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ABSTRACT

A Baxter permutation of the integers 1-N(N odd) is a permutation P such that if P(j) is between P(i) and P(i+1), then j > i + 1 or j < i depending on the parity (even or odd) of i. Baxter permutations were first studied by Glen Baxter in a 1964 paper on the common fixed points of commuting functions question. Further results on them were obtained by Baxter and J. T. Joichi, and in a 1967 paper we established additional properties and described an algorithm for generating them directly. Most recently, Chung, Graham, Hoggatt, and Kleiman have derived an explicit formula for B(n), the number of Baxter permutations for N = 2n-1.

In this paper we show that Baxter permutations have a more general significance than is apparent from their origin in the esoteric field of commuting functions theory, for they arise in a very natural way in the study of composition of arbitrary continuous functions on an interval.

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MEMORANDUM FOR FILE

Glen Baxter [1] and Baxter and Joichi [2] have studied a class of permutations associated with pairs of continuous functions f, g on a closed interval I which commute under functional composition:

$$f(g(x)) = g(f(x)) \text{ for all } x \in I.$$

(Their motivation was the "common fixed point" question, which has been settled in [4] and [5].) If the set H' of fixed points of the composite function $h = gf = fg$ is finite, then Baxter showed that the permutation ϕ' obtained by restricting f to H',

$$\phi' = f|H', \ \phi':H' \to H'$$

satisfies certain conditions and is therefore <u>s-admissable</u>, in the terminology of [2]. If we consider only the set $H \subset H'$ of <u>crossing points</u> of h, then Baxter has shown that $\phi = f|H$ is a permutation of H which in [2] is called <u>w-admissable</u>. In an earlier paper, we have renamed the w-admissable permutations as <u>Baxter permutations</u>, and we presented an algorithm for generating them directly. In this paper we show that these permutations have a more general significance than is apparent from their origin in the esoteric field of commuting functions theory, for they also arise in a very natural way in the study of composition of arbitrary continuous functions on an interval. The principal objective of this paper is to prove:

Main Theorem. A permutation P of the first N positive integers is a Baxter permutation if and only if there are continuous functions $f,g:I \to I$ such that the composite function gf has a finite number of fixed points and the image under f of the i^{th} crossing point of gf is the $P(i)^{th}$ crossing point of fg.

Note that the requirement that f and g commute is entirely eliminated. In fact, the commutativity requirement is not only unnecessary but overly restrictive, for there are Baxter permutations (e.g., (1753)(264)) which cannot be obtained from commuting functions (see Theorem 4).

The "if" part of the Main Theorem is actually implicit in Baxter's derivation [1], if in his proof one regards the "k" and "m" subscripts as identifying fixed points of gf and fg respectively. However, we give a new, geometric proof of this result. The "only if" part then follows readily from the geometric interpretation of the problem.

We shall give a formal definition of Baxter permutations following Theorem 2. First we shall establish some convenient geometric machinery.

<u>Definition</u>: If $I = [a,b]$ is a closed interval and $f,g: I \to I$ are continuous functions, let

$$F = \{(x,y) | y=f(x)\}, \quad G = \{(x,y) | x=g(y)\}.$$

The set F is the graph of f, but G is the "inverse graph" of g since we use $x = g(y)$ instead of $y = g(x)$; see Figure 1. We identify the following other subsets of $I \times I$:

$$F^+ = \{(x,y) | f(x)>y\}, \quad G^+ = \{(x,y) | g(y)>x\},$$

$$F^- = \{(x,y) | f(x)<y\}, \quad G^- = \{(x,y) | g(x)<x\}.$$

These sets are illustrated in Figure 1 also.

Given f and g as above, denote the composite functions as $h = gf$ and $k = fg$. Let H' be the set of fixed points of h and K' the set of fixed points of k. We observe that $f(H') = K'$, $g(K') = H'$, and so both are one-to-one. Thus H' and K' are homeomorphic, and if one is finite, so is the other. The main advantage obtained when studying <u>commuting</u> functions is that $H' = K'$. (When h and k have only interior fixed points, and there is an <u>order-preserving</u> homeomorphism from H' to K', it can be extended to a homeomorphism u on I such that $u(K') = H'$; then we are essentially considering uf and gu^{-1}, whose two composites have the same set of fixed points. But if H' and K' are <u>infinite</u>, we shall see that there is not necessarily an <u>order-preserving</u> homeomorphism between them.)

Definition. Let $H^+ = \{x|h(x)>x\}$, $H^- = \{x|h(x)<x\}$. An interior point of $I = [a,b]$ is a crossing point of h if it is a limit point of both H^+ and H^-. The point a is a crossing point of h if it is a limit point of H^-, while b is a crossing point of h if it is a limit point of H^+. Let H be the set of crossing points of h. Similarly define K^+, K^-, and the set K of crossing points of k. A component of H^+ or K^+ is called an up-interval (for h or k); a component of H^- or K^- is called a down-interval.

We may eliminate the special definitions for the endpoints of I by considering $I_\varepsilon = [a-\varepsilon,b+\varepsilon]$ with $\varepsilon > 0$ and defining f and g to be constant on $[a-\varepsilon,a]$ and $[b,b+\varepsilon]$. Then fixed points and crossing points are preserved, and the composite functions have only interior fixed points. Thus without loss of generality we assume that the endpoints of I are never in H' or K'.

