Modular Forms on $\mathrm{SL}_2(\mathbb{Z})$

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Let $k \in \mathbb{Z}$ and let $\mathrm{SL}_2(\mathbb{Z})$ denote the special linear group

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}.$$

A **modular form of weight** k is an analytic function f defined on the complex upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ that transforms under the action of $\mathrm{SL}_2(\mathbb{Z})$ according to the relation [1]

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

and whose Fourier series $f(z) = \sum_{n=-\infty}^{\infty} \gamma_n e^{2\pi i n z}$ satisfies $\gamma_n = 0$ for all $n < 0$. In particular, we have

$$f(z+1) = f(z), \qquad f(-1/z) = (-z)^k f(z).$$

If, additionally, we have $\gamma_0 = 0$, then f is a **cusp form of weight** k. Every nonconstant modular form has weight $k \geq 4$, where k is even, and every nonzero cusp form has weight $k \geq 12$. The set M_k of modular forms and the set S_k of cusp forms are finite-dimensional vector spaces over $\mathbb{C}$ with [2]

$$\dim(M_k) = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor & \text{if } k \equiv 2 \bmod 12, \\ \left\lfloor \frac{k}{12} \right\rfloor + 1 & \text{if } k \equiv 0, 4, 6, 8, 10 \bmod 12 \end{cases}$$

and $\dim(S_k) = \dim(M_k) - 1$ if $k \geq 12$. We will focus primarily on a specific basis element of S_{12}, leaving other aspects of this huge research area for later.

The **discriminant function** $\Delta : \mathbb{H} \to \mathbb{C}$, defined via

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{m=1}^{\infty} \tau(m) q^m$$

where $q = e^{2\pi i z}$ and $\tau : \mathbb{Z}^+ \to \mathbb{Z}$ is the **Ramanujan tau function** [3, 4, 5, 6, 7], can be proved to be a cusp form of weight 12. Nobody knows whether $\tau(m) \neq 0$

for all $m \geq 1$, but Mordell [8] proved that τ is a multiplicative function and Deligne [9, 10, 11] proved that $|\tau(p)| \leq 2\,p^{11/2}$ for any prime p. This implies that [12]

$$\tau(m) = O\left(m^{11/2+\varepsilon}\right)$$

as $m \to \infty$, for any $\varepsilon > 0$; further [13, 14, 15, 16, 17],

$$\liminf_{m\to\infty} m^{-11/2}\tau(m) = -\infty, \qquad \limsup_{m\to\infty} m^{-11/2}\tau(m) = \infty.$$

Let the **Hecke L-series** be

$$L_\Delta(z) = \sum_{m=1}^{\infty} \tau(m) m^{-z} = \prod_p \frac{1}{1-\tau(p)p^{-z}+p^{11-2z}}, \qquad \mathrm{Re}(z) > \tfrac{13}{2},$$

and its modification be

$$L_\Delta^*(z) = (2\pi)^{-z}\Gamma(z)L_\Delta(z).$$

Then $L_\Delta(z)$ can be extended to an entire function and the functional equation $L_\Delta^*(z) = L_\Delta^*(12-z)$ is satisfied everywhere. One can compute $L_\Delta(6) = 0.7921228386...$, for example, but it turns out that more can be said.

Define two constants [18, 19, 20]

$$\begin{aligned} \xi &= 30L_\Delta^*(6) = 0.0463463808... \\ &= 960(0.0000482774...) = 5(0.0092692761...), \end{aligned}$$

$$\begin{aligned} \eta &= 28L_\Delta^*(5) = 28L_\Delta^*(7) = 0.0457516089... \\ &= \tfrac{32}{15}(0.0214460667...) = \tfrac{2}{5}(0.1143790224...). \end{aligned}$$

It can be shown that the values of $L_\Delta^*(n)$ at even $2 \leq n \leq 10$ are rational multiples of ξ:

$$L_\Delta^*(4) = L_\Delta^*(8) = \tfrac{1}{24}\xi, \qquad L_\Delta^*(2) = L_\Delta^*(10) = \tfrac{2}{25}\xi,$$

and that the values of $L_\Delta^*(n)$ at odd $1 \leq n \leq 11$ are rational multiples of η:

$$L_\Delta^*(3) = L_\Delta^*(9) = \tfrac{1}{18}\eta, \qquad L_\Delta^*(1) = L_\Delta^*(11) = \tfrac{90}{691}\eta.$$

