MATHEMATICS

ON THE COEFFICIENTS OF THE MODULAR INVARIANT $J(\tau)$

BY

A. VAN WIJNGAARDEN

1. Introduction

In recent years several authors paid attention to properties and the computation of the Fourier coefficients of the modular invariant $J(\tau)$. H. S. ZUCKERMAN [9] showed a relationship with the partition function, which enabled him to compute the first 24 coefficients with relative ease, thus extending considerably the first tabulation by W. E. H. BERWICK [1], who gave the first 7 coefficients. In order to proceed in this way one needs, however, first tables of the partition function that go further than the existing ones of H. GUPTA [2], [3]. D. H. LEHMER [4] proved several congruences and relations between the coefficients themselves and between the coefficients, the divisor functions and Ramanujan's tau function, and computed, moreover, the 25-th coefficient. B. VAN DER POL [7] gave also many relations of that last type, and relations between $J(\tau)$ and the theta functions. In his paper also values of the next three coefficients are given that were computed by the author of this paper, the last of which being erroneous due to a regrettable coincidence. In the present paper a table of the first 100 coefficients is given, together with some new properties, found empirically and proved afterwards.

The author is indebted to professor VAN DER POL for many substantial contributions and to Miss G. BOTTERWEG, Miss H. C. HAGENAAR and Mr W. KLEIN for their share in the numerical work.

2. Relations between theta functions and $J(\tau)$

Let M_k be a function of the integer k and the three numbers x, y and z defined by

$$(2.1) \qquad M_k = x^k + y^k + z^k,$$

so that always $M_0 = 3$, and let, moreover, x, y and z be such that

$$(2.2) \qquad M_1 = 0.$$

Then we have

$$M_2 = x^2 + y^2 + z^2 = -2(xy + yz + zx),$$
$$M_3 = x^3 + y^3 + z^3 = 3xyz$$

so that

$$^1/_3\,M_3\,M_k = xyz\,(x^k + y^k + z^k),$$
$$^1/_2\,M_2\,M_{k+1} = -\,(xy + yz + zx)\,(x^{k+1} + y^{k+1} + z^{k+1}),$$

from which it follows by addition, and taking into account (2.1) and (2.2)

(2.3) $$M_{k+3} = {}^1/_2\,M_2\,M_{k+1} + {}^1/_3\,M_3\,M_k.$$

The solution of this recurrence relation that satisfies (2.2) can be written down in explicit form.

. Introducing

(2.4) $$I = ({}^1/_2\,M_2)^3\,({}^1/_3\,M_3)^{-2}$$

one finds after some tedious arithmetic

(2.5)
$$
\begin{cases}
M_0 = 3, \qquad M_1 = 0, \\[4pt]
M_{2k} = ({}^1/_2\,M_2)^k \displaystyle\sum_{h=0}^{[k/3]} \frac{2k}{k-h}\binom{k-h}{2h} I^{-h}, \qquad k > 0, \\[10pt]
M_{2k+1} = ({}^1/_2\,M_2)^{k+1/2}\,I^{-1/2} \displaystyle\sum_{h=0}^{[k-1/3]} \frac{2k+1}{k-h}\binom{k-h}{2h+1} I^{-h}, \qquad k > 0, \\[10pt]
M_{-k} = ({}^1/_2 M_2)^{-1/2 k} \displaystyle\sum_{h=0}^{[k/3]} (-)^h \frac{k}{k-2h}\binom{k-2h}{h} I^{1/2 k - h}, \qquad k > 0.
\end{cases}
$$

Inspecting the exponents of M_2 and I in these formulae it appears that the product $M_k M_{-k}$ takes a particularly simple form. In effect

