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COVER SHEET FOR TECHNICAL MEMORANDUM

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SUBJECT— A Catalog of Partially Ordered Systems - [redacted]

[redacted]

DATE— August 8, 1961

AUTHOR— E. N. Gilbert

FILING SUBJECT— (TO BE ASSIGNED BY AUTHOR) Combinatoric

ABSTRACT

The partially ordered systems of n elements are given for $n \leq 6$. The numbers of systems for $n = 1,2,\ldots,6$ are 1,2,5,16,63,319. These systems are of interest in the statistical design of experiments.

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A Catalog of Partially Ordered Systems - [redacted]

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August 8, 1961

MEMORANDUM FOR FILE

For an application to statistics, Prof. O. Kempthorne has posed the problem "find all partially ordered* systems of n elements." In listing partially ordered systems, two systems which are isomorphic (i.e., differ only in the way the elements are named) are to be considered identical. The number P(n) of (non-isomorphic) partially ordered systems has been known for $n \leq 5$. Birkhoff's Lattice Theory, page 4, attributes the following numbers to I. Rose

n	2	3	4	5
P(n)	2	5	16	63

.

The enumeration which follows extends this table one more step:

$$P(6) = \cancel{319}\ 318 .$$

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Since the systems themselves are of interest this memorandum catalogs them. They appear here drawn as Hasse diagrams (Birkhoff pp. 5-6). For n = 5,6, only connected diagrams are included; all the missing diagrams are combinations of two or more connected diagrams placed side by side, as in

B. p 1.

* binary relation $x \geq y \ni$

Reflexive: $x \geq x$

Antisym: $x \geq y$ & $y \geq x \Rightarrow x = y$

Trans: $x \geq y$ & $y \geq z \Rightarrow x \geq z$

with n = 6. The numbers C(n) of connected diagrams are

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n	1	2	3	4	5	6
C(n)	1	1	3	10	44	~~239~~ 238 NJAS

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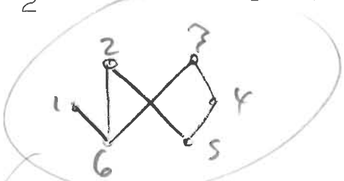
Producing the P(n) diagrams from the connected diagrams is a simple exercise in forming combinations with repetitions allowed. A generating function relationship between P(n) and C(n) is

$$1 + \sum_{n=1}^{\infty} P(n)x^n = \prod_{n=1}^{\infty} (1-x^n)^{-C(n)} .$$

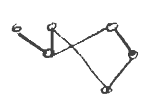
Two kinds of invariants help to classify these diagrams. One is a <u>numerical invariant</u> which exhibits the numbers of elements which appear on each level when the diagram is drawn in a certain standard way. Define a unique level for each element by the rule:

<u>an element x is at level L if the longest chain extending upward from x is of length L-1.</u>

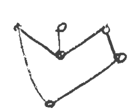
Then say that a diagram is in standard form if for all $L_1 < L_2$ all points of level L_1 are placed higher than all points of level L_2. For example, standard forms of



are



or



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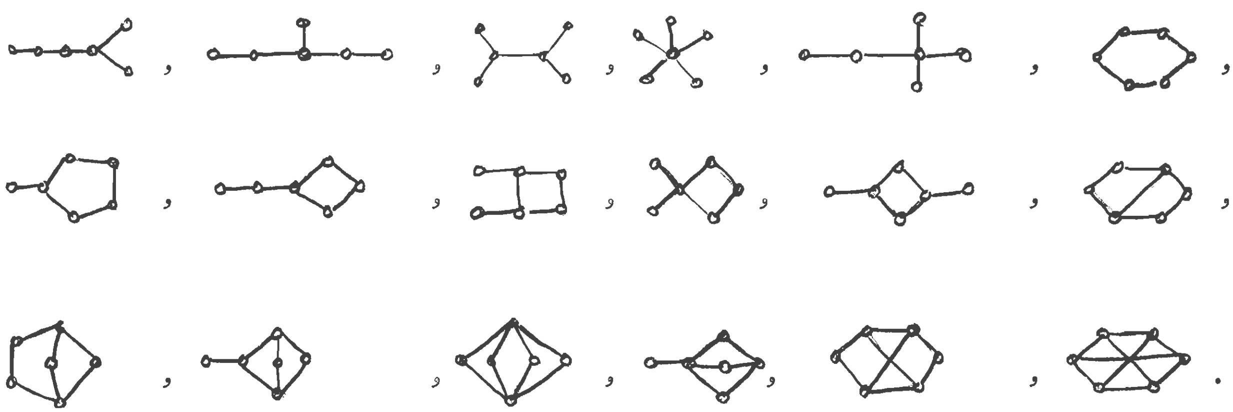
The numerical invariant is the sequence of numbers of elements on levels 1,2,... (321 for the system in the illustration). In the catalogue, diagrams appearing in the same row have the same numerical invariant.