These can alternatively be written in terms of $L_\Delta(1)$ and $L_\Delta(2)$; see Table 1. Similar collapsing occurs at integer arguments $< k$ for the unique cusp forms of weight $k = 16$ and $k = 18$ [7]. An integral expression for $L_\Delta^*(n)$ is [21]

$$L_\Delta^*(n) = \frac{1}{i^{n-1}\pi^{11}} \int_0^1 \left(\int_v^1 \frac{du}{\sqrt{u(u-1)(u-v)}} \right)^{n-1} \left(\int_1^\infty \frac{du}{\sqrt{u(u-1)(u-v)}} \right)^{11-n} v\,(1-v)\,dv$$

where $n = 1, 2, ..., 11$ and i is the imaginary unit. The product $\xi\eta = 0.0021204214...$ also appears in the following [18, 19, 22, 23, 24]:

$$\begin{aligned} \lim_{x\to\infty} \frac{1}{x^{12}} \sum_{m\leq x} \tau(m)^2 &= \frac{2^3\pi^{11}}{3^4 5^2 7\, 11}\xi\eta = 0.0320070045... \\ &= \frac{2^8\pi^{11}}{3^4 5^8 7\, 11}(1.0353620568...) = \frac{1}{12}(0.3840840544...) \end{aligned}$$

which is an interesting asymptotic mean square result. In contrast, we know that [25, 26]

$$\sum_{m\leq x} \tau(m) = O\left(x^{35/6+\varepsilon}\right)$$

as $x \to \infty$, for any $\varepsilon > 0$, and that [27, 28]

$$\liminf_{x\to\infty} x^{-23/4} \sum_{m\leq x} \tau(m) = -\infty, \qquad \limsup_{x\to\infty} x^{-23/4} \sum_{m\leq x} \tau(m) = \infty,$$

but a more precise estimate of the mean apparently remains open. Moreover [0.2],

$$\sum_{m\leq x} |\tau(m)| = o\left(x^{13/2}\right)$$

as $x \to \infty$. See also [29, 30, 31].

Table 1. Values of $L_f(1)$, $L_f(2)$; f is the unique cusp form of weight $k = 12, 16, 18$

k	12	16	18
$L_f(1)$	0.0374412812...	0.5870144080...	−3.5316483054...
$L_f(2)$	0.1463745420...	1.6654560382...	−8.6783515629...

0.1. Congruence Subgroups. Given N to be a positive integer, define the following subgroup of the full modular group $SL_2(\mathbb{Z})$:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\}$$

and define a **weight** k **modular form of level** N exactly as before, with $SL_2(\mathbb{Z})$ replaced by $\Gamma_0(N)$. Clearly the preceding discussion applies to the case $N = 1$ and k free; we focus henceforth on the case $k = 2$ and N free. The first nonzero weight 2 cusp form has level 11:

$$f(z) = q \prod_{n=1}^{\infty} (1 - q^n)^2 \left(1 - q^{11n}\right)^2$$

whose Fourier coefficients coincide [32] with those of the L-series for the elliptic curve isogeny class $11A$. The next two cusp forms have level 14 and 15, corresponding to $14A$ and $15A$. On the one hand, not all cusp forms are linked to elliptic curves: the first counterexamples have level 22 and 23. On the other hand, the Taniyama-Shimura conjecture (proved by Wiles, Taylor, Diamond, Conrad & Breuil [33]) asserts that every elliptic curve E is linked to a cusp form with level N equal to the conductor of E.

Let $S_2(N)$ denote the vector space of weight 2 cusp forms of level N. The dimension $\delta_0(N)$ of $S_2(N)$ over $\mathbb{C}$ possesses a more complicated formula than earlier [34, 35, 36, 37, 38, 39]:

$$\delta_0(N) = 1 + \frac{\psi(N)}{12} - \frac{\nu_2(N)}{4} - \frac{\nu_3(N)}{3} - \frac{\chi(N)}{2}$$

where

$$\psi(N) = N \prod_{p|N} \left(1 + \frac{1}{p}\right), \qquad \chi(N) = \sum_{d|N} \varphi\left(\gcd\left(d, \frac{N}{d}\right)\right),$$

$$\nu_2(N) = \begin{cases} 0 & \text{if } 4|N, \\ \prod\limits_{p|N} \left(1 + \left(\frac{-4}{p}\right)\right) & \text{otherwise;} \end{cases} \qquad \nu_3(N) = \begin{cases} 0 & \text{if } 9|N, \\ \prod\limits_{p|N} \left(1 + \left(\frac{-3}{p}\right)\right) & \text{otherwise;} \end{cases}$$