(2.6)
$$
\begin{cases}
M_0 M_0 = 9, \qquad M_1 M_{-1} = 0, \\[6pt]
M_{2k} M_{-2k} = I^{k-[k/3]-[2k/3]} \displaystyle\sum_{h=0}^{[k/3]} \frac{2k}{k-h}\binom{k-h}{2h} I^{[k/3]-h} \\[10pt]
\qquad\qquad \cdot \displaystyle\sum_{h=0}^{[2k/3]} (-)^h \frac{2k}{2k-2h}\binom{2k-2h}{h} I^{[2k/3]-h}, \qquad k > 0. \\[12pt]
M_{2k+1} M_{-(2k+1)} = I^{k-[k-1/3]-[2k+1/3]} \displaystyle\sum_{h=0}^{[k-1/3]} \frac{2k+1}{k-h}\binom{k-h}{2h+1} I^{[k-1/3]-h} \\[10pt]
\qquad\qquad \cdot \displaystyle\sum_{h=0}^{[2k+1/3]} (-)^{h+1} \frac{2k+1}{2k+1-2h}\binom{2k+1-2h}{h} I^{[2k+1/3]-h}, \qquad k > 0.
\end{cases}
$$

Hence, $M_k M_{-k}$ is a polynomial in I of degree $[k/2]$ with integer coefficients, that for $I = 0$ takes the value 9 if $k \equiv 0 \pmod 3$ and vanishes if $k \not\equiv 0 \pmod 3$. More specifically

(2.7) $$M_k M_{-k} = \sum_{n=0}^{[k/2]} m_k(n)\,I^n,$$

where the coefficients $m_k(n)$ for $k = 0\,(1)13$ are given in the following table:

k	$m_k(0)$	$m_k(1)$	$m_k(2)$	$m_k(3)$	$m_k(4)$	$m_k(5)$	$m_k(6)$
0	9						
1	0						
2	0	2					
3	9	— 3					
4	0	— 8	2				
5	0	25	— 5				
6	9	— 12	— 9	2			
7	0	— 49	49	— 7			
8	0	96	— 40	— 8	2		
9	9	— 27	— 135	78	— 9		
10	0	— 150	355	— 100	— 5	2	
11	0	242	— 121	— 242	110	— 11	
12	9	— 48	— 828	892	— 201	— 0	2
13	0	— 338	1521	— 507	— 338	143	— 13

Now, introducing the Jacobian theta functions [1)]

$$\theta_1' = \theta_1'(q) = \{\partial/\partial z \, \theta_1(z, q)\}_{z=0}, \quad \theta_j = \theta_j(q) = \theta_j(0, q) \quad (j = 2, 3, 4),$$

it is seen from the first of the two fundamental relations

(2.8) $$\theta_2^4 - \theta_3^4 + \theta_4^4 = 0, \qquad \theta_1' = \theta_2 \theta_3 \theta_4,$$

that a set of x, y and z satisfying (2.2) is given by $x = \theta_2^4$, $y = -\theta_3^4$, $z = \theta_4^4$.

Accordingly

(2.9) $$M_k = \theta_2^{4k} + (-\theta_3^4)^k + \theta_4^{4k},$$

and, (confer [7]),

(2.10) $$I = 2^{-2}\, 3^3\, J(\tau).$$

Hence, the equations (2.6) or (2.7) are as many relations between theta functions and the absolute modular invariant $J(\tau)$. Most of these results can be found in [7]. They are given here because of the explicit form into which they have been brought, and, moreover, because slight errors in the values of some coefficients occur in formula (28) of [7], however without any further consequences.

3. Relations between theta functions and the generating functions of the partition functions

Some classical results are needed for our purpose. Be $p(n)$ the number of partitions of the natural number n into the sum of natural numbers without restrictions. If restrictions are imposed it is denoted by an index.

[1)] Here, and in what follows, q is considered as argument rather than τ $(q = e^{\pi i \tau})$, since only arithmetical properties are examined. The argument is, moreover, suppressed everywhere when it is q, but only used in cases like $\theta_4(q^4)$, where others might prefer $\theta_4(0, 4\tau)$. It seemed too hard to be consistent and to denote by J or $J(q)$ rather than by $J(\tau)$ a function that owes its name to its modular properties.