Theorem 1. If Π_x and Π_y are the projections of $I \times I$ onto first and second components respectively, and if $L' = F \cap G$ (see Figure 2), then

$$H' = \Pi_x(L'),\ K' = \Pi_y(L'),$$

$$H^\pm = \Pi_x(F \cap G^\pm),\ K^\pm = \Pi_y(G \cap F^\pm).$$

Proof is straightforward and is omitted.

(Figure 3 illustrates an infinite L' whose projections have distinct order types, justifying our earlier remark.)

Corollary 1A. [Baxter] Let z_1, z_2, and z be points of H' such that $z_1 < z_2$ and $f(z)$ is between $f(z_1)$ and $f(z_2)$. If $U = (z_1, z_2)$ is an up-interval, then $z > z_2$; while if U is a down-interval, then $z < z_1$. A similar conclusion holds for K' and g.

Proof: Since $f(z)$ is between $f(z_1)$ and $f(z_2)$, $f(z) = f(x)$ for some $x \in U$. But $\{(x,f(x)) | x \in U\} \subset F \cap G^+$ if U is an up-interval, so $x < g(f(x)) = gf(z) = z$. Similarly, for points $(x,f(x))$ in $F \cap G^-$ we have $x > g(f(x)) = gf(z) = z$.

Corollary 1A is the Baxter-Joichi condition "γ" in [2].

Corollary 1B. A point z is a crossing point of h if and only if for every neighborhood U of $(z,f(z))$, $U \cap F$ intersects both G^+ and G^-. A similar conclusion holds for the crossing points of k.

Proof: For "only if", given a neighborhood U of $(z,f(z))$ we can find open sets V_x, V_y such that $(z,f(z)) \in V_x X V_y \subset U$ and $f(V_x) \subset V_y$. Then $F \cap (V_x X V_y)$ must meet G^+ and G^- since V_x meets H^+ and H^-. For the "if" part, note that $V \times I$ is a neighborhood of $(z,f(z))$ when V is a neighborhood of z.

We conjecture that in fact, F intersects both G^+ and G^- in each neighborhood of $(z,f(z))$ if and only if G intersects both F^+ and F^- in each neighborhood of $(z,f(z))$, but we have not proved this. This would imply that f maps the crossing points of h onto the crossing points of k. The difficulties arise when z is a limit point of fixed points of h.

<u>Definition</u>. A point p of a set S is called <u>isolated</u> if it is not a limit point of $S - \{p\}$.

<u>Corollary 1C</u>. f maps the set of isolated points of H' onto the isolated points of K'.

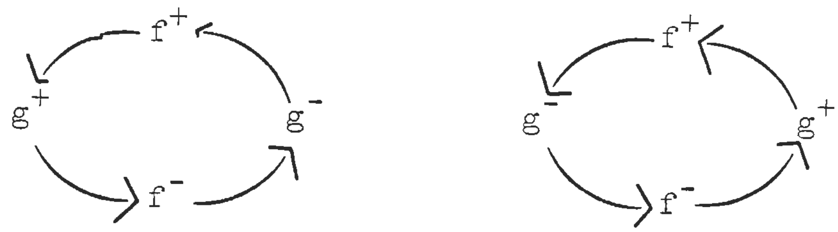
<u>Proof</u>: The isolated points of H' and K' are the images under Π_x and Π_y of the isolated points of $L' = F \cap G$.

Following Baxter, we observe that the isolated fixed points of a function can be divided into three classes or <u>types</u>. Type I points are <u>up-crossing</u> - the function is below the diagonal to the left of the point and above the diagonal to the right, so that a Type I point is the right endpoint of a down-interval and the left endpoint of an up-interval. Type II consists of the <u>down-crossing</u> points, with an up-interval on the left and a down-interval on the right. The noncrossing or <u>touching</u> points form a Type III.

<u>Theorem 2</u>. [Baxter] An isolated fixed point z of h = gf is an up-crossing point if and only if f(z) is an up-crossing point of k = fg, and z is a down-crossing point of h if

and only if $f(z)$ is a down-crossing point of k. Thus f preserves the type of isolated fixed points of h.

Proof. Let $p = (z,f(z)) \in L' = F \cap G$; since p is isolated there is a disk D, with boundary C, centered at p which contains no other point of L'. Let F_D be the component of $F \cap D$ containing p and G_D the component of $G \cap D$ containing p. Then F_D and G_D are arcs and $F_D \cap G_D = \{p\}$. Let f^- and f^+ be the points on F_D with least and greatest first coordinate respectively, and let g^- and g^+ be the points on G_D with least and greatest second coordinate. Then f^- and f^+ are the endpoints of F_D, g^- and g^+ the endpoints of G_D, and all four lie on the boundary C of D. Consider the possible arrangements of the points f^-, f^+, g^-, g^+ on the circle C. We take all directions on C counterclockwise, with angle about p increasing. Any point of $G \cap C$ on the arc from f^- to f^+ must lie in F^+, while points of $G \cap C$ on the arc from f^+ to f^- must lie in F^-. Asymmetrically, on the arc from g^- to g^+ a point of $F \cap C$ must be in G^- et cetera. If f^+ and f^- are adjacent on C then g^- and g^+ are adjacent also, and the points of $G \cap C$ are both in F^+ or both in F^-; similarly the points of $F \cap C$ are both in G^+ or both in G^-. Thus if the pairs f^+, f^- and g^+, g^- are adjacent on C both z and $f(z)$ are not crossing points. If f^+ and f^- are not adjacent, then two arrangements are possible:



In case 1, f^- lies in G^+ while f^+ lies in G^-, so z is a down-crossing point. The point g^- lies in F^+ and g^+ lies in F^-, so f(z) is a down-crossing point also. Similarly, in case 2 the points z and f(z) are both up-crossing points. Thus either both z and f(z) are up-crossing, both are down-crossing or both are touching, and the proof is complete.