$\varphi(N) = N \prod_{p|N} (1 - 1/p)$ is the Euler totient function [40], and $(-4/p)$, $(-3/p)$ are Kronecker-Jacobi-Legendre symbols [41]. We have asymptotic extreme results [36, 42]

$$\liminf_{N\to\infty} \frac{\delta_0(N)}{N} = \frac{1}{12}, \qquad \limsup_{N\to\infty} \frac{\delta_0(N)}{N \ln(\ln(N))} = \frac{e^\gamma}{2\pi^2}$$

and average behavior

$$\sum_{N \le y} \delta_0(N) = \frac{5}{8\pi^2} y^2 + o\left(y^2\right)$$

as $y \to \infty$. Similar dimension estimates can be found for the vector space $M_2(N)$ of weight 2, level N modular forms [43].

Define also the subgroup

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : a \equiv d \equiv 1 \bmod N \text{ and } c \equiv 0 \bmod N \right\}$$

and the corresponding weight 2 cuspidal vector space dimension $\delta_1(N)$. An analogous formula for $\delta_1(N)$ is known [36, 37, 43], with extreme results

$$\liminf_{N\to\infty} \frac{\delta_1(N)}{N^2} = \frac{1}{4\pi^2} < \frac{1}{24} = \limsup_{N\to\infty} \frac{\delta_1(N)}{N^2}$$

and average behavior

$$\sum_{N\leq y}\delta_1(N)=\frac{1}{72\zeta(3)}y^3+o\left(y^3\right)$$

as $y\to\infty$. Generalization to arbitrary integer weight k is also possible.

Let $D=1$ or D be a fundamental discriminant [44]. A **level** N**, weight** k **modular form** $f:\mathbb{H}\to\mathbb{C}$ with **Nebentypus character** $(D/\cdot)$ transforms according to

$$f\left(\frac{az+b}{cz+d}\right)=\left(\frac{D}{d}\right)(cz+d)^k f(z)\qquad\text{for all }\left(\begin{array}{cc}a&b\\c&d\end{array}\right)\in\Gamma_0(N).$$

The trivial case $D=1$ reduces to the earlier definition. For example, we have

$$(-15/d)|_{d=1,2,\ldots,15}=\{1,1,0,1,0,0,-1,1,0,0,-1,0,-1,-1,0\}.$$

It turns out that the vector space of cusp forms corresponding to $(N,k,D)=(15,3,-15)$ is two-dimensional, and that a certain basis element is given by [38, 45, 46, 47]

$$f(z)=q\prod_{n=1}^{\infty}\left(1-q^{3n}\right)^3\left(1-q^{5n}\right)^3+q^2\prod_{n=1}^{\infty}\left(1-q^{n}\right)^3\left(1-q^{15n}\right)^3.$$

This will be useful later [0.3]. Also, the vector space of cusp forms corresponding to $(N,k,D)=(6,4,1)$ is one-dimensional with basis element

$$g(z)=q\prod_{n=1}^{\infty}\left(1-q^{n}\right)^2\left(1-q^{2n}\right)^2\left(1-q^{3n}\right)^2\left(1-q^{6n}\right)^2,$$

which we likewise will see again.

0.2. Ramanujan Tau Function. Let us continue where we stopped earlier. It is conjectured that [48, 49, 50, 51, 52]

$$\sum_{m\leq x}|\tau(m)|\sim A\,x^{13/2}\left(\ln(x)\right)^{-1+8/(3\pi)}$$

as $x\to\infty$, for some constant $0<A<\infty$, whereas it is known that [50, 53]

$$\sum_{m\leq x}\tau(m)^4\sim B\,x^{23}\ln(x)$$

for some constant $0<B<\infty$. Numerical estimates of A and B would be good to see someday. We cannot hope for similar accuracy in estimating $\sum_{m\leq x}\tau(m)$ until the correct order of magnitude – conjectured to be $O\left(x^{23/4+\varepsilon}\right)$ – is established. Evidence that 23/4 is the best exponent includes the formula [54, 55, 56, 57, 58, 59, 60, 61]

$$\frac{1}{x}\int_1^x \left(\sum_{m\leq y}\tau(m)\right)^2 dy \sim C_\tau\, x^{23/2}$$

as $x\to\infty$, where [62, 63]

$$C_\tau = \frac{1}{50\pi^2}\sum_{k=1}^{\infty}\frac{\tau(k)^2}{k^{25/2}} = \frac{1.5882400955...}{50\pi^2}.$$

