^6$ is colmax, so there are 3 quadrangles and 1 triangle at a vertex, or 1 quadrangle and 4 triangles, say v_1 of the first and v_4 of the second. $v_1 + v_4 = 12$, $3v_1 + v_4 = 32$, $v_1 + 4v_4 = 18$ so $v_1 = 10$, $v_4 = 2$. These last 2 vertices contribute 7 or 8 triangles according as their join belongs to a triangle or not; but there are only 6 triangles. From figure 10 and a

$$\begin{array}{ccccccccccccccc|c|c|c}
1&0&0&0&0&0&1&0&0&0&1&1&1&0&0&0&0&0\\
0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0\\
0&0&1&0&0&0&0&0&1&1&1&0&0&0&1&0&0&0\\
1&1&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
1&0&1&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&1&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&1&0&0&0&0&0&0&0&1&1&1&0&0\\
0&1&0&0&1&0&0&0&0&0&0&1&0&1&0&0&1&0\\
0&0&1&1&0&0&0&0&0&0&0&1&1&0&0&0&0&1\\
0&0&0&1&1&0&1&0&0&1&0&0&0&0&0&1&0&0\\
0&0&0&1&0&1&0&1&0&0&1&0&0&0&0&0&1&0\\
0&0&0&0&1&1&0&0&1&0&0&1&0&0&0&0&0&1
\end{array}$$

FIGURE 10

$$\begin{array}{ccccccccccccccccccc}
1&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
0&1&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0\\
0&1&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0\\
0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0\\
0&1&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1\\
0&0&1&1&0&0&0&0&0&0&1&0&0&1&0&0&1&0&0\\
0&0&1&0&1&0&0&1&0&0&0&0&0&0&1&0&0&1&0\\
0&0&1&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&1\\
0&0&1&0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0
\end{array}$$

FIGURE 11

similar argument, $k(12,15) = 52$, since $v_1 + v_4 = 12$, $3v_1 + v_4 = 28$, $v_1 + 4v_4 = 24$ so $v_1 = 8$, $v_4 = 4$ and we cannot have 8 triangles, 4 at each of 4 vertices. $k(12,16) = 54$ since $4^6 3^{10}$ is colmax, giving $v_1 = v_4 = 6$, six vertices with only one triangle at each, so at least 3 edges from each of the 6 quadrangles join these vertices, while $k(6) < 18$. By C' with $c = 4$, $k(12,17) = 56$ and $k(12,18) = 58$. Figure 11 and C' with $c = 5$ gives $k(12,19) = 61$. If we replace each of columns 1 and 2 by a 3-col and 3 2-cols, we have $k(12,20) = 62$, $k(12,21) = 64$, $k(12,22) = 66$ and $k(12,23) = 67$, $k(12,24) = 69$, $k(12,25) = 71$, since $4^2 3^{18}$ is colmax, but at least 4 vertices are not occupied by quadrangles, so at least 2 edges are not occupied; $4 \cdot 3^{20}$ is colmax also; 66 by C' with $c = 5$; $3^{21} 2^2$ but $t_{12} < 21$ and $4 \cdot 3^{19} 2^3$ is colmax but contains unoccupied edges; 69 by C' with $c = 5$ and 71 by theorem 2.

$$
\begin{array}{ccccccccccccc}
1&0&0&1&0&1&0&0&0&0&0&0&1\\
0&0&1&0&1&0&0&0&0&0&0&1&1\\
0&1&0&1&0&0&0&0&0&0&1&1&0\\
1&0&1&0&0&0&0&0&0&1&1&0&0\\
0&1&0&0&0&0&0&0&1&1&0&0&1\\
1&0&0&0&0&0&0&1&1&0&0&1&0\\
0&0&0&0&0&0&1&1&0&0&1&0&1\\
0&0&0&0&0&1&1&0&0&1&0&1&0\\
0&0&0&0&1&1&0&0&1&0&1&0&0\\
0&0&0&1&1&0&0&1&0&1&0&0&0\\
0&0&1&1&0&0&1&0&1&0&0&0&0\\
0&1&1&0&0&1&0&1&0&0&0&0&0\\
1&1&0&0&1&0&1&0&0&0&0&0&0
\end{array}
$$

FIGURE 12

$$
\begin{array}{ccccccccccccccccc|c}
1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0\\
0&0&0&1&0&1&0&1&0&1&0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&1&0&1&0&1&0&0&1&0&0&0&0\\
0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0\\
0&1&0&1&0&0&0&0&0&0&1&0&1&0&1&0&0&0\\
0&1&0&0&0&0&1&0&0&1&0&1&0&0&0&1&0&0\\
0&0&1&0&1&0&0&1&0&0&0&0&1&0&0&1&0&0\\
1&1&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&1\\
0&0&1&1&0&0&1&0&0&0&0&0&0&0&0&0&1&1\\
1&0&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0\\
1&0&0&1&1&0&0&0&0&0&0&1&0&0&0&0&0&0\\
1&0&0&0&0&1&1&0&0&0&0&0&1&0&0&0&0&0\\
0&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&1&0
\end{array}
$$

FIGURE 13

For $m = 13$, figure 12, which is saturated and colmax, shows that $k(13) = 53$. If we replace a 4-col by a 3-col and 3 2-cols, it follows by C' with $c = 4$, that $k(13,14) = 54$, $k(13,15) = 56$ and $k(13,16) = 58$. Figure 14, and C' with $c = 4$, show $k(13,17) = 60$, $k(13,18) = 62$.

Figures 14 and 15 are colmax so $k(13, 19) = 65$ and $k(13,20) = 67$ by A.

$$\begin{matrix}
1&0&0&1&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0\\
1&0&0&0&1&0&0&0&0&0&1&1&1&0&0&0&0&0&0\\
1&0&0&0&0&1&0&0&0&0&0&0&0&1&1&1&0&0&0\\
1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&1\\
0&1&0&1&0&0&0&0&0&0&1&0&0&1&0&0&1&0&0\\
0&1&0&0&1&0&0&1&0&0&0&0&0&0&1&0&0&0&1\\
0&1&0&0&0&1&0&0&1&0&0&0&1&0&0&0&0&1&0\\
0&1&0&0&0&0&1&0&0&1&0&1&0&0&0&1&0&0&0\\
0&0&1&1&0&0&0&0&0&0&0&1&0&0&1&0&0&1&0\\
0&0&1&0&1&0&0&0&1&0&0&0&0&0&0&1&1&0&0\\
0&0&1&0&0&1&0&0&0&1&1&0&0&0&0&0&0&0&1\\
0&0&1&0&0&0&1&1&0&0&0&0&1&1&0&0&0&0&0\\
0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0
\end{matrix}$$