1≤6, 2≤6, 2≤5, 3≤6, 3≤4, 4≤5

The second invariant is a graphical invariant, the diagram itself considered just as a graph. On p. II of the catalog the connected partially ordered systems of 5 elements are arranged in columns having the same graphical invariant. For n = 6 there are too many possible graphs to allow a columnar arrangement. However, the diagrams corresponding to a numerical invariant are separated into bunches having the same graphical invariant. Within a row the graphs appear in the order ,



Note that these graphs include none having cycles of length 3 since no such graph can represent a partially ordered system.

The catalog was produced by a long calculation involving many special cases and requiring much testing by inspection for isomorphism. No proof will be given that the catalog is correct. Since the two invariants subdivide the diagrams into small bunches, there seems to be little chance that a diagram appears twice. The catalog has been checked for omissions in several ways. J. B. Kruskal did the case n = 5 independently. For n = 6, one check verified that the dual (diagram turned upside down) of every diagram in the catalog also belonged to the catalog. This is a good check because the dual diagram must usually be redrawn to bring it into a standard form. The numerical invariant of the dual is not simply the reverse of the original invariant (the example above which had invariant 321 has a dual with invariant 231). Another check for n = 6 verified that all diagrams having exactly one element at level 1 are present. Since removing the element at level 1 leaves a 5 element diagram, there are exactly 63 such systems. Finally, an independent enumeration was made of 6 element connected partially ordered systems having only one element at the lowest level.

To make guesses about P(7),P(8),... from known values of P(1),...,P(6) is admittedly risky. However, the following heuristic argument is offered for whatever it may be worth. The total number of binary relations satisfying the reflexive and antisymmetric laws is known to be approximately

$$\frac{3^{\binom{n}{2}}}{n!} = e^{O(n^2)} .$$

Partially ordered systems are relations of this kind which also satisfy the transitive law. Certainly then, log P(n) cannot grow faster than $O(n^2)$. This suggests the possibility of extrapolating log P(n) approximately by means of a quadratic in n; equivalently

$$(1) \qquad P(n+3) = P(n) \left\{ P(n+2)/P(n+1) \right\}^3$$

is an extrapolation formula for P(n). As support for this idea, a difference table shows $\Delta^3 \log P(n)$ is small for $n \leq 6$. Formula (1) would have estimated P(5) as

$$2 \times (16/5)^3 \approx 66 \text{ and } P(6) \text{ as } 5(63/16)^3 \approx 306 .$$

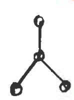
Continuing, (1) estimates P(7) as 2085 and estimates that there are several million partially ordered systems of 10 elements.

