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[GU4]

A THEOREM IN PARTITIONS

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The following theorem is due to Glaisher [1,2], but it does not seem to be as widely known as it deserves to be. Four proofs are given, two combinatory and two algebraic, some of which may be new.

Let $p_e(n)$ be the number of partitions of a non-negative integer n into an even number of parts, or the number of partitions of n of which the largest part is even, and let $p_o(n)$ be the number of partitions of n into an odd number of parts, or the number of partitions of n of which the largest part is odd. Then

$$p_e(n) - p_o(n) = (-)^n p_w(n),$$

where $p_w(n)$ is the number of 'self-conjugate' partitions of n, or the number of partitions of n into odd, unequal parts. (Define $p_e(0) = p_w(0) = 1$, $p_o(0) = 0$.)

The first proof is similar to Franklin's [3,4,5,6] of Euler's analogous theorem [7,8] concerning partitions into *unequal* parts. The *Ferrers diagrams* [9,10] of all the partitions of n are considered and a 1-1 correspondence established between those with an even number of parts and those with an odd number of parts, except for an excess of one kind or the other, which correspond to the self-conjugate partitions, whose Ferrers diagrams are symmetrical about their leading diagonals.

Define the *face*, f, of a partition of n as the number of parts equal in size to the largest part, and the *base*, b, as the size of the smallest part. *I.e.*, b and f are positive integers.

If $b \neq f$, then one of them (considered as the last row or column) may be transferred to the other, but not *vice versa*; *e.g.* in the first diagram below, the base, $b = 2$, may be transferred to the face. This establishes a correspondence between two partitions, one with an even number of parts, the other with an odd number. Assume for the present that, after transfer, the new base is not equal to the new face.

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0 0 0 0 0 0 0 0 0            0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0    f = 3   0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0            0 0 0 0 0 0 0 0 0
0 0 0 0 0 0                  0 0 0 0 0 0
0 0 0 0 0 0                  0 0 0 0 0 0
0 0 0                        0 0 0
0 0                          0 0
                             0 0
  b = 2                      0 0
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$b = f = 2,\ h = 5,\ l = 3,\ s = 8,$

$b_2 = f_2 = 3,\ h_2 = 3,\ l_2 = 1,\ s_2 = 4,$

$b_3 = 6,\ f_3 = 5,\ i = 3,\ e_3 = 1,\ d = 5.$

If $b = f$ (before or after transfer) define the *head*, h, as the amount by which the largest parts (f of them) exceed the next largest part; the *leg*, l, as the number of parts equal in size ($b = f$) to the smallest part; and the *extremity*, s, as $h + l$. Then continued transfers produce $s + 1$ partitions (with heads $0,1,2, \ldots, s$ and legs $s, s-1, \ldots, 1, 0$ respectively). If s is odd, these may be put into corresponding pairs. Note that the preceding paragraph, in which $b \neq f$, may be included here as the case $s = 1$.

If $b = f$ and s is even, say $s = 2k$, then either k is even or odd. If k is even, the correspondences $(h,l) = (0,2k)$ with $(1, 2k-1)$, $(2, 2k-2)$ with $(3, 2k-3)$, $\ldots$, $(k-2, k+2)$ with $(k-1, k+1)$, $(k+1, k-1)$ with $(k+2, k-2)$, $\ldots$, $(2k-1, 1)$ with $(2k, 0)$, can be set up, leaving the partition with $(h,l) = (k,k)$, *i.e.* with symmetrical extremity, unpaired.

If k is odd, the partitions can be paired as before, except that the middle *three*, with $(h,l) = (k-1, k+1)$, (k,k) and $(k+1, k-1)$ now remain. There is an excess of one partition, with numbers of parts $k \pm 1$ in head and leg.

If the extremity is removed, leaving a *body*, the parity of the number of nodes, n, is not changed, since an even number, $2kb$, is removed. Moreover the extra, unpaired partition has the same parity of number of *parts* as the body, since k parts are removed when k is even, and $k \pm 1$ when k is odd.