FIGURE 14

$$\begin{matrix}
0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&1&0&0&0&0&0&1&1&0&0&0&0&0&0\\
1&0&1&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&0&0\\
1&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&1&1&0&0\\
1&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&1&1\\
0&1&0&1&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0\\
0&1&0&0&1&0&0&0&0&0&0&1&0&0&0&1&1&0&0&0\\
0&1&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1\\
0&0&1&1&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1\\
0&0&1&0&1&0&0&0&0&0&1&0&0&1&0&0&0&1&0&0\\
0&0&1&0&0&1&1&0&0&0&0&0&0&0&0&0&1&0&1&0\\
0&0&0&1&0&1&0&0&0&1&0&0&0&1&0&1&0&0&0&0\\
0&0&0&0&1&1&0&0&1&0&0&0&1&0&1&0&0&0&0&0
\end{matrix}$$

FIGURE 15

By F with $a = 1$, $k(13,21) = 68$, since $4^5 3^{16}$ is colmax; if there were a 4-row, it could be deleted, and $k(12,21) = 64$ while D shows there is no 7-row, so the rows are $6^3 5^{10}$. The 6-rows occupy at least $3 \times 6 - \binom{3}{2} = 15$ columns, so the other 10 rows and 6 columns contain $68 - (15 \times 3) = 23 = k(10,6)$, so there is a grid. In figure 15, replace column 1 by a 3-col and 3 2-cols. $70 = 4^4 3^{18}$ is colmax, and as in the previous argument, the rows are $6^5 5^8$. The 5 6-rows occupy at least $5 \times 6 - \binom{5}{2} = 20$ columns, and as a 6-row intersects only 3-cols, there is no room for 4 4-cols, and $k(13,22) = 70$. By C' with $c = 5$, $k(13,23) = 72$. Figure 16 and C' with $c = 5$ show that $k(13,24) = 74$ and $k(13,25) = 76$.

For $m = 14$, figure 17 shows $k(14) = 57$ since 57 contains a 5-col, and the other 9 rows and 13 columns then contain at least $57 - (5+13) > k(9,13)$. Similarly $k(14,15) = 59$, a 5-col being excluded as before, while $4^{14}3$ is not possible, since $u_{14} = 14$ and 14 quadrangles can be accommodated d only with 7 disjoint unused edges, not with a triangle. To see that $k(14,16) = 61$, note that a 5-col is again excluded, while a 2-col would

0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	1	0	0	1	0	0	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	1	0	0	1	0	1	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	1	0	0	1	0	1	1	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0
1	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1	0
1	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1
1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1	1	0	0
0	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0
0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	1	0	1	0
0	1	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	1

FIGURE 16

0	0	0	0	0	0	0	1	1	0	0	1	0	1	1	0
0	0	0	0	0	0	1	1	0	0	1	0	1	0	0	1
0	0	0	0	0	1	1	0	0	1	0	1	0	0	0	0
0	0	0	0	1	1	0	0	1	0	1	0	0	0	0	0
0	0	0	1	1	0	0	1	0	1	0	0	0	0	0	0
0	0	1	1	0	0	1	0	1	0	0	0	0	0	0	0
0	1	1	0	0	1	0	1	0	0	0	0	0	0	0	0
1	1	0	0	1	0	1	0	0	0	0	0	0	0	1	0
1	0	0	1	0	1	0	0	0	0	0	0	0	1	0	1
0	0	1	0	1	0	0	0	0	0	0	0	1	1	0	0
0	1	0	1	0	0	0	0	0	0	0	1	1	0	0	0
1	0	1	0	0	0	0	0	0	0	1	1	0	0	0	0
0	1	0	0	0	0	0	0	0	1	1	0	0	1	0	0
1	0	0	0	0	0	0	0	1	1	0	0	1	0	0	0

FIGURE 17

imply a grid, since $k(14,15) = 59$; so $61 = 4^{13}3^{3}$. A 3-row implies a grid, since $k(13,16) = 58$, and a 6-row would intersect at least 3 4-cols, forming at least 15 pairs, only 13 of which can be distinct, so the rows are $5^{5}4^{9}$. If a 5-row intersects 4 4-cols, it forms $12 + 2 > 13$ pairs, so each 5-row intersects at least 2 3-cols, and there are $\geq 5 \times 2/3$ 3-cols, which is false.

1	1	1	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
0	1	0	1	1	0	0	1	0	0	0	0	1	0	0	0	0	0
0	1	0	0	0	1	1	0	1	0	0	0	0	1	0	0	0	0
0	0	0	1	0	0	0	0	0	1	1	0	0	1	1	0	0	0
0	0	0	0	1	0	0	0	1	0	0	1	0	0	1	1	0	0
0	0	0	0	0	1	0	0	0	1	0	1	1	0	0	0	1	0
0	0	0	0	0	0	1	1	0	0	1	0	0	0	0	1	1	0
1	0	0	1	0	1	0	0	0	0	0	0	0	0	0	1	0	1
0	0	1	0	1	0	1	0	0	1	0	0	0	0	0	0	0	1
1	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1	0
1	0	0	0	0	0	1	0	0	0	0	0	1	0	1	0	0	0
1	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0	1	0	0	0	0	0	0	0	1	0
0	0	1	0	0	1	0	1	0	0	0	0	0	0	1	0	0	0

FIGURE 18

1	0	0	0	1	0	0	0	0	0	1	1	1	0	0	0	0	0	0
1	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	0	0	0
1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1
0	1	0	0	1	0	0	1	0	0	0	0	0	1	0	0	1	0	0
0	1	0	0	0	1	0	0	1	0	1	0	0	0	0	0	0	1	0
0	1	0	0	0	0	1	0	0	1	0	1	0	0	1	0	0	0	0
0	0	1	0	0	0	1	1	0	0	1	0	0	0	0	1	0	0	0
0	0	1	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	1
0	0	1	0	0	1	0	0	0	1	0	0	1	0	0	0	1	0	0
0	0	0	1	0	1	0	1	0	0	0	1	0	0	0	0	0	0	1
0	0	0	1	0	0	1	0	1	0	0	0	1	1	0	0	0	0	0
0	0	0	1	1	0	0	0	0	1	0	0	0	0	0	1	0	1	0
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0

FIGURE 19

Figure 18 and C' with $c = 4$ show that $k(14,17) = 64$ and $k(14,18) = 66$, and similarly figure 19 shows that $k(14,19) = 69$. Figure 20 and A show