The following scheme is proposed. Restrictions on the character of the terms are denoted by a lower index, restrictions on the number of terms by an upper index. Moreover, e stands for *even*, o for *odd*, d for *different*, s for *same*, p for *prime*. A number, if need be together with algebraical symbols, has an obvious meaning, and other symbols may be introduced by additional conventions. Hence, $p_d(n)$ is the number of partitions of n into different (unequal) terms; $p_{do}(n)$ is the number of partitions of n into unequal odd terms; $p_d^o(n)$ is the number of partitions of n into an odd number of unequal terms; $p_{d>6}^3(n)$ is the number of partitions of n into three unequal terms each greater than six; $p_p^2(2n) > 0$, $n > 1$, expresses the conjecture of GOLDBACH, and so on. By definition, a value is attributed to a partition function of the argument 0, viz. the one that suits its generating function.

Some generating functions are in classical notation

$$
(3.1) \quad
\begin{cases}
q_0 = q_0(q) = \prod_{n=1}^{\infty} (1-q^{2n}) = \sum_{n=-\infty}^{\infty} (-)^n q^{n(3n+1)}, \\
q_1 = q_1(q) = \prod_{n=1}^{\infty} (1 + q^{2n}) = \sum_{n=0}^{\infty} p_d(n) q^{2n}, \\
q_2 = q_2(q) = \prod_{n=1}^{\infty} (1 + q^{2n-1}) = \sum_{n=0}^{\infty} p_{do}(n) q^n, \\
q_3 = q_3(q) = \prod_{n=1}^{\infty} (1-q^{2n-1}) = \sum_{n=0}^{\infty} (-)^n p_{do}(n) q^n,
\end{cases}
$$

where apparently by definition $p_d(0) = p_{do}(0) = 1$.

From the product expansions of the theta functions it follows that

$$
(3.2) \quad
\begin{cases}
\theta_1' = 2q^{1/4} q_0^3, & \theta_2 = 2q^{1/4} q_0 q_1^2, \\
\theta_3 = q_0 q_2^2, & \theta_4 = q_0 q_3^2.
\end{cases}
$$

Again, the functions of section 2 take the following form

$$
(3.3) \quad
\begin{cases}
M_2 = 2q_0^8 q_1^{-8}(2^8 q^2 q_1^{24} + 1), & M_3 = -2^4 3 q q_0^{12}, \\
I = 2^{-8} q^{-2} q_1^{-24}(2^8 q^2 q_1^{24} + 1)^3,
\end{cases}
$$

whereas the relations (2.8) read

$$
(3.4) \quad 2^4 q q_1^8 = q_2^8 - q_3^8, \qquad q_1 q_2 q_3 = 1.
$$

From (3.1) and (3.4) one derives easily

$$
(3.5) \quad
\begin{cases}
q_0(q^2) = q_0(q)q_1(q) = q_0(iq) q_1(iq), \\
q_2(q^2) = q_1^{-1}(iq), & q_3(q^2) = q_1^{-1}(q).
\end{cases}
$$

Hence, from (3.2) and (3.5) it follows

$$
(3.6) \quad \theta_3(q^2) = q_0(iq) q_1^{-1}(iq), \qquad \theta_4(q^2) = q_0(q) q_1^{-1}(q).
$$

At last a special combination will be examined, viz.

$$\theta_2^{-8}(q) + \theta_2^{-8}(iq) = 2^{-8}\, q^{-2}\, q_0^{-8}(q^2)\, \{q_1^{-8}(q) - q_1^{-8}(iq)\}$$
$$= 2^{-8}\, q^{-2}\, q_0^{-8}(q^2)\, \{q_3^{8}(q^2) - q_2^{8}(q^2)\}$$
$$= -\,2^{-4}\, q_0^{-8}(q^2)\, q_1^{8}(q^2) = -\,2^{-4}\, \theta_4^{-8}(q^4),$$

whence

$$(3.7) \qquad\qquad \theta_4^{-8}(q^4) = -\,2^4\, \{\theta_2^{-8}(q) + \theta_2^{-8}(iq)\}.$$