There are analogous formulas [55, 64, 65, 66, 67, 68, 69] for the error terms in the divisor and circle problems [70]:

$$\frac{1}{x}\int_1^x \left(\sum_{m\leq y}d(m) - y\ln(y) - (2\gamma-1)y\right)^2 dy \sim C_d\, x^{1/2},$$

$$\frac{1}{x}\int_1^x \left(\sum_{m\leq y}r(m) - \pi y\right)^2 dy \sim C_r\, x^{1/2}$$

where

$$C_d = \frac{1}{6\pi^2}\sum_{k=1}^{\infty}\frac{d(k)^2}{k^{3/2}} = \frac{\zeta(3/2)^4}{6\pi^2\zeta(3)} = 0.6542839775...,$$

$$C_r = \frac{1}{3\pi^2}\sum_{k=1}^{\infty}\frac{r(k)^2}{k^{3/2}} = \frac{16\zeta(3/2)^2\beta(3/2)^2}{3\left(1+2^{-3/2}\right)\pi^2\zeta(3)} = 1.6939569917...$$

and $\zeta(z) = L_1(z)$, $\beta(z) = L_{-4}(z)$ denote the Riemann zeta and Dirichlet beta functions, respectively [71, 72].

Returning finally to the problem of estimating $\tau(m)$ itself, we ask about the values of constants c_+, c_- for which [17]

$$0 < \limsup_{m\to\infty} m^{-11/2}\exp\left(\frac{-c_+\ln(m)}{\ln(\ln(m))}\right)\tau(m) < \infty,$$

$$-\infty < \liminf_{m\to\infty} m^{-11/2}\exp\left(\frac{-c_-\ln(m)}{\ln(\ln(m))}\right)\tau(m) < 0.$$

Is there a reason to doubt that $c_+ = c_-$?

0.3. Mahler's Measure. Before beginning, we observe that the Laurent polynomial equation

$$1 + x + \frac{1}{x} + y + \frac{1}{y} = 0$$

is isomorphic to the elliptic curve $15A8$ via the change of coordinates [73, 74]

$$(x, y) \mapsto \left(\frac{y}{x}, \frac{x^3 - y^2 - xy}{xy} \right).$$

Similarly, the equation

$$1 + x + \frac{1}{x} + y + \frac{1}{y} + xy + \frac{1}{xy} = 0$$

is isomorphic to the curve $14A4$, and the equation

$$-1 + x + \frac{1}{x} + y + \frac{1}{y} + xy + \frac{1}{xy} = 0$$

is isomorphic to the curve $30A1$. Such representations of elliptic curves (as polynomials in x, x^{-1}, y, y^{-1}) are especially attractive when symmetric in x, y as shown.

The **(logarithmic) Mahler measure** of a Laurent polynomial $P(x_1, x_2, \ldots, x_n) \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ is defined to be

$$m(P) = \int_0^1 \int_0^1 \cdots \int_0^1 \ln \left| P(e^{2\pi i \theta_1}, e^{2\pi i \theta_2}, ..., e^{2\pi i \theta_n}) \right| \, d\theta_1 d\theta_2 \cdots d\theta_n.$$

We studied $\exp(m(P))$ for univariate P in [75]; our focus here will be on the case $n \geq 2$. Smyth [76, 77] proved that

$$\begin{aligned} m(1 + x_1 + x_2) &= L'_{-3}(-1) = \frac{3\sqrt{3}}{4\pi} L_{-3}(2) = 0.3230659472... \\ &= \ln(1.3813564445...), \end{aligned}$$

$$\begin{aligned} m(1 + x_1 + x_2 + x_3) &= 14\zeta'(-2) = \frac{7}{2\pi^2}\zeta(3) = 0.4262783988... \\ &= \ln(1.5315470966...) \end{aligned}$$

and Rodriguez-Villegas [78, 79, 80] conjectured that

$$m(1 + x_1 + x_2 + x_3 + x_4) = -L'_f(-1) = \frac{675\sqrt{15}}{16\pi^5} L_f(4) = 0.5444125617...,$$