0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1
0	0	0	0	0	0	1	1	0	1	0	0	1	0	0	0	0	0	0	1	0
0	0	0	0	0	1	0	0	1	1	0	0	0	1	0	0	0	0	1	0	0
0	0	0	0	1	0	1	0	1	0	0	1	0	0	0	0	0	1	0	0	0
0	0	0	1	0	1	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0
0	0	1	0	1	0	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
0	1	0	1	0	0	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0
1	0	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	1	0	0	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0
1	0	0	1	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0
0	0	1	1	0	0	0	0	0	1	1	0	0	0	0	0	0	1	0	0	0
0	1	1	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0
1	1	0	0	0	0	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0
1	0	0	0	0	0	0	1	1	0	1	0	0	0	1	0	0	0	0	0	0

FIGURE 20

1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0
0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	1	1	0	0	0	0
0	1	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	1	1	0	0
0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	1	0	1	0	0	0
0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0	1	0	0
0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	1	0
0	0	0	1	0	0	0	0	0	0	1	0	1	0	0	1	0	0	0	0	1
0	0	0	0	1	0	0	0	0	1	0	0	1	0	0	0	1	0	0	1	0
0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	1
0	0	0	0	0	1	0	0	1	0	0	0	0	0	1	1	0	0	0	1	0
0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	0	0	0	1	0	1
0	0	0	0	0	0	1	0	0	0	1	1	0	0	0	0	0	1	0	1	0
0	0	0	0	0	0	1	0	1	0	0	0	0	1	0	0	1	0	0	0	1

FIGURE 21

that $k(14,20) = 71$ and $k(14,21) = 73$. It is probable that $k(14,n) = 2n + 30$ $(22 \leqslant n \leqslant 30)$, but proofs become diffuse and drift into the realm of strong

1	0	0	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0
0	1	1	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0
1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0
0	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0
0	0	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	0	0	0
0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	1	0	0
0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	1	0
0	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	1
1	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1

FIGURE 22

conjecture. Figure 22 shows that the last five values are at least so large, and proofs are reliable in the last 3 cases. That $k(14,n) = 2n + 29$ $(31 \leqslant n \leqslant 35)$ follows from theorems 6(c) and 2.

For $k(16,20) = 81$ and $k(21) = 106$, see [16]. On deleting a row from the 16 by 20 matrix it remains colmax and $k(15,20) = 76$. Probably

$k(15,n) = 3n + 16$ $(15 \leq n \leq 19)$, the last value being correct since $4^{16}3^2$ is colmax and $u_{15} = 15$. Figure 21 is colmax, so $k(15,21) = 78$.

6. Special values for $a = 2$, $b = 3$.

In this section $k(m,n)$ means $k_{2,3}(m,n)$ and grid means 2,3-grid. The values of $k(m,n)$ outside the indicated line in table 3 are given by (10) and by theorems 5 and 7, using (19) if necessary.

$k_{2,3}(m,n)$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
2	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
3	8	10	11	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
4	10	13	14	16	17	19	20	22	23	25	26	27	28	29	30	31	32	33	34	35	36	37
5	12	15	17	19	21	23	24	26	27	29	30	32	33	35	36	38	39	41	42	43	44	45
6	14	17	20	22	25	26	28	31	32	34	35	37	38	40	41	43	44	46	47	49	50	52
7	16	19	23	25	29	30	32	35	37	38	40	43	44	46	47	49	50	52	53	55	56	58
8	18	21	25	28	32	34	36	39	41	43	45	47	49	51	53	55	57	59	60	62	63	65
9	20	23	28	31	35	37	40	43	46	47	50	52	55	57	58	60	62	65	67	69	70	73
10	22	25	31	34	38	41	43	47	51	52	55	57	61	62	64	67	68	71	73			79
11	24	27	33	37	41	45	47	51	56	57												
12	26	29	35	39	44	49	51	54														
13	28	31	37	42	47	53	55	57														
14	30	33	39	45	50	57	59															
15	32	35	41	47	52	59	62															
16	34	37	43	50	55	62	65															
17	36	39	45	53	58	65	69															
18	38	41	47	55	60	67	73															
19	40	43	49	58	63	70	76															
20	42	45	51	61	66	73	79															
21	44	47	53	63	68	75	82															

TABLE 3

```
1 1 1 1 0 0
1 1 0 0 1 1
0 0 1 1 1 1
1 0 1 0 1 0
0 1 0 1 0 1
1 0 1 0 0 1
0 1 0 1 1 0
1 0 0 1 0 1
```

FIGURE 23

```
1 1 1 1 0 0 0
1 1 0 0 1 1 0
1 0 1 0 1 0 1
1 0 0 1 0 1 1
0 1 1 0 0 1 1
0 1 0 1 1 0 1
0 0 1 1 1 1 0
1 1 0 0 0 0 1
```

FIGURE 24

```
1 1 0 1 0 0 1 0
1 0 1 0 0 1 1 0
0 1 0 0 1 1 1 0
1 0 0 1 1 1 0 0
0 0 1 1 1 0 1 0
0 1 1 1 0 1 0 0
1 1 1 0 1 0 0 0
0 0 1 0 1 1 0 1
0 1 0 1 1 0 0 1
1 0 1 1 0 0 0 1
0 1 1 0 0 0 1 1
1 1 0 0 0 1 0 1
1 0 0 0 1 0 1 1
0 0 0 1 0 1 1 1
```

FIGURE 25

```
1 0 0 1 1 1 1 0 0
0 1 0 1 1 0 0 1 1
0 0 1 0 0 1 1 1 1
1 1 0 0 0 1 0 1 0
1 1 0 0 0 0 1 0 1
1 0 1 1 0 0 0 1 0
1 0 1 0 1 0 0 0 1
0 1 1 1 0 1 0 0 0
0 1 1 0 1 0 1 0 0
```

FIGURE 26

```
1 1 1 1 0 0 1 0 0
1 1 0 0 1 1 0 1 0
0 0 1 1 1 1 0 0 1
```

FIGURE 27

```
1 1 1 1 0 0 0 0 1
0 0 0 0 1 1 1 1 1
1 1 0 0 1 1 0 0 0
1 1 0 0 0 0 1 1 0
1 0 1 0 1 0 1 0 0
1 0 1 0 0 1 0 1 0
1 0 0 1 1 0 0 1 0
1 0 0 1 0 1 1 0 0
0 1 1 0 0 1 1 0 0
0 1 1 0 1 0 0 1 0
0 1 0 1 0 1 0 1 0
0 1 0 1 1 0 1 0 0
0 0 1 1 0 0 1 1 0
0 0 1 1 1 1 0 0 0
1 0 0 0 1 0 0 0 1
0 1 0 0 0 1 0 0 1
```

FIGURE 28

$n = 6$. Figure 23 and C' with $c = 3$ show $k(5,6) = 19$. Similarly $k(6,6) = 22$, $k(7,6) = 25$ and $k(8,6) = 28$.