E. N. GILBERT

Att.
Figs. I-VII

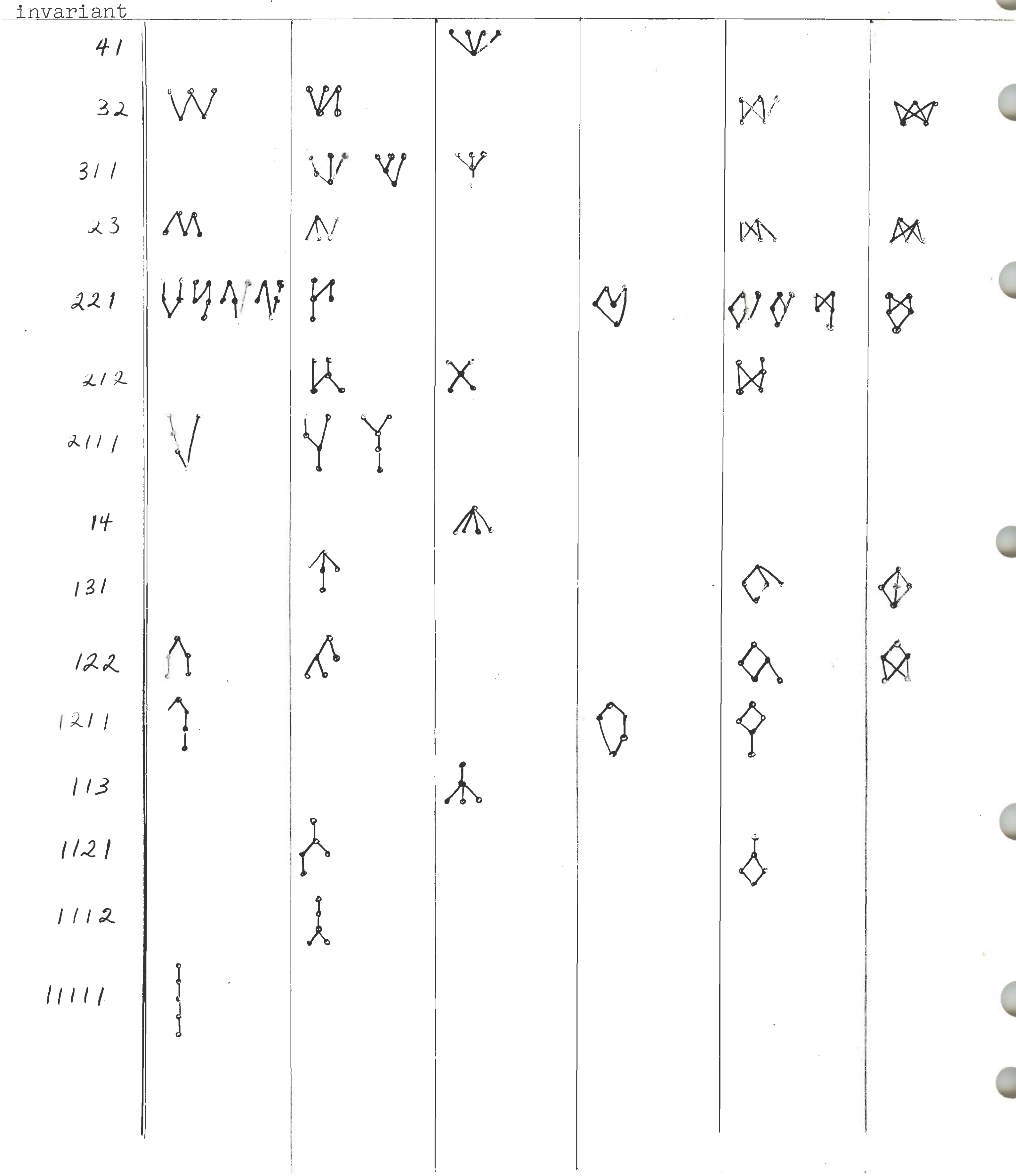
Red for labelled versions

PARTIALLY ORDERED SYSTEMS OF n ELEMENTS

	invariant	disconnected	connected	# 112	608	1035	1927
$n = 1$	1			1	1	1	1
$n = 2$	2	 1					
	11		 2	2	1	3	2
$n = 3$	3	 1					
	21	 6	 3				
	12		 3				
	111		 6	5	3	19	12
$n = 4$	4	 1					
	31	 12  12	 4				
	22	 12  12	 24  6				
	211	 24	 12  24				
	13		 4				
	121		 24  12				
	112		 12				
	1111		 24	16	10	219	146