$$m(1+x_1+x_2+x_3+x_4+x_5) = -8L_g'(-1) = \frac{648}{\pi^6}L_g(5) = 0.6273170748...$$

where f, g are the cusp forms defined at the end of [0.1]. Deninger [81] conjectured that

$$\begin{aligned} m\left(1+x+\frac{1}{x}+y+\frac{1}{y}\right) &= L_{15A}'(0) = \frac{15}{4\pi^2}L_{15A}(2) = 0.2513304337... \\ &= \ln(1.2857348642...) \end{aligned}$$

and Boyd [74] conjectured that

$$\begin{aligned} m\left(1+x+\frac{1}{x}+y+\frac{1}{y}+xy+\frac{1}{xy}\right) &= L_{14A}'(0) = \frac{7}{2\pi^2}L_{14A}(2) = 0.2274812230... \\ &= \ln(1.2554338662...). \end{aligned}$$

The latter is the smallest known measure of bivariate polynomials; the former is the second-smallest known. Both conjectures can be rephrased in completely explicit terms [74]: If

$$\sum_{n=1}^{\infty} a_n q^n = q\prod_{k=1}^{\infty}\left(1-q^k\right)\left(1-q^{3k}\right)\left(1-q^{5k}\right)\left(1-q^{15k}\right),$$

$$\sum_{n=1}^{\infty} b_n q^n = q\prod_{k=1}^{\infty}\left(1-q^k\right)\left(1-q^{2k}\right)\left(1-q^{7k}\right)\left(1-q^{14k}\right)$$

then

$$\int_0^{2\pi}\int_0^{2\pi} \ln\left|1+2\cos(s)+2\cos(t)\right| ds\, dt = 15\sum_{j=1}^{\infty}\frac{a_j}{j^2},$$

$$\int_0^{2\pi}\int_0^{2\pi} \ln\left|1+2\cos(s)+2\cos(t)+2\cos(s+t)\right| ds\, dt = 14\sum_{j=1}^{\infty}\frac{b_j}{j^2}.$$

These integrals bear some resemblance to certain constants in [82]. Trivariate analogs of these two examples are [83, 84, 85]

$$m\left(1+x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}\right) = 0.3703929298... = \ln(1.4483035845...),$$

$$m\left(1+x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}+xy+\frac{1}{xy}+yz+\frac{1}{yz}+xyz+\frac{1}{xyz}\right) = 0.4798982839...$$

but no relation to special L-series values has yet been proposed. Other variations include [74, 85]

$$m\left(-1+x+\frac{1}{x}+y+\frac{1}{y}+xy+\frac{1}{xy}\right)=L'_{30A}(0)=\frac{15}{2\pi^2}L_{30A}(2)=0.6168709387...,$$

$$m\left(-1+x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}+xy+\frac{1}{xy}+yz+\frac{1}{yz}+xyz+\frac{1}{xyz}\right)=0.8157244463....$$

The third-smallest known measure of bivariate polynomials is [74, 84, 86]

$$m\left(-1+x+\frac{1}{x}-y-\frac{1}{y}+x^2y^2+\frac{1}{x^2y^2}\right)=0.2693386412...=\ln(1.3090983806...)$$

and the fourth-smallest known is [74, 84, 87]

$$\begin{aligned} m\left(1+x^2+\frac{1}{x^2}+y^2+\frac{1}{y^2}+xy+\frac{1}{xy}+x^2y^2+\frac{1}{x^2y^2}+\frac{y}{x}+\frac{x}{y}\right) &= 0.2743632972... \\ &= \ln(1.3156927029...). \end{aligned}$$

We emphasize that, of all the $m(P)$ formulas exhibited here, only Smyth's results are rigorously proved.

0.4. Klein's Modular Invariant. The only modular form $f:\mathbb{H}\to\mathbb{C}$ of weight 0 is a constant. (Assume, as at the beginning, that f is of level 1 and has trivial character.) What happens if we weaken our hypotheses on f? A **modular function** f is an $\mathrm{SL}_2(\mathbb{Z})$-invariant meromorphic function on $\mathbb{H}$ whose Fourier series $f(z)=\sum_{n=-\infty}^{\infty}\gamma_n q^n$ has at most finitely many $\gamma_n\neq 0$ for $n<0$. The set of modular functions can be proved to be a field, $\mathbb{C}(j)$, generated by Klein's **j-invariant** or **Hauptmodul** [1, 88, 89, 90, 91, 92]