$n = 7$. Figure 24 and C' with $c = 4$ show $k(6,7) = 25$ and $k(7) = 29$. Two 5-rows form a grid, as do a 5-row and more than 2 4-rows, and since $T_m = 7$, with 7 unused faces (triads), $k(m,7) = 3m + 8$ $(7 \leq m \leq 14)$.

$n = 8$. $k(6,8) = 26$ by F with $a = 2$, since $4^2 3^6$ is colmax while there are 2 rows which contain at least 10. A 5-row intersects only 3-cols, and there can be no longer rows. There are insufficient 3-cols to intersect 2 5-rows, so we arrive at a contradiction. $k(7,8) = 30$ by C' with $c = 4$, and F with $a = 2$ applied to figure 24. Add a 2-col to this figure with ones in rows 7 and 8, and $k(8) = 34$ by C' with $c = 4$. Figure 25 shows $k(m,8) \geq 4m + 1$ $(9 \leq m \leq 14)$. Equality follows inductively by C' with $c = 4$ if we first show $k(9,8) = 37$. We may assume there is no 3-row by E. If there were a 6-row, other rows could contain at most 2 in these 6 columns, leaving at least $37 - 6 - (2 \times 8)$ in the other

2 columns, so that the most level partition of columns is $8\cdot 7\cdot 4^4 3^2$, while $\binom{8}{2}+\binom{7}{2}+4\binom{4}{2}+3\binom{3}{2} > 2\binom{9}{2}$, so the rows are $5\cdot 4^8$. As in the last sentence, the rows other than the 5-row contain at least $37-5-(2\times 8) = 16$ in the 3 columns not occupied by the 5-row. In only one row can all 3 columns be occupied, so there are at least 3 coincident pairs, i.e. 2 of the 3 columns which are both occupied in at least 3 rows. These 3 contain 6 ones in the other 5 columns, so 2 lie in the same column and form a grid.

$n = 9$. $k(m,9) = 4m + 4$ $(m \leq 9)$ by (10), theorem 7, figure 26 and C' with $c = 4$. Similarly, since we may add a 2-col to figure 25, $k(m,9) = 4m + 3$ $(10 \leq m \leq 14)$ if we first prove $k(10,9) = 43$. By E we may assume there is no 3-row, and we may dispose of the possibility of a 7-row as in the last paragraph. Two 6-rows would form a grid, and if the rows were $6\cdot 5\cdot 4^8$, then the 6-row and 5-row, either overlap in 2-cols, so that the other 7 columns contain a grid, or any row which has 1 in a column in which the 6-row and 5-row overlap forms a grid with one of them. So the rows are $5^3 4^7$ and each pair of 5-rows overlaps in just 2 columns, so they are essentially as shown in figure 27. The only 4-rows now available are those lying in columns $1i89$, $2i89$, $1j79$, $2j79$, $k789$, $i578$, or $i678$, where $i = 3,4$; $i = 5,6$ and $k = 1,2,3,4,5$ or 6. At most one can be selected from each of these 7 sets, yet $k789$ $(1 \leq k \leq 4)$ excludes one of the first 2 sets and 5789, 6789 exclude one of the last two.

We next show $k(15,9) = 62$. It is at least this by figure 28, while 62 does not contain a 3-row by E. If there is a 6-row, then as before, the other 3 columns contain at least $62-6-(2\times 14) = 28$, and 2 of them at least 19, so they overlap in at least 5 rows. These 5 rows contain more than 6 other ones, so two occur in the same column and form a grid. Hence the rows are $5^2 4^{13}$. The 2 5-rows overlap in 2 columns as before, say they occupy columns 12345 and 12678. Columns 1 and 2 are not both occupied in any other row. If column 1 is a 6-column or longer, then there are at least 4 rows in which columns 1 and 8 are both occupied, together with a 1 in each of the sets of columns 3,4,5 and 6,7,8. This implies grids, so columns 1 and 2 are at most 5-cols. There are at least 7 rows and 7 columns not occupied by these, and since $T_7 = 7$, there are just 7 such rows. The 7 tetrahedra occupy all but 7 faces of K_7^3 and no 2 of these are disjoint. But the faces corresponding to the parts of the 5-rows in columns 345 and 678 are disjoint, and we have a contradiction.

Figure 28 also shows $k(16,9) = 65$, since again there are no 3-rows or 6-rows, so the rows are $5 \cdot 4^{15}$. The 4 columns not occupied by the 5-row contain at least 15 (horizontal) pairs, only 6 of which are distinct, so there are at least 2 sets of 3 rows with pairs in the same 2 columns, and since one of these contains 6 in the columns of the 5-row, there are 2 in some column, forming a grid. It now follows by C' with $c = 4$ that $k(17,9) = 69$ and $k(18,9) = 73$ since $T_9 = 18$. Similarly, with $c = 3$, $k(m,9) = 3m + 19$ $(18 \leq m \leq 30)$.

$n = 10$. Delete from figure 42 a column and $11 - m$ of the 5 resulting 4-rows. Then C' with $c = 4$ shows that $k(m,10) = 4m + 7$ $(6 \leq m \leq 11)$, since it is true for $m = 6$ by theorem 5. Since $k(6,10) = 31$, no 6 rows may contain more than 30; so $k(12,10) = 54$, since the rows are $5^6 4^6$ and the 5-rows are essentially as in figure 29 which is colmax. The only 4-rows which can now be added are those in columns 148X, 1579, 237X, 2469 and 3568; which are not 6 in number. On the other hand, these 5 and a 3-row can be added. A further 3-row can be added to show $k(13,10) = 57$, since by E, 57 does not contain a 3-row, and if the rows are $5^5 4^8$, the

```
1 1 1 1 1 0 0 0 0 0
1 1 0 0 0 1 1 1 0 0
1 0 1 0 0 1 0 0 1 1
0 0 1 1 0 0 1 1 1 0
0 0 0 1 1 1 1 0 0 1
0 1 0 0 1 0 0 1 1 1
```