Extending the "type" designations to non-isolated points of H' appears to be feasible but not particularly informative. Because of the possibility of H' containing open intervals, one can have "crossings" of the diagonal for which none of the fixed points are "crossing points" as we have defined the term. Also, Figure 3 indicates that the definition of classifications for the non-isolated points of H' which would be preserved by f would not be a straightforward matter. Thus for the rest of the paper we restrict ourselves to composite functions which have <u>only</u> isolated fixed points, that is, the cases in which H' (and K') are finite.

When H' is finite, then Theorem 2 applies to each point of H', so the image of each crossing point is a crossing point and the image of each touching point is a touching point. If H and K are the crossing points of h and k, then $f(H) = K$ and $g(K) = H$. Up-crossing points and down-crossing points must alternate, with the left-most and right-most crossing points being down-crossing, or Type II. Thus when H' is finite, H has an odd number of elements. If we number the points of H from 1 to N beginning with the smallest, the odd-numbered points will be Type II and the even-numbered ones Type I. If we similarly number the points of K in increasing order, then f induces a permutation P on the integers $J_N = \{1,2,\ldots,N\}$ by which the image under f of the i^{th} crossing point of $h = gf$ is the $P(i)^{th}$ crossing point of $k = fg$. By Theorem 2, P maps even integers onto even and odd integers onto odd, and by Corollary 1A, if $P(j)$ is between $P(i)$ and $P(i+1)$ then j is either greater than $i + 1$ or less than i depending on whether i is even or odd. These properties of P define what Baxter and Joichi called a <u>w-admissable</u> permutation [2], and what we have called a <u>Baxter permutation</u> [3].

<u>Definition</u>. If N is odd, and $J_N = \{1,2,\ldots,N\}$, then $P:J_N \to J_N$ is a <u>Baxter permutation</u> (of order N) if and only if:

(1) P maps even integers onto even integers and odd onto odd;

(2) if $P(j)$ is between $P(2i)$ and $P(2i+1)$, then $j > 2i + 1$;

(3) if $P(j)$ is between $P(2i-1)$ and $P(2i)$, then $j < 2i - 1$.

In the definition of w-admissable, our condition (1) is called (α) and our conditions (2) and (3) are called (γ). In [2] there is also a condition (β) which for w-admissable permutations is a trivial consequence of (α) and (γ).

With this definition we obtain the following corollary to Theorem 2, which is half of the Main Theorem:

<u>Corollary 2A</u>. [Baxter] If $f,g:I \to I$ are continuous and $h = gf$ has a finite number of fixed points including N crossing points, then $P:J_N \to J_N$, defined by $P(i) = j$ if and only if the image under f of the i^{th} crossing point of h is the j^{th} crossing point of $k = fg$, is a Baxter permutation.

For the converse we shall prove the following result:

<u>Theorem 3</u>. If N is odd and $P:J_N \to J_N$ is a Baxter permutation, then there are continuous functions $f,g:[0,N+1] \to [0,N+1]$ such that $f(i) = P(i)$ for each $i \in J_N$ and $H' = H = K' = K = J_N$.

<u>Proof</u>. We give a construction for f; the same construction produces g, if one uses P^{-1} instead of P. Define $f(x) = P(1)$ on $[0,1]$ and $f(x) = P(N)$ on $[N,N+1]$. Since our construction will produce an f (and g) such that $f([0,N+1] \subset [1,N]$, this insures that the first interval $[0,1)$ is an up-interval and the last a down-interval, as is required.

For all intervals $(i,i+1)$ but the first and last, f will be defined to be strictly between $P(i)$ and $P(i+1)$. If $P(i)$ and $P(i+1)$ are not adjacent, then f simply interpolates linearly between $P(i)$ and $P(i+1)$:

$$f(i+\theta) = (1-\theta)\, P(i) + \theta P(i+1) \quad \text{for} \quad 0 \leq \theta \leq 1.$$

We shall show that if $P(i)$ and $P(i+1)$ are not adjacent, then $(i,i+1)$ contains no points of H' and $(i,i+1) \subset H^+$ or H^- depending on whether i is even or odd. There are four cases to consider, i even or odd and $P(i+1)$ greater or less than $P(i)$. The proofs are similar; we shall consider only i even (an up-interval) and $P(i+1) < P(i)$ (see Figure 4). Since P is a Baxter permutation, $gf(j) > i+1$ when $P(j)$ is between $P(i)$ and $P(i+1)$. Since $g(y)$ is between $g(j)$ and $g(j+1)$ when y is between j and $j+1$, $h(x) = gf(x) > i+1 > x$ when $x \varepsilon (i,i+1)$ and $f(x)$ is between $P(i+1)$ and $P(i) - 1$. The only problem is that F and G might intersect somewhere in the square $[i,i+1] \times [P(i)-1,P(i)]$ besides at $(i,P(i))$. But f has an absolute slope on $[i,i+1]$ greater than 1, and g has an absolute slope on $[P(i)-1,P(i)]$ greater than 1 since $P(i+1) \neq P(i) - 1$ implies that $P^{-1}(P(i)-1)$ cannot be adjacent to $P^{-1}P(i) = i$. Thus there is a positive angle from F to G in the square $[i,i+1] \times [P(i)-1,P(i)]$, and F and G meet only at $(i,P(i))$. The other three cases are all similar in that the restriction that g lie between