n = 5

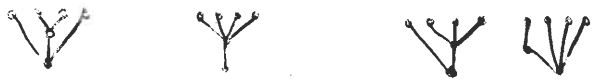
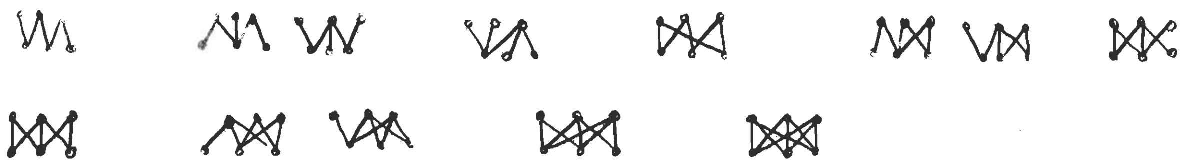
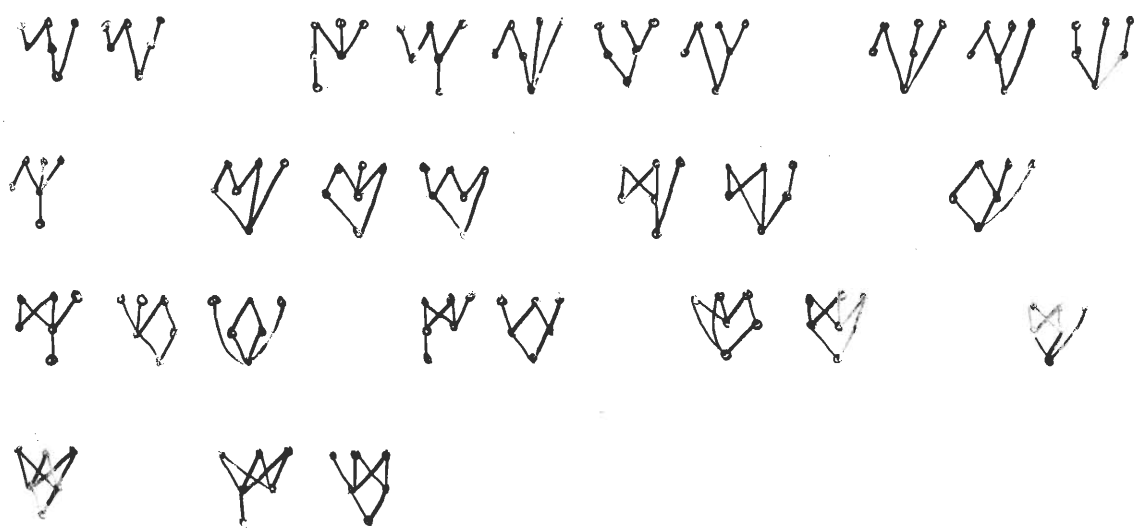
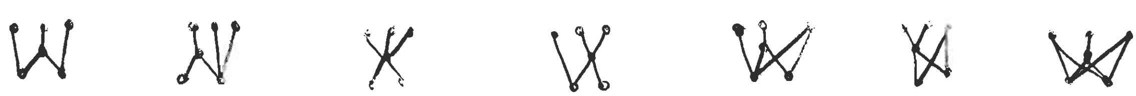
CONNECTED PARTIALLY ORDERED SYSTEMS OF 5 ELEMENTS