FIGURE 29

```
0 1 1 1 0 0 1 0 1 1
1 1 0 0 1 1 1 0 0 0
1 0 1 0 1 0 0 1 1 0
1 0 0 1 0 1 0 1 0 1
```

FIGURE 30

```
0 0 0 0 0 1 1 1 1 1 1
1 1 0 1 0 1 0 1 0 0 0
1 0 1 0 1 1 0 0 1 0 0
0 1 1 0 1 0 1 1 0 0 0
0 1 1 1 0 0 0 0 1 1 0|1|
1 0 1 1 0 0 1 0 0 0 1|0|
1 1 0 0 1 0 0 0 0 1 1|0|1
---------------------
0 0 0 1 1 1 1 0 0 0 0|1|1
```

FIGURE 31

```
1 1 0 0 0 0 1 1 1 1 0 0
0 0 1 1 0 0 1 1 0 0 1 1
1 0 1 0 1 1 1 0 0 0 0 0
0 1 0 1 1 1 0 1 0 0 0 0
1 0 0 1 1 0 0 0 1 0 1 0|1
0 1 1 0 1 0 0 0 0 1 0 1|1
0 1 1 0 0 1 0 0 1 0 1 0|0|1
1 0 0 1 0 1 0 0 0 1 0 1|0|1
```

FIGURE 32

5-rows are essentially as all but one of figure 29 and deletion of a row allows the addition of at most one 4-row, while if the rows are $6\cdot5^34^9$, the first 4 are essentially as in figure 30, and again mot more than 6 4-rows can be added. From $T_{10} = 30$, we may deduce $k(30,10) = 121$ and $k(29,10) = 117$, by A'.

$m = 7$. By A and figure 31, $k(7,11) = 37$. To see that $k(7,12) = 38$, note that 4^23^{10} is colmax. By E we may assume there is no 4-row, and by D, no 7-row, so the rows are 6^35^4. A 6-row intersects only 3-cols, by D, so they are confined to the 10 3-cols; however $k(3,10) < 18$. By C', with $c = 5$, $k(7,13) = 40$, since $t_{2,7} > 13$.

```
0 0 0 1 1 1 1 1 1 0 0 0 0 0 0|1|1
0 0 0 0 0 0 0 0 0 1 1 1 1 1 1|1|1
1 1 0 1 0 0 1 0 0 1 1 0 0 0 0|
1 1 0 0 1 0 0 1 0 0 0 0 1 1 0
1 0 1 0 1 0 0 0 1 1 0 1 0 0 0
1 0 1 0 0 1 1 0 0 0 0 0 1 0 1
0 1 1 0 0 1 0 1 0 0 1 1 0 0 0
0 1 1 1 0 0 0 0 1 0 0 0 0 1 1
```

FIGURE 33

```
1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0
1 1 0 0 0 0 0 1 1 1 1 1 0 0 0 0 0 0
0 0 1 1 0 0 0 1 1 0 0 0 1 1 1 0 0 0
0 0 0 0 1 1 0 0 0 1 1 0 1 1 0 1 0 0
0 0 0 1 1 0 0 0 0 1 0 1 0 0 1 0 1 1
0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 1 1 1|
1 0 1 0 0 0 0 0 0 0 1 1 0 0 1 1 0|1|1
0 1 0 0 0 0 1 0 0 0 1 0 0 1 1 0 0 1|1|1
```

FIGURE 34

$m = 8$. From figures 31 to 34, to each of which 2 2-cols may be added, it follows by A that $k(8,n) = 2n + 19$ $(10 \leq n \leq 20)$.

$m = 9$. From figure 42 with 2 rows deleted, $k(9,11) = 46$, by A. We next show $k(9,12) = 47$. By E we may assume no 4-row, and a 7-row implies the other 5 columns contain at least $47 - 7 - (2 \times 8) = 24$, and the most level column partition would be $5^4 4^3 3^5$, which fails by A. The rows are thus $6^2 5^7$. If the 6-rows did not overlap, the 5-rows would form grids with one of them. If they overlap in only one column, no other row intersects it, but a 2-col implies a grid elsewhere by E. Suppose the 6-rows occupy columns 123456 and 12789X. Columns 1 and 2 contain at most 1 more in each, else a grid is formed with columns 11 and 12 or with a 6-row. So 5 rows contain 1 in column 11 or 12, say at least 3 in column 12. These 3 rows contain 12 other points in columns 3 to 10. Two of these points lie in each of at least 4 columns, so 2 pairs coincide and form a grid with column 12. Figure 35, and C' with

1	1	1	1	1	1	0	0	0	0	0	0	0	
1	1	0	0	0	0	1	1	1	1	0	0	0	
0	0	1	1	0	0	1	1	0	0	1	1	0	
0	0	0	0	1	1	0	0	1	1	1	1	0	
1	0	1	0	0	0	0	0	1	0	1	0	1	
1	0	0	0	1	0	0	1	0	0	0	1	1	
0	1	0	0	0	1	1	0	0	0	1	0	1	
0	0	1	0	1	0	1	0	0	1	0	0	1	1
0	0	0	1	0	1	0	1	1	0	0	0	1	1

FIGURE 35

1	1	1	1	0	0	0	0	0	1	0	0	1	0	0
1	0	0	0	1	1	1	0	0	1	0	0	0	1	0
0	1	0	0	1	0	0	1	1	1	0	0	0	0	1
0	0	1	0	1	1	0	1	0	0	1	0	1	0	0
0	1	0	1	0	0	1	1	0	0	1	0	0	1	0
1	0	0	1	0	1	0	0	1	0	1	0	0	0	1
0	0	0	1	1	0	1	0	1	0	0	1	1	0	0
0	1	1	0	0	1	0	0	1	0	0	1	0	1	0
1	0	1	0	0	0	1	1	0	0	0	1	0	0	1

FIGURE 36

$c = 5$, show that $k(9,13) = 50$ and $k(9,14) = 52$. Figures 36 and 37 and C' with $c = 6$, show that $k(9,15) = 55$ and $k(9,16) = 57$. It may be shown by arguments too diffuse to give here that $k(9,17) = 58$, $k(9,18) = 60$ and $k(9,19) = 62$ (see figure 38). By A and figures 39, 40 and 41, we

```
1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0
1 1 0 0 0 0 0 0 1 1 1 1 0 0 0 0
1 1 0 0 0 0 0 0 0 0 0 0 1 1 1 1
0 0 1 1 0 0 0 0 1 1 0 0 1 1 0 0
0 0 1 1 0 0 0 0 0 0 1 1 0 0 1 1
0 0 0 0 1 1 0 0 1 0 1 0 1 0 1 0
0 0 0 0 1 1 0 0 0 1 0 1 0 1 0 1
0 0 0 0 0 0 1 1 1 0 0 1 0 1 1 0
0 0 0 0 0 0 1 1 0 1 1 0 1 0 0 1
```