$P^{-1}(j)$ and $P^{-1}(j+1)$ on $(j,j+1)$ prevents F and G from intersecting except in a single square, and in that square the functions are linear with slopes greater than 1.

When $P(i)$ and $P(i+1)$ are adjacent the proof is more intricate, but still straightforward. We cannot simply interpolate linearly, for that would make F and G coincide and thus create an interval of fixed points of h instead of an up-interval or down-interval. We shall rely on the property that if $P(i+1) = P(i) + 1$, the intervals $(i,i+1)$ and $(P(i),P(i+1))$ are the same type (up or down), while if $P(i+1) = P(i) - 1$, the intervals $(i,i+1)$ and $(P(i+1),P(i))$ are of opposite type. (This property is the condition (β) of [2].) We choose a continuous δ on $[0,1]$ such that $\delta(0) = \delta(1) = 0$ and $0 < \delta(\theta) < \min(\theta,1-\theta)$ for $0 < \theta < 1$, e.g., $\delta(\theta) = \frac{1}{2}\min(\theta,1-\theta)$. There are again four cases to consider: i even or odd (corresponding to $(i,i+1)$ being a down-interval or an up-interval) and $P(i+1) = P(i) \pm 1$. With $0 \leq \theta \leq 1$, we define f on $[i,i+1]$ as follows (see Figure 5):

type	$P(i+1) =$	$f(i+\theta) =$
up	$P(i) + 1$	$P(i) + \theta + \delta(\theta)$
up	$P(i) - 1$	$P(i) - \theta - \delta(\theta)$
down	$P(i) + 1$	$P(i) + \theta - \delta(\theta)$
down	$P(i) - 1$	$P(i) - \theta + \delta(\theta)$.

Also in Figure 5 we show as a dotted line the location of G when g is constructed by the same rule. For i even, F lies to the right of G and so $h(x) > x$ on $(i,i+1)$, while for i odd, F is to the left of G, $h(x) < x$, and $(i,i+1)$ is a down-interval. In either case $H' \cap (i,i+1) = \phi$, so $H' = H = J_N$ as required. The same analysis applied to $k = fg$ shows that $K' = K = J_N$, so the proof is complete. Figure 6 shows F and G, h, and k constructed in the proof for the Baxter permutation (153)(24).

Corollary 3A. [Baxter-Joichi] The inverse of a Baxter permutation is a Baxter permutation also.

Proof. Given P, by the theorem we can construct a pair of functions f,g which induce P. But then the pair g,f induce P^{-1}, which by Corollary 2A is a Baxter permutation.

Corollary 3B. If $P = P^{-1}$, then there are commuting continuous functions $f,g: I \to I$ which induce P if and only if P is a Baxter permutation.

Proof: Only the "if" part remains to be proved. But if $P = P^{-1}$, the constructions of f and g specified in the theorem yield the same function, so f and $g = f$ commute. By the theorem P is induced by f and $g = f$.

The next result shows that the connection between Baxter permutations and commuting functions does not extend to an equivalence.

Theorem 4. Not all Baxter permutations can be induced by commuting functions; in particular, the Baxter permutation (1753)(264) cannot arise from commuting functions.

Proof. To set the stage, in Figure 7 we graph the functions which Theorem 3 constructs to induce (1753)(264). Now assuming that there actually exist functions $f', g' : I \to I$ which commute and for which $f'|H$ may be represented as (1753)(264), we may choose a homeomorphism $u : I \to [0,8]$ such that $u(H) = J_7$ and consider the functions $f = uf'u^{-1}$ and $g = ug'u^{-1}$, which commute, map $[0,8]$ into $[0,8]$, and for which $H = K = J_7$. Since 5 is between $f(2) = 6$ and $f(3) = 1$, and $g(5) = 7$, we have $7 \varepsilon gf([2,3])$. But $g([2,3]) \subset [4,8]$, and $f([4,7]) < 6$, so in order to have $7 \varepsilon fg([2,3])$ we must have $7 \varepsilon f([7,8])$ and $7 \varepsilon g([2,3])$ (which is indicated by the dotted line in Figure 8). Similarly $1 \varepsilon gf([5,6])$, so that in order to have $1 \varepsilon fg([5,6])$, we must have $1 \varepsilon g([0,1])$ and $1 \varepsilon f([5,6])$ (shown as the solid line in Figure 8). But in order to have both $7 \varepsilon g([2,3])$ and $1 \varepsilon f([5,6])$, there must be additional points of intersection of F and G in $[5,6] \times [2,3]$ which produce additional crossing points of fg and gf. Thus no commuting functions can induce (1753)(264).

Next we present some results on the s-admissable permutations introduced by Baxter and Joichi in [2].

