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# The Number of topologies on n points.

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Abstract: Upper and lower bounds are obtained for the number of all topologies and the number of $T_0$-topologies on n points. Various conjectures regarding the magnitudes of these numbers are also stated.

## § 1 Introduction:

Let p(n) be the total number of distinct (possibly homeomorphic) topologies on n points and let $p_0(n)$ be the number of $T_0$-topologies on n points. In this paper, I have tried to obtain reasonable upper and lower bounds for p(n) and $p_0(n)$. These are obtained and discussed in the next section. Clearly, the bounds are unsatisfactory but seem to be much better than what is known (compare [2]). I show that p(n) and $p_0(n)$ vary like $e^{cn^2}$. This is in sharp contrast to $B_n$(notation determined historically), the number of Borel-fields on n elements, since $B_n = o(e^{cn^2})$ for any $c>0$. See [5].

I consider the case of only finite cardinals n in this paper. If $\alpha$ is any infinite cardinal, it is known that $p(\alpha) = 2^{2^\alpha}$ (see Sierpinski [4] pp. 82) but I do not know of any discussion of the number of $T_0$, $T_1$,...,topologies. For a finite cardinal, of course, there is only one $T_1$, or better separated topology.

After the present article was completed, Mr. S. Johansen attracted my attention to the recently published article of Evans, Harary and

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Lynn "On the Computing enumeration of finite topologies" in Communications of ACM Vol. 10, No. 5, May, 1967. The present paper has much overlap with that article. My methods, however, seem more straight-forward. The values of $p(n)$ and $p_0(n)$ for $n=5,6,7$ in the table in the next section are taken from the above-mentioned paper. The other values for $n=1$ to $4$ were obtained by systematic enumeration following the discussion developed in the next section and tally with the figures in the above-mentioned paper. See also [2].

I should like to thank Mr. Richard Piotrowski for various helpful numerical calculations and Professor de Bruijn for several penetrating observations.

§ 2

Let S be a finite set containing n distinct elements labelled say, for convenience, by the integers 1,2,... n. Of all the possible equivalent ways of defining a topology on S, I shall choose the one by means of closure operators satisfying Kuratowski's postulates. Because of the finiteness of S, a closure operator is uniquely and completely determined by a function F from S to P(S), the set of all subsets of S, which satisfies the following properties:

(a) for all $x \epsilon S$, $x \epsilon F(x)$ and (b) if $y \epsilon F(x)$ then $F(y) \subset F(x)$.

If now $F(A)$, for $A \subset S$, is defined to be $\bigcup\{F(x) : x \epsilon A\}$ then F satisfies the usual closure postulates viz. $F(\emptyset) = \emptyset$ ($\emptyset$ = empty set), $F(A) \supset A$, $F(F(A)) = F(A)$, $F(A \cup B) = F(A) \cup F(B)$. The problem then is to count the number $p(n)$ of distinct functions F satisfying (a)

and (b) above. The function F induces a $T_0$-topology (definition: given $x,y$, $x \neq y$, $\exists$ an open set containing $x$ and not containing $y$ or vice versa) if and only if given $x,y,x \neq y$, either $x \notin F(y)$ or $y \notin F(x)$ or both. Let $p_0(n)$ be the number of $T_0$-topologies on S.

If we write now $x \mathrel{R} y$ if $x \in F(y)$, it is immediately verified that R defines a reflexive and transitive relation on S. Conversely, given a reflexive and transitive relation R on S, the function $F(x) = \{y | y \mathrel{R} x\}$ is a mapping of S to $\overline{P}(S)$ satisfying (a) and (b) above. Hence $p(n)$ is the same as the number of reflexive and transitive relations on S. Similarly it follows that $p_0(n)$ is the number of reflexive, anti-symmetric and transitive relationships on S, i.e. the number of partial orders on S. This latter fact is well-known see e.g. Birkhoff [1] pp. 14.