n = 6

III

CONNECTED PARTIALLY ORDERED SYSTEMS OF 6 ELEMENTS

invariant	
51	
42	
411	
33	
321	
312	
3111	

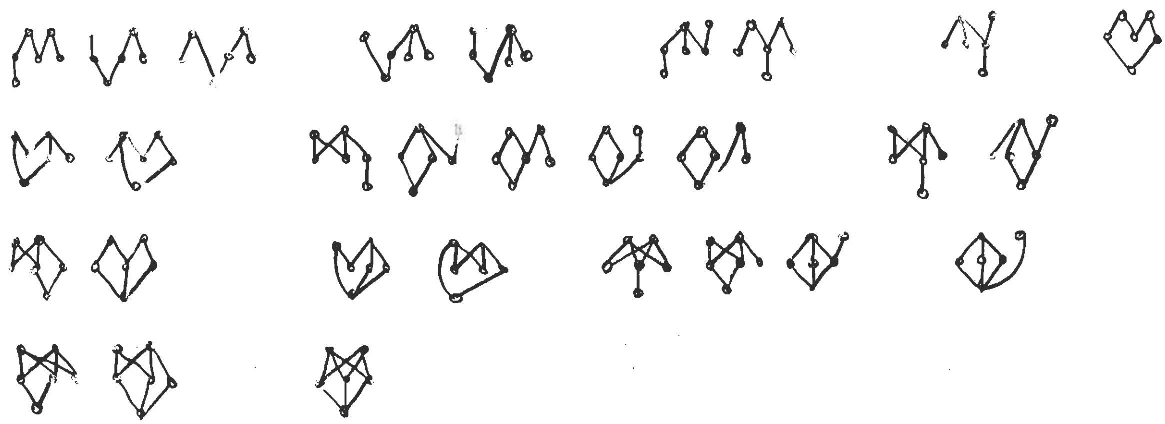
IV

CONNECTED PARTIALLY ORDERED SYSTEMS OF 6 ELEMENTS

invariant

24

231



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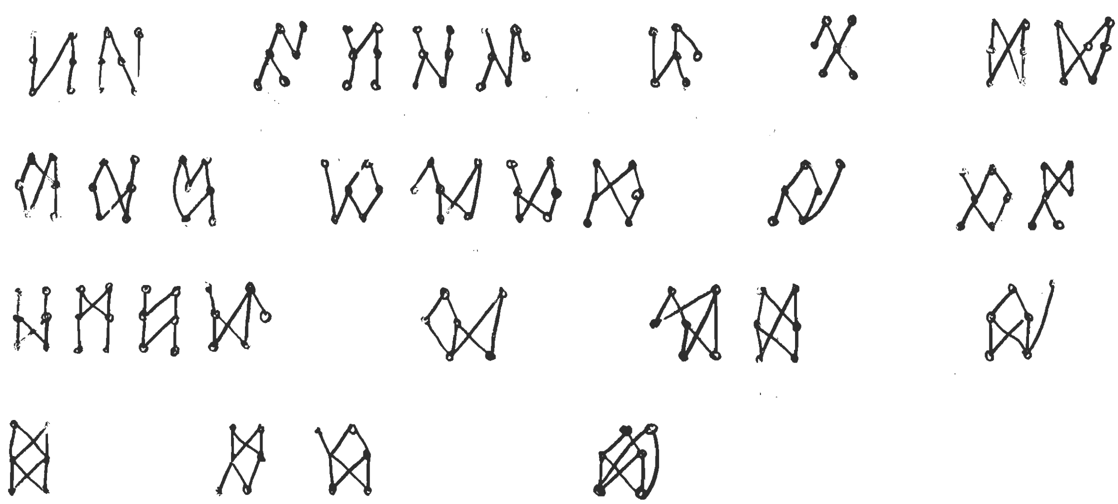

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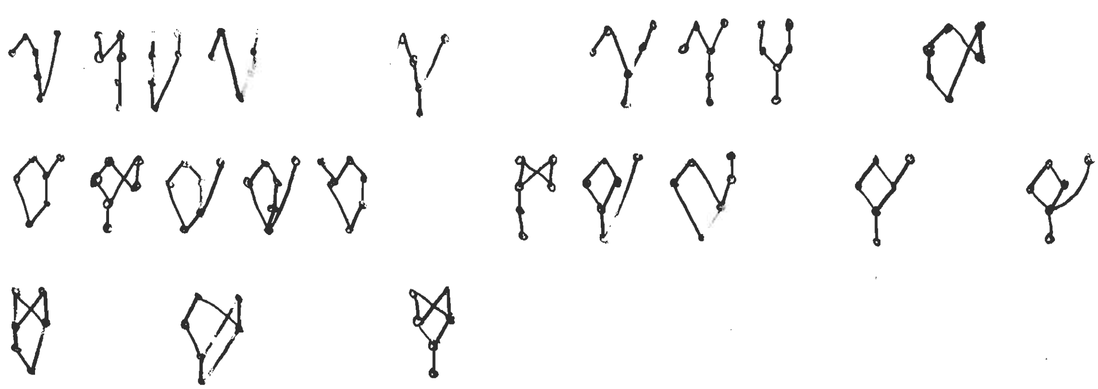