4. The coefficients of $J(\tau)$

As well known, the coefficients $c(n)$ in the expansion

$$(4.1) \qquad\qquad j(\tau) = 2^8\, I = 12^3\, J(\tau) = \sum_{n=-1}^{\infty} c(n)\, q^{2n}$$

are integers. There are many ways to compute the coefficients, each of one resulting in rather lengthy calculations if n is not very small. This is rather natural since the coefficients grow very rapidly with n, and the digits have to come from somewhere. LEHMER [4] has derived multiplicative relations that for special values of n involve a moderate amount of arithmetical operations but all on large numbers. LEHMER [4] and VAN DER POL [7] have given recurrence relations implying divisor functions and Ramanujan's tau-function. Typical examples are the following ones ([7], formula 54a)

$$(4.2) \qquad \begin{cases} \displaystyle\sum_{k=-1}^{n-1} c(k)\, \tau(n-k) = \frac{720}{691}\, \{91\, \sigma_{11}(n) + 600\, \tau(n)\}, \\[2.5ex] \displaystyle\sum_{k=-1}^{n-1} k c(k)\, \tau(n-k) = 24\, \sigma_{13}(n). \end{cases}$$

Here $\tau(n)$ is Ramanujan's function and $\sigma_{2m+1}(n)$ is the sum of the $(2m+1)$th powers of the divisors of n, if $m = 1, 2, 3, \ldots$, whereas by definition $\sigma_{2m-1}(0) = \frac{1}{2}\zeta(1-2m) = -B_{2m}/(4m)$ and B_{2m} are the Bernoullian numbers, $B_0 = 1$, $B_2 = 1/6$, $B_4 = -1/30\ \ldots$. The τ-function is well tabulated and the σ-functions are easily computed. Moreover, they are not very large. From both formulae $c(n)$ is found by repeating the same procedure. The second formula has moreover an advantage over the first one in that it yields $(n-1)c(n-1)$, whereas the first one yields $c(n-1)$ directly $(\tau(1) = 1)$, so that in the second case one has an important numerical check in the remainderless division by $n-1$.

Yet, another type of formulae exists, one of which will be investigated now. The third formula (2.7) together with (4.1) yields

$$(4.3) \qquad\qquad j(\tau) = 2^7\, M_2\, M_{-2}.$$

Now, M_2 is a very simple function. In effect (confer [7])

$$(4.4) \qquad\qquad M_2 = 480 \sum_{n=1}^{\infty} \sigma_3(n)\, q^{2n},$$

where it should be remembered that $\sigma_3(0) = 1/240$. Be analogously

$$(4.5) \qquad M_{-2} = \theta_2^{-8} + \theta_3^{-8} + \theta_4^{-8} = 2^{-8} \sum_{n=-1}^{\infty} d(n) q^{2n},$$

where all $d(n)$ are positive integers $(n \geqslant -1)$. Also,

$$(4.6) \qquad \theta_4^{-8} = 2^4 \sum_{n=0}^{\infty} s(n) q^{n},$$

where $s(0) = 2^{-4}$ and $s(n)$ is a positive integer for $n > 0$. Since $\theta_4(q) = \theta_3(-q)$, also

$$(4.7) \qquad \theta_3^{-8} = 2^4 \sum_{n=0}^{\infty} (-)^n s(n) q^n,$$

Similarly

$$(4.8) \qquad \theta_2^{-8} = -2^{-8} \sum_{n=-1}^{\infty} t(n) q^{2n},$$

where all $t(n)$ are integers $(n \geqslant -1)$ with alternating signs. Hence

$$(4.8) \qquad d(n) = 2^{13} s(2n) - t(n), \qquad n \geqslant -1,$$

where, of course, $s(-2) = 0$.