FIGURE 37

```
1 1 0 0 0 1 1 1 1 1 0 0 0 0 0 0 0 0
0 0 1 1 0 1 1 0 0 0 1 1 1 0 0 0 0 0
1 0 0 0 1 1 0 0 0 0 1 0 0 1 1 1 0 0
1 0 0 0 0 0 0 1 0 0 0 1 1 1 0 0 1 1
0 1 1 0 0 0 0 0 1 0 1 0 0 0 1 0 1 1
0 0 0 1 1 0 1 0 0 1 0 0 0 0 0 1 1 1
0 0 1 1 0 0 0 1 0 1 0 0 0 1 1 0 0 0 | 1
0 1 0 0 1 0 0 0 1 0 0 1 1 0 0 1 0 0 | 1
1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 |
```

FIGURE 38

have $k(9,20) = 65$, $k(9,21) = 67$ and $k(9,22) = 69$. Also, $k(9,23) = 70$, since $4\cdot 3^{22}$ is colmax and the single quadrangle requires edges which are not occupied by triangles.

```
1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 1 0 0 0 0 1 0 1 1 0 0 0 0 1 0 0 1 0 0
0 0 1 1 0 0 0 1 1 1 0 0 0 0 0 1 0 0 1 0
1 0 0 1 0 0 1 0 0 0 1 1 0 0 0 0 1 0 1 0
0 1 1 0 0 0 0 1 0 0 1 1 0 0 1 0 0 0 0 1
1 0 1 0 0 0 1 0 0 0 0 0 1 1 0 1 0 0 0 1
0 1 0 1 0 0 0 1 0 0 0 0 1 1 0 0 1 1 0 0
0 0 0 0 1 1 0 0 1 0 1 0 1 0 1 1 1 0 0 0
0 0 0 0 1 1 0 0 0 1 0 1 0 1 0 0 0 1 1 1
```

FIGURE 39

```
1 1 0 1 0 0 0 0 0 1 0 0 0 1 0 0 0 1 0 1 0
1 1 0 0 1 0 0 0 0 0 1 0 0 0 1 0 0 0 1 0 1
1 0 1 0 0 1 0 0 0 0 0 1 0 1 0 1 0 0 1 0 0
1 0 1 0 0 0 1 0 0 0 0 0 1 0 1 0 1 1 0 0 0
0 1 1 0 0 0 0 1 0 1 0 1 0 0 0 0 1 0 0 0 1
0 1 1 0 0 0 0 0 1 0 1 0 1 0 0 1 0 0 0 1 0
0 0 0 1 1 1 1 0 0 1 1 1 1 0 0 0 0 0 0 0 0
0 0 0 1 1 0 0 1 1 0 0 0 0 1 1 1 1 0 0 0 0
0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 1 1 1 1
```

FIGURE 40

```
1 1 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0
1 1 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0
1 1 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0
1 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1
0 0 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0
0 0 1 1 0 0 0 0 0 0 1 1 1 1 0 0 1 1 0 0 0 0
0 0 0 0 1 1 0 0 0 0 1 1 0 0 1 1 0 0 0 0 1 1
0 0 0 0 0 0 1 1 0 0 0 0 1 1 0 0 0 0 1 1 1 1
0 0 0 0 0 0 0 0 1 1 0 0 0 0 1 1 1 1 1 1 0 0
```

FIGURE 41

```
1 0 0 0 1 0 0 1 0 1 1
0 0 0 1 0 0 1 0 1 1 1
0 0 1 0 0 1 0 1 1 1 0
0 1 0 0 1 0 1 1 1 0 0
1 0 0 1 0 1 1 1 0 0 0
0 0 1 0 1 1 1 0 0 0 1
0 1 0 1 1 1 0 0 0 1 0
1 0 1 1 1 0 0 0 1 0 0
0 1 1 1 0 0 0 1 0 0 1
1 1 1 0 0 0 1 0 0 1 0
1 1 0 0 0 1 0 0 1 0 1
```

FIGURE 42

$m = 10$. A, and deletion of a row of figure 42, show that $k(10,11) = 51$. Then C' with $c = 5$ shows $k(10,12) = 52$. Similarly, using figure 43,

$k(10,13) = 55$ and $k(10,14) = 57$. Replace the (first) 6-col in figure 43 by 3 4-cols in rows 5678, 589X, 679X, and $k(10,15) = 61$ by C' with

0	1	1	1	1	1	1	0	0	0	0	0	0	
0	1	1	0	0	0	0	1	1	1	1	0	0	
0	0	0	1	1	0	0	1	1	0	0	1	1	
0	0	0	0	0	1	1	0	0	1	1	1	1	
1	1	0	1	0	0	0	0	0	1	0	1	0	
1	1	0	0	1	0	0	0	0	0	1	0	1	
1	0	1	0	0	1	0	0	1	0	0	1	0	
1	0	1	0	0	0	1	1	0	0	0	0	1	
1	0	0	1	0	1	0	1	0	0	1	0	0	1
1	0	0	0	1	0	1	0	1	1	0	0	0	1

FIGURE 43

1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	
1	1	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	
0	0	1	1	0	0	1	1	0	0	1	1	0	0	0	0	0	
0	0	0	0	1	1	0	0	1	1	1	1	0	0	0	0	0	
1	0	1	0	0	0	0	0	1	0	1	0	1	1	1	0	0	
1	0	0	1	0	0	0	0	0	1	0	1	1	0	0	1	1	
0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	1	1	
0	1	0	0	0	1	1	0	0	0	0	1	1	1	0	0	0	1
0	0	1	0	1	0	1	0	0	1	0	0	0	0	1	1	0	1
0	0	0	1	0	1	0	1	1	0	0	0	0	0	1	0	1	1

FIGURE 44

$c = 6$. Next $k(10,16) = 62$, since $4^{14}3^2$ is colmax, and 62 contains a 7-row or longer, which intersects at least 3 3-cols, while there are only 2. Figure 44 and C' with $c = 6$ shows $k(10,17) = 64$, $k(10,18) = 67$. Similarly $k(10,19) = 68$. On permuting 0135 and 014 cyclically, mod 10, columns $4^{10}3^{10}$ are possible and colmax, so $k(10,20) = 71$. Columns 7, 10 and 12 of figure 44 (2389, 2469, 3468) may be replaced by 239, 246, 289, 348, 368 and 469 so $4^9 3^{12}$ is possible and colmax, so $k(10,21) = 73$. We may deal similarly with columns 8, 9 and 11, and $k(10,24) = 79$. Since $t_{2,10} = 30$, $k(10,30) = 91$, and $k(10,29) = 88$, because $4 \cdot 3^{28}$ is colmax, while the quadrangle implies unused edges. However, $4^2 3^{26}$ are possible columns, since we may use the trefoil and associated triangle as a quadrangle, twice, so $k(10,28) = 87$.