Now treat the body in the same way as the original partition, defining a new base, b_2, and a new face, f_2. Either this leads to a complete pairing, or there is an excess of one, with symmetrical extremities, of the same parity of number of parts as the next body. This process is carried out i times ($i \geq 1$) until the final base and face intersect. If these are still equal, $b_i = f_i = d$, say, the residue is the *Durfee square* [11,12] of the original partition, *i.e.* the largest square, d^2, of nodes occurring in the (top left of the) Ferrers diagram. If $b_i \neq f_i$, then $e_i = |b_i - f_i|$ transfers are possible, resulting in $e_i + 1$ possible shapes for the final body, which may be paired, or almost paired, as before.

Conversely, any self-conjugate partition corresponds to an excess of one over this method of pairing. The excess will always have the same parity as the number of nodes in the Durfee square, and so the same parity as n itself. Hence the result.

A second combinatorial proof is suggested on noting that a partition of n into odd, unequal parts must have an odd number of parts if n is odd and an even number if n is even. A 1-1 correspondence between partitions of n with odd and even numbers of parts, except for such partitions, may be set up as follows.

In any partition, look for the largest even part, say r, and for the two largest equal parts, say q, q. It may happen that $r = q$;

this does not affect the argument. If $2q > r$ (including the case $r = 0$, where there is no even part) combine the two equal parts to produce a partition with a larger even part, and one less part. If $r \geqslant 2q$ (including the case $q = 0$, where the parts are all unequal) split the largest even part into two equal parts, producing a partition with at least as large equal parts and one more part. The only partitions to which the process does not apply are those with $r = q = 0$, *i.e.* with no even or equal parts. It can be seen

(a) that the process is unique,

(b) that it produces a partition with a number of parts of opposite parity from the original, and

(c) that when it is applied to the resulting partition, it restores the original one.

The required correspondence is established.

The two proofs are illustrated by the correspondences for $n = 9$, $p(9) = 30$, $p_w(9) = 2$, so there are $\frac{1}{2}(30 - 2) = 14$ pairs in each case. The self-conjugate partitions of 9 are 3^3 and 51^4, so these are not paired in the first proof:

9	71^2	41^5	21^7	72	521^2	321^4	63	431^2	52^2	32^21^2	54	432	32^3
81	61^3	31^6	1^9	621	421^3	2^21^5	531	3^21^3	42^21	2^31^3	4^21	3^221	2^41

In the second proof the partitions into odd unequal parts are 9 and 531 and the pairings in this case are

81	72	63	621	61^3	54	521^2	432	431^2	42^21	421^3	41^5	321^4	21^7
4^21	71^2	3^3	3^221	3^21^3	52^2	51^4	32^3	32^21^2	2^41	2^31^3	2^21^5	31^6	1^9 .

An immediate corollary is that $p(n)$ is of the same parity as $p_w(n)$, but this is obvious by pairing partitions with their conjugates. This does not, of course, give a pairing between partitions with odd and even numbers of parts.

Some algebraic identities are also implied. By removing the Durfee square, the generating function for $p_w(n)$ may be seen [13] to be

$$\sum_{n=0}^{\infty} p_w(n)x^n = 1 + \frac{x}{1-x^2} + \frac{x^4}{(1-x^2)(1-x^4)} + \frac{x^9}{(1-x^2)(1-x^4)(1-x^6)} +$$

$$+ \ldots = \sum_{n=0}^{\infty} x^{n^2} \prod_{m=1}^{n} (1-x^{2m})^{-1}, \text{ so that}$$

$$\sum_{n=0}^{\infty} (-)^n p_w(n)x^n = 1 - \frac{x}{1-x^2} + \frac{x^4}{(1-x^2)(1-x^4)} -$$

$$- \frac{x^9}{(1-x^2)(1-x^4)(1-x^6)} + - + - \ldots = \sum_{n=0}^{\infty} (-)^n x^{n^2} \prod_{m=1}^{n} (1-x^{2m})^{-1} \qquad \text{(A)}$$

or, since $p_w(n)$ is also the number of partitions into unequal odd parts,

$$\sum_{n=0}^{\infty} p_w(n)x^n = (1+x)(1+x^3)(1+x^5) \ldots = \prod_{m=1}^{\infty} (1+x^{2m-1}), \text{ so that}$$

$$\sum_{n=0}^{\infty} (-)^n p_w(n)x^n = (1-x)(1-x^3)(1-x^5) \ldots = \prod_{m=1}^{\infty} (1-x^{2m-1}). \qquad \text{(B)}$$