<u>Definition</u>. Let $I = [a,b]$ and $H' = \{x_1, x_2, \ldots, x_n\} \subset I$ such that $a < x_1$, $x_i < x_{i+1}$, $x_n < b$. Arbitrarily denote each of the intervals (x_i, x_{i+1}) as an up-interval or down-interval, and let $[a, x_1)$ be an up-interval and $(x_n, b]$ a down-interval. Then H' with the intervals so specified is called an <u>s-set</u>. Each point of an s-set can be assigned a <u>type</u> (I, II, or III) depending on the type (up or down) of the adjacent intervals. A permutation $P: J_n \to J_n$ is called <u>s-admissable</u> (relative to the s-set H') if it satisfies:

(α) $x_{P(i)}$ has the same type as x_i;

(β) if $P(i+1) = P(i) + 1$, then (x_i, x_{i+1}) and $(x_{P(i)}, x_{P(i+1)})$ have the same type, while if $P(i+1) = P(i) - 1$, then (x_i, x_{i+1}) and $(x_{P(i+1)}, x_{P(i)})$ have opposite type;

(γ) if $P(j)$ is between $P(i)$ and $P(i+1)$, then $j > i+1$ if (x_i, x_{i+1}) is an up-interval and $j < i$ if (x_i, x_{i+1}) is a down-interval.

Baxter's result may be stated in terms of s-admissable permutations as follows: let f and g be continuous functions on I such that $h = gf$ and $k = fg$ have finitely many fixed points, all interior to I. Make H' an s-set using h to define up- and down-intervals, and K' an s-set similarly using k. Let the permutation P be defined by $f(x_i) = y_{P(i)}$ for $x_i \varepsilon H'$, $y_i \varepsilon K'$. Then P is s-admissable relative to H', and P^{-1} is s-admissable relative to K'.

For the converse, we generalize Theorem 3 as follows:

Theorem 5. If H' is an s-set and P is s-admissable, then there are continuous functions $f,g:I \to I$ such that $f(x_i) = x_{P(i)}$ and the up-intervals, down-intervals, fixed points, and crossing points of $h = gf$ and $k = fg$ agree with those specified by H'.

Proof. Without loss of generality we may assume that $I = [0,n+1]$ and $H' = J_n$. The functions f and g are specified exactly as in Theorem 3, interpolating linearly except when P(i) and P(i+1) are adjacent, in which case a function δ is used. The same arguments show that the up-intervals and down-intervals behave as required and that no new fixed points are created.

Corollary 5A. [Baxter-Joichi] The inverse of an s-admissable permutation is also s-admissable.

Proof. One simply reverses the order of f and g, which changes P to P^{-1}.

Since analogous theorems are obtained for both s-admissable permutations and w-admissable (Baxter) permutations, a legitimate question is why bother making the restriction from fixed points to crossing points, and why should this paper treat the Baxter permutations as the primary concern. One answer lies in the topological origin of the subject: the touching points are unstable, in that they can be removed by arbitrarily small alterations to the functions. Thus they may be considered as topologically inessential, and should be disregarded. Another answer is that the Baxter permutations are simpler and somehow more basic, depending as they do only on a particular odd integer, while for s-admissable permutations up- and down-designations must be specified in addition. It seems mathematically unsatisfactory that the identity permutation on $2n + 1$ points can be s-admissable in 2^{2n} different ways, depending on the designation of the intervals. Also, as is brought out in the next theorem, each s-admissable permutation has a Baxter permutation "skeleton" which determines most of its basic topological properties.

A final interesting question on s-admissable permutations is whether property (β) actually imposes restrictions beyond those of (α) and (γ); we have previously observed that (β) is a trivial consequence of (α) and (γ) for Baxter permutations. In the interest of brevity we shall only sketch the proof that for s-admissable permutations also, (β) follows from (α) and (γ).

<u>Theorem 6</u>. Let H' be an s-set and P a permutation satisfying conditions (α) and (γ). Then P satisfies (β) as well and is thus s-admissable.

Proof. Let (x_i, x_{i+1}) be an interval such that $P(i+1) = P(i) \pm 1$ and the type of the interval between $x_{P(i)}$ and $x_{P(i+1)}$ may violate (β). If either x_i or x_{i+1} is a crossing point (β) is easily verified, so the only problem is when both x_i and x_{i+1} are type III (touching). Let x_ℓ and x_m be the crossing points on either side of x_i, with $x_\ell < x_m$, so that all intervals between x_ℓ and x_m have the same type as (x_i, x_{i+1}). Since all the intervals are the same type, the points of $H' \cap (x_\ell, x_m)$ may be divided into two sets, S^+ and S^-, so that on S^+, $P(j)$ is increasing, and on S^-, $P(j)$ is decreasing. Again, four cases must be considered, depending on the interval type and the sign of $P(m) - P(\ell)$. With up-intervals and and $P(\ell) > P(m)$, the situation is as depicted in Figure 9. For $i \varepsilon S^+$, $P(i) < P(m)$, and for $i \varepsilon S^-$, $P(\ell) > P(i) > P(m)$. The point x_ℓ is Type I so the interval directly below $x_{P(\ell)}$ is a down-interval. Also the interval above $x_{P(m)}$ is a down-interval and the interval below is an up-interval. To have a violation of (β) for the points in S^- there would have to be crossing points and up-intervals bounded by touching points between $x_{P(m)}$ and $x_{P(\ell)}$. But the crossing points would have to be to the right of x_m, while the touching points would have to be between x_m and x_ℓ, and this would put x_m in the range of an up-interval, a violation of (γ). Similar arguments can be made for S^+ and the other three cases.