$m = 11$. Figure 42 is colmax, so $k(11) = 56$. Also, by C' with $c = 5$, $k(11,12) = 57$.

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7. Special values for $a = b = 3$.

In this section, $k(m,n)$ means $k_3(m,n)$ and grid means 3-grid. The values of $k(m,n)$ above the line in table 4 are given by (10) and theorem 8.

$k_3(m,n)$

m \ n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
3	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49	51
4		14	17	19	22	25	27	29	31	33	35	37	39	41	43	45	47	49	51	53	55	57
5			21	23	26	29	31	34	37	39	42	45	47	50	53	55	58	61	63	65	67	69
6				27	30	33	37	40	43	46	49	51	54	57	59	62	65	67	70	73	75	78
7					34	38	41	45	48	51	54	57	61	64	67	70	73	76	79	82	85	88
8						43	46	51	54	58	61	65										
9							50	55	60	65												
10								61														

TABLE 4

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Sierpiński [18] showed that $k(6) = 27$. See also figure 45, which shows $k(6,7) = 30$, since $5^2 4^5$ is colmax, while the number of faces at a vertex of K_6^3 is 10, and these cannot all be used twice each to form only K_5^3 and K_4^3, since these have respectively 6 and 3 faces at a vertex. Figure 45 also shows $k(6,8) = 33$, since there is no 3-col by E, and no 6-col since $33 - 6 > k_{3,2}(6,7)$; hence the columns are $5 \cdot 4^7$. Thus there is no 7-row by D, and no 4-row by E, so the rows are $6^3 5^3$. The 6-rows are essentially 123456, 123478, 125678. If a 5-row intersects column 1 or 2 it forms a grid, and if not, columns 1 and 2 are 3-cols, a contradiction. Figure 46 shows $k(6,n) = 3n + 10$ $(9 \leq n \leq 13)$ on using $T_{2,6} < 10$ for $n = 10$ and A otherwise.

$$\begin{array}{cccccc|c|c}
0&1&1&1&1&1&1&1\\
1&0&1&1&1&1&0&0\\
1&1&0&1&1&0&1&0\\
1&1&1&1&0&0&0&1\\
1&1&1&0&0&1&1&0\\
1&1&0&0&1&1&0&1
\end{array}$$

FIGURE 45

$$\begin{array}{ccccccccc|cccc}
1&1&1&1&1&1&0&0&0&1&1&0&0\\
1&1&1&0&0&0&1&1&1&1&1&0&0\\
0&0&0&1&1&1&1&1&1&1&1&0&0\\
0&1&1&0&1&1&0&1&1&0&0&1&1\\
1&0&1&1&0&1&1&0&1&0&0&1&1\\
1&1&0&1&1&0&1&1&0&0&0&1&1
\end{array}$$

FIGURE 46

$$\begin{array}{ccccccc|c|c}
0&1&1&1&1&1&1&1&0\\
1&0&1&1&1&0&1&0&1\\
1&1&0&1&1&1&0&0&1\\
1&1&1&0&0&1&1&0&1\\
1&1&1&0&1&0&0&1&0\\
1&0&1&1&0&1&0&1&0\\
1&1&0&1&0&0&1&1&0
\end{array}$$

FIGURE 47

Brzeziński [19] showed $k(7) = 34$. A proof follows from C' and figure 47, which also show $k(7,8) = 38$. That $k(7,9) = 41$ also follows from figure 47: by E there is no 3-col; and no 7-col, since $41 - 7 > k_{3,2}(7,8)$, so the columns are $5^5 4^4$ or $6 \cdot 5^3 4^5$. If there were 5 5-cols, 2 would overlap in 4 rows and either a third forms a grid with them, or the other 3 themselves form a grid. If there are columns $6 \cdot 5^3$ they do not have a common row since $k_{2,3}(6,4) = 21 - 4$, so they are essentially in rows 123456, 12347, 12567, 34567. The only 4-cols which do not form a grid are then $ijk7$ ($i = 1,2$; $j = 3,4$; $k = 5,6$), and 5 of these must contain 2 which form a grid with a 5-col.

$$\begin{array}{cccccccccc|c|c|c|c}
1&1&1&0&1&1&1&1&0&0&0&0&0&0\\
1&1&0&1&1&1&0&0&1&1&0&0&0&0\\
1&0&1&1&0&1&1&0&1&0&1&0&1&0\\
1&0&1&1&1&0&0&1&0&1&1&0&0&1\\
0&1&1&1&1&0&1&0&1&0&0&1&0&1\\
0&1&1&1&0&1&0&1&0&1&0&1&1&0\\
1&1&0&0&0&0&1&1&1&1&1&1&1&1
\end{array}$$

FIGURE 48

$$\begin{array}{ccccccccccccccc|ccccc}
1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0\\
1&0&0&1&0&1&1&1&0&0&1&1&1&0&0&1&1&0&1&0\\
0&0&1&0&1&0&1&1&1&0&0&1&1&1&0&0&1&1&0&1\\
0&1&0&1&0&0&0&1&1&1&0&0&1&1&1&1&0&1&1&0\\
1&0&1&0&0&1&0&0&1&1&1&0&0&1&1&0&1&0&1&1\\
0&1&0&0&1&1&1&0&0&1&1&1&0&0&1&1&0&1&0&1
\end{array}$$

FIGURE 49

Figure 48 and B show $k(7,10) = 45$, since 5 5-cols form a grid as in the previous paragraph. Similarly $k(7,11) = 48$, since columns $5^4 4^7$ have 6 or 3 faces each at a vertex of K_7^3 out of $2\binom{6}{2} = 30$ possible. So if some faces are unoccupied, they are at least 3 in number, while