If we multiply and divide this last expression by

$$(1+x)(1+x^2)(1+x^4)(1+x^8) \ldots$$

and also by the same infinite product with x replaced in turn by x^3, x^5, x^7, x^9, ..., we may also write it in the form

$$\sum_{n=0}^{\infty} (-)^n p_w(n)x^n = \frac{1}{(1+x)(1+x^2)(1+x^3)(1+x^4)\ldots}$$

$$= \prod_{m=1}^{\infty} (1+x^m)^{-1}. \qquad \text{(C)}$$

Now the number of partitions of n into exactly m parts is equal to the number of partitions of $n-m$ into at most m parts, and so is the coefficient of x^{n-m} in the expansion of

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ \ldots\ (1-x^m)},$$

i.e. the coefficient of x^n in

$$\frac{x^m}{(1-x)(1-x^2)(1-x^3)\ \ldots\ (1-x^m)}.$$

So $\displaystyle\sum_{n=0}^{\infty} p_e(n)x^n = 1 + \frac{x^2}{(1-x)(1-x^2)} + \frac{x^4}{(1-x)(1-x^2)(1-x^3)(1-x^4)} +$

$$+ \ldots = \sum_{j=0}^{\infty} x^{2j} \prod_{m=1}^{2j} (1-x^m)^{-1},$$

and $\displaystyle\sum_{n=0}^{\infty} p_o(n)x^n = \sum_{j=1}^{\infty} x^{2j-1} \prod_{m=1}^{2j-1} (1-x^m)^{-1}$, so that

$$\sum_{n=0}^{\infty} \{p_e(n) - p_o(n)\}x^n = \sum_{n=0}^{\infty} (-x)^n \prod_{m=1}^{n} (1-x^m)^{-1} = 1 - \frac{x}{1-x} +$$

$$+ \frac{x^2}{(1-x)(1-x^2)} - \frac{x^3}{(1-x)(1-x^2)(1-x^3)} + \frac{x^4}{(1-x)(1-x^2)(1-x^3)(1-x^4)} -$$

$$- \frac{x^5}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)} + - + \ldots \qquad \text{(D)}$$

The theorem implies that (A) ≡ (B) ≡ (C) ≡ (D), so we have bases for algebraic proofs. For a third proof we show that (C) ≡ (D). Let

$$F(a) = \prod_{m=1}^{\infty} (1 + ax^m)^{-1} = \frac{1}{(1 + ax)(1 + ax^2)(1 + ax^3)\ \dots} =$$

$$\sum_{n=0}^{\infty} c_n a^n \prod_{m=1}^{n} (1 - x^m)^{-1} = 1 + \frac{c_1 a}{1 - x} + \frac{c_2 a^2}{(1 - x)(1 - x^2)} +$$

$$+ \frac{c_3 a^3}{(1 - x)(1 - x^2)(1 - x^3)} + \dots \qquad (c_0 = 1).$$

Then $F(ax) = \prod_{m=1}^{\infty} (1 + ax^{m+1})^{-1} = \frac{1}{(1 + ax^2)(1 + ax^3)(1 + ax^4)\ \dots} =$

$$= (1 + ax)F(a) = 1 + \frac{c_1 ax}{1 - x} + \frac{c_2 a^2 x^2}{(1 - x)(1 - x^2)} + \frac{c_3 a^3 x^3}{(1 - x)(1 - x^2)(1 - x^3)} +$$

$+ \dots = \sum_{n=0}^{\infty} c_n a^n \prod_{m=1}^{n} (1 - x^m)^{-1}$. Equating coefficients of a^n,

$$\frac{x c_{n-1}}{(1 - x)(1 - x^2)\ \dots\ (1 - x^{n-1})} + \frac{c_n}{(1 - x)(1 - x^2)\ \dots\ (1 - x^n)} =$$

$$= \frac{c_n x^n}{(1 - x)\ \dots\ (1 - x^n)} \qquad (n = 1,2,\dots), \quad c_n = -x c_{n-1}, \quad c_n = (-x)^n$$