It is useful at this point to note that there is a simple relationship between $p(n)$ and $p_0(n)$ viz.

$$p(n) = \sum_{k=1}^{n} a_{n,k} p_0(k)$$

where $a_{n,k}$ = the number of distinct partitions of S into k disjoint, non-empty subsets. E.g. $a_{n,1}=1$, $a_{n,2}= 2^{n-1}-1, \ldots$ $a_{n,n}=1$. To prove this formula, one can argue as follows. Given a reflexive and transitive R, define $x \sim y$ if $y \mathrel{R} x$ and $x \mathrel{R} y$. This is clearly an equivalence relationship and on the partition induced by this equivalence, R induces a partial order. From here, the formula follows immediately. Explicit formulae for $a_{n,k}$ can be written

down but only the sum $B_n = \sum_{k=1}^{n} a_{n,k}$ will be needed. Then $B_n$'s are the so-called exponential numbers (also called Bell numbers or Euler numbers) and equal the number of distinct partitions of S (or equivalence relationships on S or algebras of sets on S). An estimate of $B_n$ will be mentioned later. For a recent report on the $B_n$'s see Rota [3].

I shall now obtain upper and lower bounds for $p_0(n)$. From these, bounds for $p(n)$ will be deduced.

I shall use the familiar device of Hasse diagrams slightly elaborated, for getting information about $p_0(n)$, the number of partial orders on S. Given a partial order $\leq$, define for $x \in S$,

$$d(x) = \max \{k | \quad x_1 < x_2 < \ldots < x_{k-1} < x\}.$$

Clearly (the inequality will not be used later) $1 \leq d(x) \leq |F(x)|$, where $|A|$ = cardinal number of A and F is the function introduced before which induces the $T_0$-topology which gives the partial order. If $d(x) = d(y)$, then x must be unrelated to y. Now define a I b (a immediately below b) if $a<b$ but for no c, $a<c<b$. Clearly, if a I b, then $d(b) \geq d(a) + 1$. Both strict equality or inequality are possible. Further, given $x \in S$, $d(x) \geq 2$ there is a $y \in S$ such that y I x, $d(y) = d(x) - 1$. Form now the Hasse diagram as follows. Let $m = \max \{d(x) | x \in S\}$, $1 \leq m \leq n$. Arrange the points of S in m rows, the x with $d(x) = k$, $1 \leq k \leq m$, being put in the kth now. Join x with y if and only if x I y or y I x. Such a diagram is fully characterized by the following description. Points are arranged in m non-empty

rows, points of the same row are never joined, each point of the ith row, $i \geq 2$, is joined to at least one point of the $(i-1)$st row and a point of the ith row can be joined to a point of the jth row, $j>i$, if and only if there is no path going through intermediate rows which joins them. Let $h_m(n)$ be the number of such m-rowed Hasse diagrams. Then clearly,

$$p_0(n) = \sum_{m=1}^{n} h_m(n) .$$

It is easy to see that $h_1(n) = 1$, $h_n(n) = 1$. On the other hand,

$$h_2(n) = \sum_{r=1}^{n-1} \binom{n}{r}(2^{n-r}-1)^r .$$

It is easily seen, by considering the maximum term in the sum for $h_2(n)$ that

$$h_2(n) > C \cdot 2^{n^2/4} \cdot \binom{n}{[\frac{n}{2}]}, \quad 0<C<1 .$$

Hence

$$p_0(n) > h_2(n) > C \cdot 2^{n^2/4} \binom{n}{[\frac{n}{2}]} .$$

It is useless to try to improve the lower bound for $p_0(n)$ by this method by considering $h_3(n)$, $h_4(n)$ etc. since they are all easily proved to be $\underline{o}(2^{\alpha n^2})$ $0<\alpha<\frac{1}{4}$. The further major contributions seem to come from $h_{k(n)}(n)$ where $k(n)$ is a suitable function of $n$.

I shall now obtain some upper bounds for $p_0(n)$. An easy one is $3^{\frac{n(n-1)}{2}}$, derived from the fact that this is the number of reflexive and anti-symmetric relationships on S. A better one will now be derived by using Hasse diagrams. The number of Hasse diagrams is clearly less than n! times the total number of undirected graphs

on n points which is $2^{\frac{n(n-1)}{2}}$. The multiplication by n! is necessary to allow for arranging the points in different rows ("colours"). Hence $p_0(n)<n!\cdot 2^{\frac{n(n-1)}{2}}$. Numerical evidence, (see table later) seems to indicate that $p_0(n)> 2^{\frac{n(n-1)}{2}}$ for n>1 but I have not been able to come even close to this.