In closing, we note that very recently Chung, Graham, Hoggatt, and Kleiman [6] have succeeded in determining B(n), the number of Baxter permutations for a given odd integer 2n - 1. It is given by

$$B(n) = \binom{n+1}{1}^{-1}\binom{n+1}{2}^{-1} \sum_{k=1}^{n} \binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}.$$

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Att.
References 1-6
Figures 1-9

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4. William M. Boyce, Commuting functions with no common fixed point, Trans. Amer. Math. Soc. 137 (1969), 77-92.

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6. F. R. K. Chung, R. L. Graham, V. E. Hoggatt, and M. Kleiman, The number of Baxter permutations, to appear.

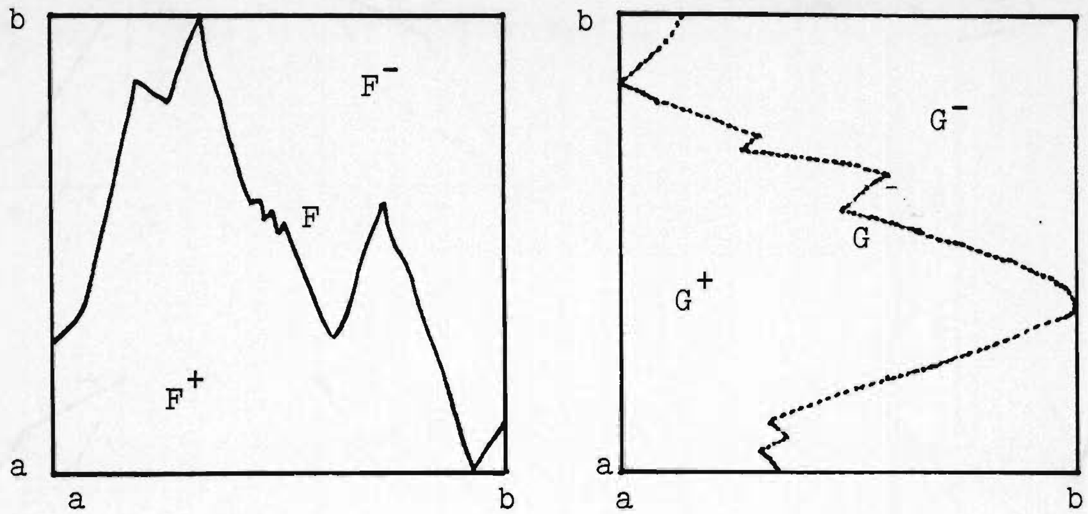


Figure 1

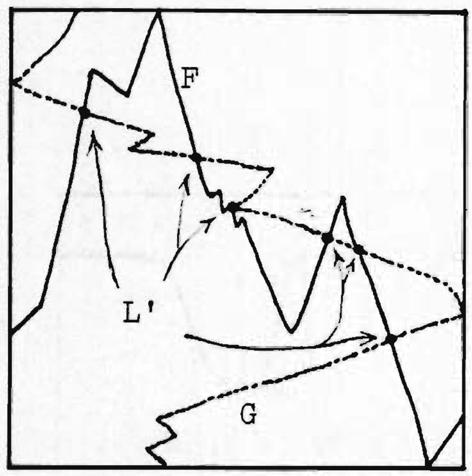


Figure 2

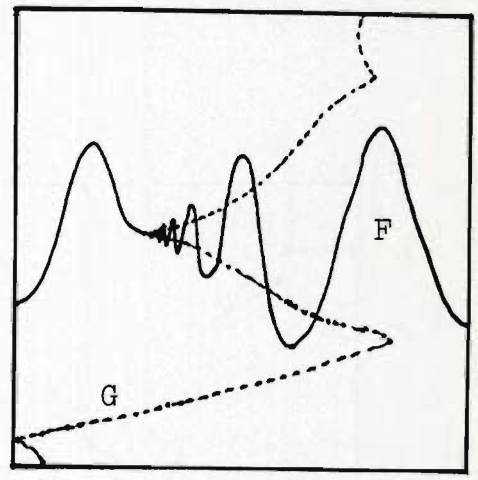
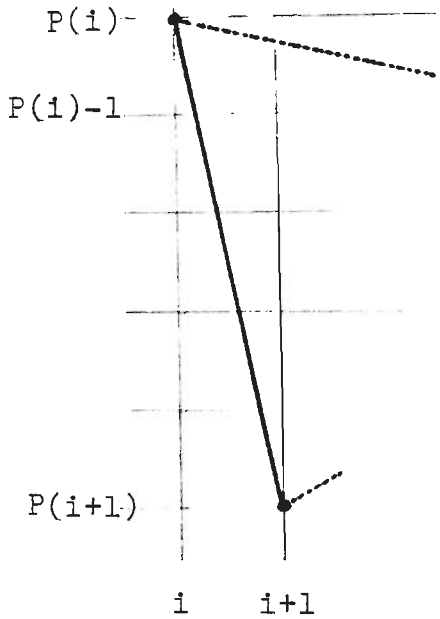