The coefficients $s(n)$ and $t(n)$ can be found from the expansions

$$(4.9) \qquad \theta_2 = 2q^{1/4} \sum_{n=0}^{\infty} q^{n(n+1)}, \qquad \theta_4 = \sum_{n=-\infty}^{\infty} (-)^n q^{n^2},$$

by applying the following well known artifice. Be $a(n)$ given coefficients for $n \geqslant 0$, and $a(0) \neq 0$. Then the coefficients $b(n)$ satisfying

$$\sum_{n=0}^{\infty} b(n) x^n = \left\{ \sum_{n=0}^{\infty} a(n) x^n \right\}^{m-1}$$

can be found by differentiating logarithmically, multiplying by the arising denominators and equating the coefficients of equal powers of x. Then one gets the recurrence relation

$$n\, a(0)\, b(n) = \sum_{k=1}^{n} (mk-n)\, a(k)\, b(n-k), \qquad n > 0, \quad b(0) = a(0)^{m-1}.$$

This device is particularly useful if the $a(n)$ form a lacunary sequence. In this way one obtains the recurrence relations

$$(4.10) \quad \begin{cases} n\, s(n) = -2 \sum_{k=1} (-)^k (n + 7k^2)\, s(n-k^2), \quad n > 0, \quad s(0) = 2^{-4}, \\ (n+1)\, t(n) = -\sum_{k=1} \left\{ n + 1 + 7\frac{k(k+1)}{2} \right\} t\left\{ n - \frac{k(k+1)}{2} \right\}, \quad n > -1, \; t(-1) = -1, \end{cases}$$

where k takes such values that $n - k^2 \geqslant 0$ resp. $n - k(k+1)/2 \geqslant -1$.

Both formulae allow numerical checks by the division by n, resp. $n + 1$.

Having obtained $s(n)$ and $t(n)$, $d(n)$ follows from (4.8) and hence $c(n)$ from (4.3), (4.4) and (4.5)

$$(4.11) \quad c(n) = 240 \sum_{k=0}^{n+1} \sigma_3(k)\, d(n-k) = d(n) + 240 \sum_{k=1}^{n+1} \sigma_3(k)\, d(n-k). \qquad \text{type 2}$$

The final summation is therefore rather simple, all terms being and $\sigma_3(k)$ being only a small factor. Of course, first $s(n)$, $t(n)$ and d to be computed but these functions are of some interest in th Moreover it will be shown that s can be expressed in t. To that end on combines (3.7) with (4.6) and (4.8):

$$\theta_4^{-8}(q^4) = 2^4 \sum_{n=0}^{\infty} s(n) q^{4n},$$

$$\theta_2^{-8}(q) + \theta_2^{-8}(iq) = -2^{-8} \left\{ \sum_{n=-1}^{\infty} t(n) q^{2n} + \sum_{n=-1}^{\infty} (-)^n t(n) q^{2n} \right\} =$$

$$= -2^{-8} \sum_{n=0}^{\infty} t(2n) q^{4n},$$

whence

(4.12)
$$s(n) = 2^{-8} t(2n), \qquad n \geqslant 0,$$

or together with (4.8)

(4.13)
$$d(n) = 2^5 t(4n) - t(n),$$

so that the computation of $c(n)$ may be completely based on that of $t(n)$ and $\sigma_3(n)$. In actual practice, if one wants to compute $c(n)$ up to a fixed upper value of n, N say, it is easier to compute $t(n)$ for $-1 \leqslant n \leqslant N$ and $s(n)$ for $N/2 \leqslant n \leqslant 2N$ directly from (4.10) than to compute $t(n)$ for $-1 \leqslant n \leqslant 4N$ and then to apply (4.12).

Actually (4.12) was found empirically. From the conjecture that it should hold followed then (3.7) which was proved afterwards in the given way.

This method of computing $c(n)$ has the disadvantage, however, that after the formation of $s(n)$ and $t(n)$ no internal checks occur in the formulae. It is therefore advisable to have an efficient numerical check on the values of $c(n)$ for all values of n.