Finally, the formula for p(n) in terms of $p_0(n)$ gives us, that $p_0(n)<p(n)<n!\cdot 2^{\frac{n(n-1)}{2}} B_n \quad n>1$ .

The general orders of magnitude of p(n) and $p_0(n)$ will now be expressed in the following statement which is merely a weak deduction from the above.

<u>Theorem</u>:

$$C\cdot 2^{n^2/4}\binom{n}{[\frac{n}{2}]}<p_0(n)<p(n)<n!\; 2^{\frac{n(n-1)}{2}} B_n \quad (n>1)\quad 0<C<1.$$

$$\lim_{n\to\infty} p_0(n)/2^{n^2/4}=\lim_{n\to\infty} p(n)/2^{n^2/4} = \infty$$

$$\lim_{n\to\infty} p_0(n)/2^{(\frac{1}{2}+\varepsilon)n^2} = \lim_{n\to\infty} p(n)/2^{(\frac{1}{2}+\varepsilon)n^2} = 0$$

for any $\varepsilon>0$.

The last statement follows from the fact that $B_n= o(e^{\alpha n^2})$ for any $\alpha>0$ (See Szekeres and Binet [5]). Of course, what one should like to have is that there is a constant $\alpha>0$ such that $2^{(\alpha+\varepsilon)n^2}$ is too large and $2^{(\alpha-\varepsilon)n^2}$ too small for $p_0(n)$ and p(n). It is to be strongly suspected that $\alpha=\frac{1}{2}$ is the right answer and actually the following table suggests the even stronger conjecture that for n>1

$$(n-1)\cdot 2^{\frac{n(n-1)}{2}}<p_0(n)<p(n)<n!\cdot 2^{\frac{n(n-1)}{2}} .$$

Table:

| n | A1035<br>$p_0(n)$ | A798<br>$p(n)$ | A110<br>$B_n$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 3 | 4 | 2 |
| 3 | 19 | 29 | 5 |
| 4 | 219 | 355 | 15 |
| 5 | 4,231 | 6,942 | 52 |
| 6 | 130,023 | 209,527 | 203 |
| 7 | 6,129,859 | 9,535,241 | 877 |

As a check for the accuracy of the figures in the table, note that $p_0(n)$ is always an odd number and $p(n)$ has the same parity as $B_n$. This follows by noting that if R is a partial order relationship on S, then the opposite partial order R' (i.e. x R' y iff y R x) is distinct from R in all cases except for the trivial partial order where x R y iff x = y. Hence $p_0(n) \equiv 1 \pmod 2$ and so $p(n) = \sum_{k=1}^{n} a_{n,k} \equiv B_n \pmod 2$. Following a conjecture of the author, Professor de Bruijn has shown that $B_n$ is even if and only if $n \equiv 2 \pmod 3$. This shows that $p(n)$ is even if $n \equiv 2 \pmod 3$ and odd otherwise. See [5] for references to tables of $B_n$. I should like to point out that the orders of magnitude for the non-homeomorphic topologies are still in the same neighbourhood as $p(n)$ or $p_0(n)$ (viz. $2^{\alpha n^2}$, $\frac{1}{4}<\alpha\leq\frac{1}{2}$) since they differ at most by a factor of n!.

It is easy to see that $p(n)>np(n-1)$ as also that $p_0(n)>np_0(n-1)$. Both these follow (and can be considerably improved) by the following argument, given for $T_0$-topologies. Given n points, choose any

one point (in n ways) and define its closure to be the whole set. For the rest of the points choose any $T_0$-topology. The whole still gives a $T_0$-topology and $p_0(n)>np_0(n-1)$ is proved. Similarly for $p(n)$. Hence

$$\lim_{n\to\infty} \frac{p(n)}{p(n-1)} = \lim_{n\to\infty} \frac{p_0(n)}{p_0(n-1)} = +\infty.$$

It also follows from $p(n)= \sum_{k=1}^{n} a_{nk}p_0(k)$ that $\lim_{n\to\infty} \{p(n)-p_0(n)\} = +\infty$.

Whether $\lim_{n\to\infty} \frac{p(n)}{p_0(n)} = +\infty$ or not, I do not know. I suspect that the limit is finite and between 1 and 2.

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