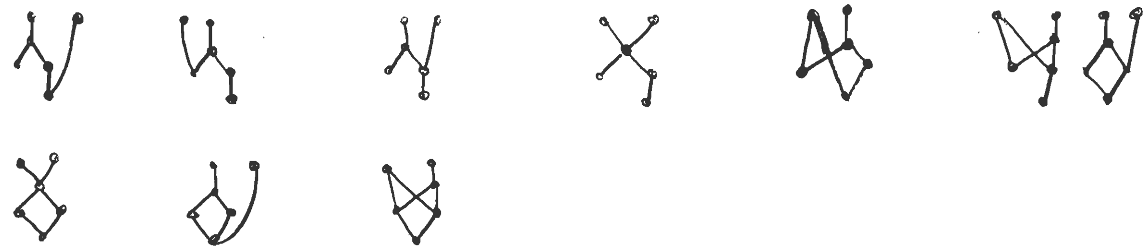
V

CONNECTED PARTIALLY ORDERED SYSTEMS OF 6 ELEMENTS

invariant

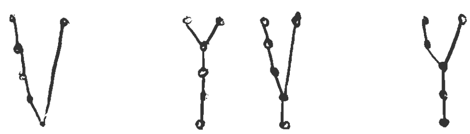
213

2121



2112

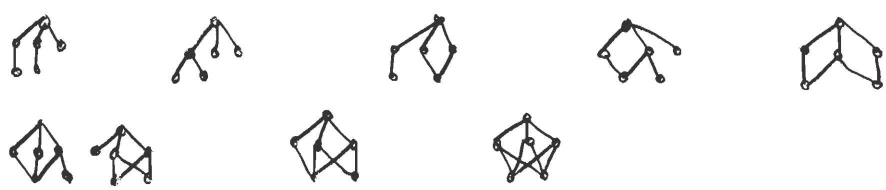
21111



15

141

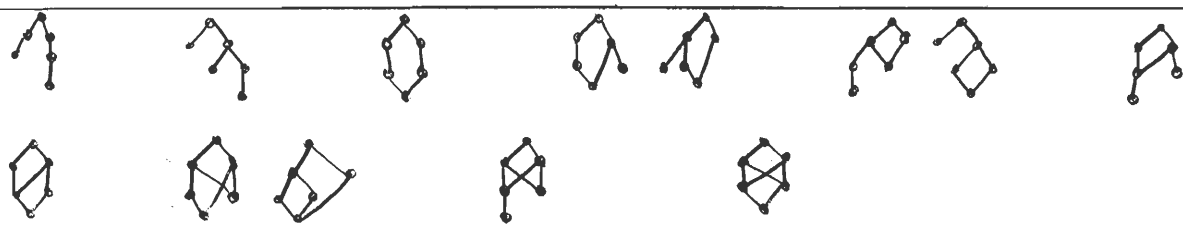
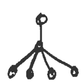
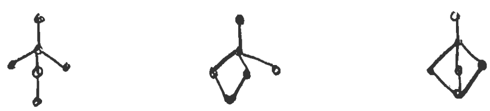
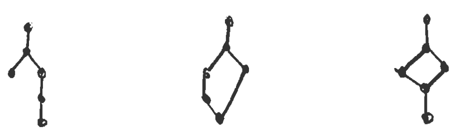
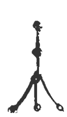
132



1311

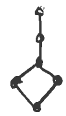
123

CONNECTED PARTIALLY ORDERED SYSTEMS OF 6 ELEMENTS

invariant	
1221	
1212	
12111	
114	
1131	
1122	
11211	
1113	

VII

CONNECTED PARTIALLY ORDERED SYSTEMS OF 6 ELEMENTS

invariant	
11121	 
11112	
111111