Figure 3



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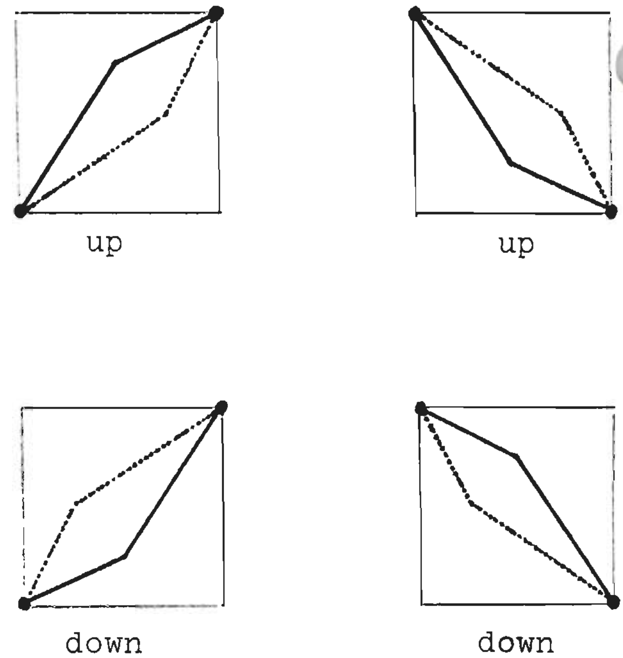