$$j(z)=\frac{1}{Q}(1+256Q)^3=\frac{1}{R}\left(1+250R+3125R^2\right)^3=\sum_{m=-1}^{\infty}c(m)q^m$$

where

$$Q=q\prod_{n=1}^{\infty}\left(\frac{1-q^{2n}}{1-q^n}\right)^{24}=\frac{\Delta(2z)}{\Delta(z)},$$

$$R=q\prod_{n=1}^{\infty}\left(\frac{1-q^{5n}}{1-q^n}\right)^{6}=\left(\frac{\Delta(5z)}{\Delta(z)}\right)^{1/4}$$

and $c(-1)=1$, $c(0)=744$, $c(1)=196884$, $c(2)=21493760$, ... Moreover, j is the unique modular function having a simple pole with residue 1 at $q=0$. Closed-form expressions and asymptotics for $c(m)$ are known [93, 94, 95], akin to those for the number $p(m)$ of partitions of m [96]. Special values include

$$j(i) = 12^3, \quad j\left((1+i\sqrt{3})/2\right) = 0, \quad j\left((1+i\sqrt{163})/2\right) = (-640320)^3;$$

the latter, plus the fact that $j(z) \approx q^{-1} + 744$, is responsible for the surprising consequence that $e^{\pi\sqrt{163}}$ misses being an integer by less than 10^{-12}. More special values include

$$j\left((1+i\sqrt{15})/2\right) = x, \quad j\left((1+i\sqrt{23})/2\right) = y$$

where x, y have minimal polynomials $x^2 + 191025x - 121287375$ and $y^3 + 3491750y^2 - 5151296875y + 12771880859375$, respectively. (The class numbers $h_{-1} = h_{-3} = h_{-163} = 1$, $h_{-15} = 2$ and $h_{-23} = 3$ play a role here [44].) Schneider [97] proved that, if $j(z)$ is algebraic, then z is algebraic if and only if z is imaginary quadratic. It is also known that, if $q \in \mathbb{Q}$ is algebraic and $0 < |q| < 1$, then $j(z)$ is transcendental [98, 99, 100]. A connection between sporadic simple group theory and modular functions (on $\Gamma_0(N)$ and extensions) is beyond the scope of our study [101, 102, 103].

0.5. Addendum. The constants A and B, associated with $|\tau(m)|$ and $\tau(m)^4$, were estimated to be 0.0996 and 0.0026 respectively by Fel [104]. Rogers & Zudilin [105, 106] proved Deninger's conductor 15 conjecture and Brunault [107] & Mellit [108] proved Boyd's conductor 14 conjecture. We await word on Rodriguez-Villegas' two conjectures involving $L_f(4)$ and $L_g(5)$.

Here is a seemingly unrelated calculus problem. Let $f(x) = (\pi/4 - x)\ln(g(x))$ be integrable on $[0, \pi/4]$, then

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} \left(1 - \frac{2k}{n}\right) \ln\left[g\left(\frac{\pi k}{2n}\right)\right] &= \frac{8}{\pi^2} \lim_{n\to\infty} \sum_{k=1}^{\lfloor n/2 \rfloor} f\left(\frac{\pi k}{2n}\right)\left(\frac{\pi(k+1)}{2n} - \frac{\pi k}{2n}\right) \\ &= \frac{8}{\pi^2} \int_0^{\pi/4} f(x)dx\end{aligned}$$

(a limit of Riemann sums). As a simple example,

$$\lim_{n\to\infty} \prod_{k=1}^{\lfloor n/2 \rfloor} \left(\frac{n}{2k}\right)^{\frac{1}{n}\left(1-\frac{2k}{n}\right)} = e^{\frac{3}{8}}$$

after setting $g(x) = \pi/(4x)$ and exponentiating. As a more complicated example,

$$\begin{aligned}\lim_{n\to\infty} \prod_{k=1}^{\lfloor n/2 \rfloor} \cot\left(\frac{\pi k}{2n}\right)^{\frac{1}{n}\left(1-\frac{2k}{n}\right)} &= e^{\frac{7\zeta(3)}{2\pi^2}} = \exp(0.4262783988...) \\ &= \sqrt{2}\exp(0.0797048085...)\end{aligned}$$

after setting $g(x) = \cot(x)$. The latter appears in the asymptotics of what is called the Atiyah determinant from quantum physics [109].

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