$(n = 0,1,2,\dots)$. Therefore

$$\frac{1}{(1+ax)(1+ax^2)(1+ax^3)\ \ldots} = 1 - \frac{ax}{1-x} + \frac{a^2x^2}{(1-x)(1-x^2)} -$$

$$\frac{a^3x^3}{(1-x)(1-x^2)(1-x^3)} + \frac{a^4x^4}{(1-x)(1-x^2)(1-x^3)(1-x^4)} - + - + \quad (1)$$

Put $a = 1$ and the result follows. $a = -1$ gives the classical identity of Euler concerning the generating function for $p(n)$. In fact, since (C) is the generating function for $p(n)$, apart from the signs in the factors, we have a fourth and even more direct proof. Write it as

$$(1 - x^1 + x^{1+1} - x^{1+1+1} + x^{1+1+1+1} - \ldots)(1 - x^2 + x^{2+2} - x^{2+2+2} +$$
$$+ x^{2+2+2+2} - \ldots)(1 - x^3 + x^{3+3} - x^{3+3+3} + x^{3+3+3+3} - \ldots)\ \ldots$$

and we see that this enumerates the partitions of n, except that, whenever an odd number of any one size of part occurs, a minus sign is included, so that the contribution to x^n will be positive or negative according as the number of parts in the partition is even or odd. *I.e.* it is the generating function

$$\sum_{n=0}^{\infty} \{p_e(n) - p_o(n)\}x^n$$

in which we are interested.

If, in (1), we write $a = x^g$, we have two forms for the generating function

$$\sum_{n=0}^{\infty} \{p_e^{(g)}(n) - p_o^{(g)}(n)\}x^n,$$

where $p_e^{(0)}(n) = p_e(n)$ and $p_e^{(g)}(n)$ is the number of partitions of n into an even number of parts, each greater than g, and similarly for

$p_o^{(g)}(n)$, but this does not appear to lead to any simply stated generalization of the original theorem.

Table of values of $p_w(n)$ from $n = 0$ to 100.

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n	0	10	20	30	40	50	60	70	80	90	100
0	1	2	7	18	46	98	209	408	784	1433	2574
1	1	2	8	20	49	107	223	437	833	1523	
2	0	3	8	23	52	117	236	471	881	1621	
3	1	3	9	25	57	125	255	501	939	1717	
4	1	3	11	26	63	133	276	530	1004	1814	
5	1	4	12	29	68	144	294	568	1065	1925	
6	1	5	12	33	72	157	312	609	1126	2048	
7	1	5	14	35	78	168	335	647	1199	2166	
8	2	5	16	37	87	178	361	686	1279	2286	
9	2	6	17	41	93	192	385	732	1355	2425	

REFERENCES

1. J.W.L. Glaisher, *Proc. Roy. Soc.*, London, 24(1875-6), 250-259.
2. L.E. Dickson, History of the Theory of Numbers, Vol. 2 (1934), 129.
3. F. Franklin, *Comptes Rendus*, Paris, 92(1881), 448-450.
4. J.J. Sylvester, Collected Mathematical Papers, Vol. IV, 11-13.
5. E. Netto, Lehrbuch der Combinatorik, (1927), 165-167.
6. G.H. Hardy and E.M. Wright, An Introduction to the Theory of Numbers, (1954), 286-288.
7. L. Euler, *v.*, *e.g.*, P. Bachmann, Niedere Zahlentheorie, Part 2(1910), 163.
8. G.H. Hardy and E.M. Wright, *loc. cit.*, theorem 358.
9. N.M. Ferrers, *v.*, *e.g.*, J.J. Sylvester, *Phil. Mag.*, (4), 5(1853), 199-202 and Coll. Math. Papers, I, 595-598.

10. P.A. MacMahon, Combinatory Analysis, Vol. 2 (1916), 3.

11. W.P. Durfee, *John Hopkins Univ. Circular*, 2(1882), 23.

12. G.H. Hardy and E.M. Wright, *loc. cit.*, 281.

13. G.H. Hardy and E.M. Wright, *loc. cit.*, 279.

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