$4\binom{5}{3} + 7\binom{4}{3} = 2\binom{7}{3} - 2$. Similarly $k(7,12) = 51$ since $6{\cdot}5{\cdot}4^{10}$ is colmax, while at a vertex of K_6^3 (the 6-col) there remain $2\binom{6}{2} - \binom{5}{2} = 20$ faces, only a multiple of 3 of which are used in the other columns, so the columns are $5^3 4^9$. As before, unoccupied faces occur at least 3 at a vertex, so 4 form a tetrahedron, and it suffices to show $5^3 4^{10}$, which is colmax, to be impossible. If 2 of the 5-cols coincide, the third forms a grid with them. If they overlap in 4 rows, the third intersects just 2 of these, say they occupy rows 12345, 12346, 12567, and now the face 127 cannot occur in any tetrahedron. Finally if each pair of 5-cols overlap in just 3 rows, say they occupy rows 12345, 12367, 14567. In order to use faces 124, 125, 126, 127 a second time, we may without loss take 2 4-cols to be 1246, 1257. We must then choose 1347 and 1356, and then $234i$, $23jk$ where i, j, k are 5,6,7 in some order. In each case an odd number of triangles involving 24 are left, which cannot all be used in tetrahedra. Similar arguments show that $k(7,13) = 54$. A similar but much longer argument shows $k(7,14) = 57$. Figure 49 and C with $c = 4$ show that $k(7,15) = 61$. It also follows that $T_{2,7} = 15$, if we show the impossibility of 4^{16}. Since $16\binom{4}{3} = 2\binom{7}{3} - 6$, there are 6 unoccupied faces, which occur in multiples of 3 at a vertex and of 2 at a 2-edge. They cannot occur at only 3 vertices, since there are only 2 faces there. They cannot occur at just 4 or 5 vertices, since all 6 then occur at one of them, and one or other of the congruence conditions is violated. If they occur at 6 vertices, there are just 3 at each, and they occupy just 3 2-edges at each; again impossible. This, with figure 49 (in which the last 5 columns may be repeated) shows that $k(7,n) = 3n + 16$ $(15 \leq n \leq 25)$, by C with $c = 3$ for $n > 16$.

0	1	1	1	1	1	1	1	0
1	1	1	1	1	0	0	0	1
1	1	1	0	0	1	1	0	1
1	1	0	1	0	1	0	1	0
1	1	0	0	1	0	1	1	0
1	0	1	1	0	0	1	1	0
1	0	1	0	1	1	0	1	0
1	0	0	1	1	1	1	0	1
0	1	0	1	0	0	1	0	1

FIGURE 50

1	1	1	1	1	1	1	0	0	0	0
1	1	1	1	0	0	0	1	1	1	0
1	1	0	0	1	1	0	1	1	0	1
1	0	1	0	1	0	1	1	0	1	1
1	0	0	1	0	1	1	0	1	1	1
0	0	1	1	1	1	0	1	1	0	0
0	1	0	1	1	0	1	1	0	1	0
0	1	1	0	0	1	1	0	1	1	0

FIGURE 51

1	1	1	0	0	0	1	0	1	1	1	0
1	1	0	1	0	1	0	1	0	1	1	0
1	0	1	1	1	0	0	1	1	0	1	0
0	1	1	1	1	1	1	0	0	0	1	0
1	1	1	0	1	1	0	1	0	0	0	1
1	1	0	1	1	0	1	0	1	0	0	1
1	0	1	1	0	1	1	0	0	1	0	1
0	1	1	1	0	0	0	1	1	1	0	1
0	0	0	0	1	1	1	1	1	1	1	1

FIGURE 52

Figure 50 and C' with $c = 5$ show that $k(8) = 43$ [v. also 2] and $k(8,9) = 46$. Figure 51 and C' with $c = 6$ show that $k(8,10) = 51$ and $k(8,11) = 54$. If we delete a 7-row from figure 52, and use C' with $c = 7$, we see that $k(8,12) = 58$; similarly we can add a 3-col and $k(8,13) = 61$. From figure 53 and C' with $c = 7$, $k(8,14) = 65$.

1	0	0	1	0	1	1	1	1	0	0	1	1	0
1	1	0	0	1	0	1	1	1	1	0	0	0	1
1	1	1	0	0	1	0	1	0	1	1	0	1	0
1	1	1	1	0	0	1	0	0	0	1	1	0	1
0	1	1	1	1	0	0	1	1	0	0	1	1	0
1	0	1	1	1	1	0	0	1	1	0	0	0	1
0	1	0	1	1	1	1	0	0	1	1	0	1	0
0	0	1	0	1	1	1	1	0	0	1	1	0	1

FIGURE 53

1	1	1	1	0	1	0	0	1	0
1	1	1	0	1	0	0	1	0	1
1	1	0	1	1	0	1	0	1	0
1	0	1	1	1	1	0	1	0	0
0	1	1	1	1	0	1	0	0	1
0	1	1	0	0	1	1	1	1	0
1	1	0	0	0	1	1	1	0	1
1	0	0	0	1	1	1	0	1	1
0	0	0	1	1	1	0	1	1	1
0	0	1	1	0	0	1	1	1	1

FIGURE 54

Figure 50 shows $k(9) \geq 50$. We demonstrate equality. If a 9 by 9 matrix has total 50 and contains a 9-row, it forms a grid, since $50 - 9 > k_{2,3}(8,9)$. An 8-row also forms a grid since $50 - 8 - 8 = k_{2,3}(8)$. Suppose there is a 7-row. If the 2 columns it does not meet contain less than 12, it forms a grid, since $k_{2,3}(8,7) = 50 - 7 - 11$. If these 2 columns contain $11 + x$ ($x = 1,2,3$), they overlap in at least $3 + x$ rows, and these then contain at most 2 in each of the 7 columns occupied by the 7-row. So these 7 columns and the other $5 - x$ rows contain at least $50 - 7 - 14 - (11 + x) = 18 - x \geq k_{2,3}(5-x, 7)$, so a grid is again formed. If there is a 4-row, $k(8,9) = 50 - 4$ implies a grid elsewhere; so the rows (and columns) are $6^5 5^4$. The 5 6-cols do not meet a common row, since $k_{2,3}(8,5) = 30 - 5$; in fact they occupy, essentially, rows 123456, 123789, 124578, 134679, 235689. The only 5-cols which do not form a grid with these are 15689, 24679, 34578 and any in the last 6 rows; but at most 3 of these can be chosen without forming a grid. Figure 52 and C with $c = 5$ show that $k(9,10) = 55$, $k(9,11) = 60$ and $k(9,12) = 65$. We may add a 3-col (*e.g.* 159) and then C' with $c = 7$ shows that $k(9,13) = 68$.

T. Jenkyns has recently shown $k(10) = 61$, by figure 54 and C with $c = 6$. The configuration of figure 54 has the symmetry of the Petersen graph.

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