Figure 5

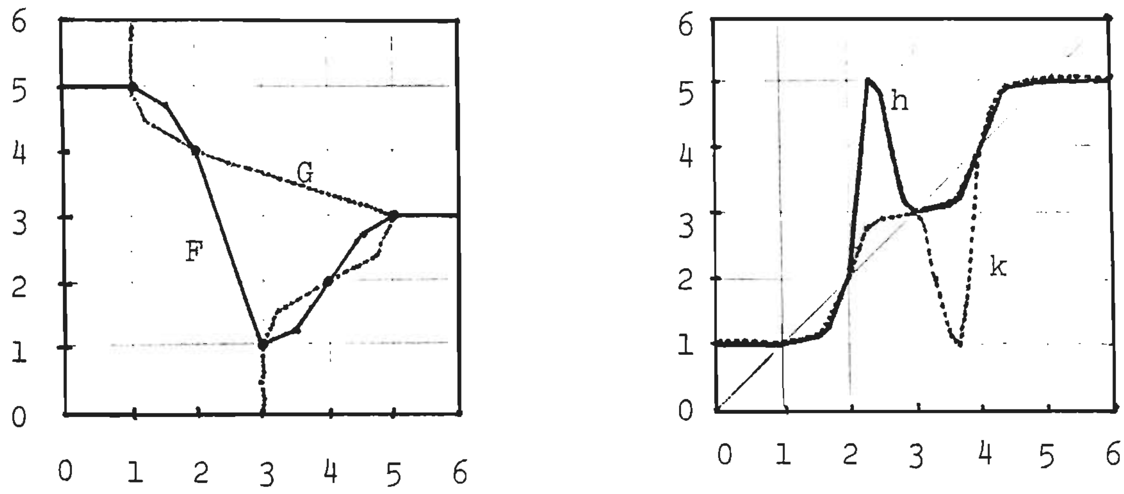

Figure 6

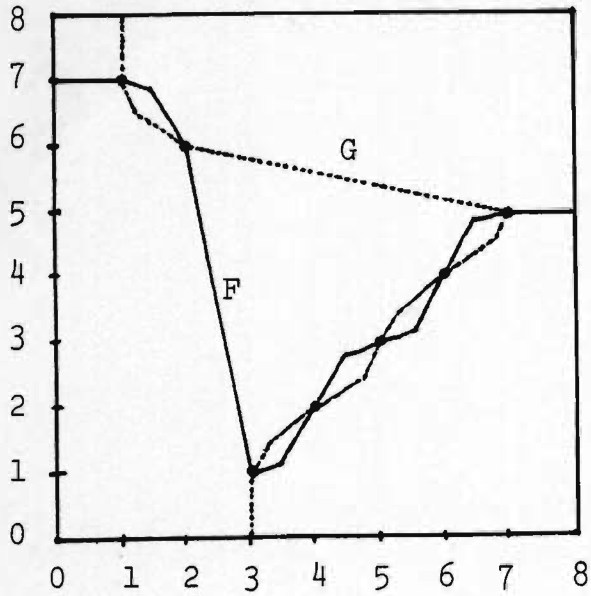

Figure 7

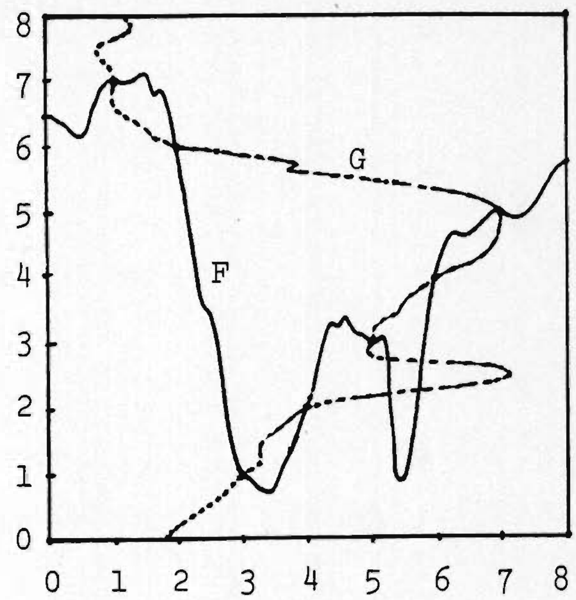

Figure 8

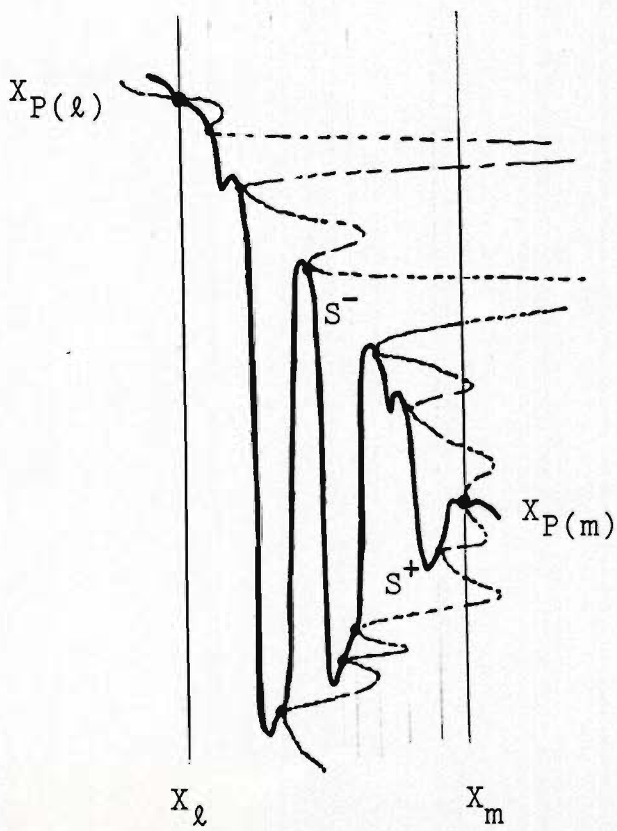

Figure 9

1181
-1185
(Cont)

NOTE: Please substitute the following pages for page 25.

The most complicated routine is the computation of $P:O_N \rightarrow O_N$. As written, this routine includes two explicit DO loops, for P(1) and P(3), and a general-purpose loop which cycles the current value of P(2i+1) for $i > 1$. After the choice of a value for P(2i+1), the $M_j(2i+3)$ are computed and a vector of admissable values for P(2i+3) established. Then the least admissable value is chosen, and the process continues. The algorithm given in Theorem 4 has no "dead ends" except when P(N) is defined. Then the process is kicked back up to the lowest level for which an admissable value remains.

The following table gives some of the quantitative results.

As is | 1181 | | 1183 | | 1185
over 2 | 1182 | | 1184

A N	B ((N+1)/2)!	C # Baxter p. 1181	1182	D Ratio C/B	E Nontrivial 1183	1184	F Ratio E/C	G Eq.cl. of NT
3	2	2	1	1.00	0		.0	0
5	6	6	3	1.00	2	1	.33	1
7	24	22	11	.92	2	1	.09	1
9	120	92	46	.77	18	9	.20	7
11	720	422	211	.59	66	33	.156	21
13	5,040	2,074	1037	.41	374	187	.180	112
15	40,320	10,754	5377	.27	1,694	847	.1575	456
17	362,880	58,202	29101	.16	9,822	4911	.169	2,603
19	3,628,800	326,240	163120	.09	51,698	25849	.1585	13,203

The most interesting feature of this table is the behavior of the ratio of nontrivial Baxter permutations of order N to the total number, as shown in column F. If we denote this ratio by $R(N)$, we have the suggested relationships

$$R(4i-1) < R(4i+1)$$

$$R(4i-1) < R(4i+3)$$

$$R(4i+1) > R(4i+5)$$

and it appears that the sequence $\{R(2i+1)\}$ is approaching a limit L with $.1585 < L < .169$. R. L. Graham has suggested that perhaps

$$L = 1/2\pi = .159\ldots$$

since the $R(4i-1)$ seem to be increasing more slowly than the $R(4i+1)$ are decreasing. A theoretical verification of these relationships would be interesting. For unrestricted permutations the ratio of fixed-point-free permutations is known to approach $1/e$ with increasing N.

The counterexample found by the author was developed by analytic techniques from a permutation with $N = 13$, namely $\downarrow\begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 \\ 11 & 9 & 1 & 3 & 7 & 13 & 5 \end{pmatrix}$. For $N = 11$ there are three equivalence classes, represented by

$$\downarrow\begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 9 & 11 & 1 & 3 & 5 & 7 \end{pmatrix}, \qquad \downarrow\begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 9 & 11 & 3 & 1 & 7 & 5 \end{pmatrix}, \qquad \downarrow\begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 11 & 7 & 3 & 1 & 5 & 9 \end{pmatrix}$$

which can possibly be developed into "smaller" counterexamples. For each Baxter permutation with $N \le 9$, and for each Baxter permutation for 11 and 13 except the classes just mentioned, if f and g commute and induce it, then f and g must have a common fixed point.