5. Congruences satisfied by $c(n)$

One knows many congruences satisfied by the coefficients $c(n)$. LEHMER [4] proved (more than) that

(5.1)
$$(n+1) c(n) \equiv 0 \pmod{24},$$

(5.2)
$$c(n) \equiv 0 \pmod{5} \text{ if } n \equiv \pm 2 \pmod{5}.$$

J. LEHNER [5], [6] proved that

(5.3)
$$\begin{cases} c(n) \equiv 0 \pmod{2^{3\alpha+8} 3^{2\beta+3} 5^{\gamma+1} 7^{\delta} 11^{\varepsilon}} \text{ if } 1 < n \equiv 0 \pmod{2^{\alpha} 3^{\beta} 5^{\gamma} 7^{\delta} 11^{\varepsilon}}; \\ \alpha, \beta, \gamma, \delta \geqslant 1, \quad \varepsilon = 1, 2, 3. \end{cases}$$

Moreover he remarked that from the numerical values of $c(n)$ available to him ($n \leqslant 25$) followed that in the case that n is one of the five primes 2, 3, 5, 7 and 11 or a power of one of these primes (thus if $n = 2, 3, 4, 5,$

7, 8, 9, 11, 16, 25), the congruences (5.3) predict the exact power of that prime dividing $c(n)$, $(n \leqslant 25)$.

We shall now prove some other remarkable congruences. From the formulae of section 2 it follows that

$$J(\tau) = {}^1/_6 \, M_2^3 \, M_3^{-2} = {}^1/_6 \, (M_2^3 \, M_3) \, M_3^{-3} = 2^2 \, 3^{-2} \, M_3^{-3} \, M_9 - 2^2 \, 3^{-4}.$$

Combining this with the formulae of section 3 one gets

$$J(\tau) + 2^2 \, 3^{-4} = 2^2 \, 3^{-2} \, M_3^{-3} \, M_9 = 2^{-10} \, 3^{-5} \, q^{-3} \, (q_2^{72} - q_3^{72} - 2^{36} \, q^9 \, q_1^{72}),$$

and after introducing (4.1), with $c(-1) = 1$, $c(0) = 744$,

$$24 \, q^3 \left\{ 3q^{-2} + 35.71 + 3 + 3 \sum_{n=1}^{\infty} c(n) \, q^{2n} \right\} = {}^1/_2 \, (q_2^{72} - q_3^{72}) - 2^{35} \, q^9 \, q_1^{72}.$$

Now, in virtue of (3.1) it holds

$$q_h^{72} = q_h(q) \, q_h^{71}(q) \equiv q_h(q) \, q_h(q^{71}) \qquad (\mathrm{mod}\ 71), \quad (h = 1, 2, 3),$$

and moreover $2^{35} \equiv 1 \ (\mathrm{mod}\ 71)$, so that

$$q + q^3 + \sum_{n=1}^{\infty} c(n) \, q^{2n+3} \equiv {}^1/_2 \left\{ q_2(q) \, q_2(q^{71}) - q_3(q) \, q_3(q^{71}) \right\}$$
$$- q^9 \, q_1(q) \, q_1(q^{71}) \qquad (\mathrm{mod}\ 71).$$

Writing, for a moment, $c'(n) = c(n)$ if $n \neq 0$, $c'(0) = 1$, then together with (3.1) again, one gets

$$\sum_{n=1}^{\infty} c'(n) \, q^{2n+3} \equiv {}^1/_2 \left\{ \sum_{n=0}^{\infty} p_{do}(n) \, q^n \sum_{n=0}^{\infty} p_{do}(n) \, q^{71n} \right.$$
$$\left. - \sum_{n=0}^{\infty} (-)^n \, p_{do}(n) \, q_3^n \sum_{n=0}^{\infty} (-)^n \, p_{do}(n) \, q^{71n} \right\}$$
$$- q^9 \sum_{n=0}^{\infty} p_d(n) \, q^{2n} \sum_{n=0}^{\infty} p_d(n) \, q^{142n} \qquad (\mathrm{mod}\ 71).$$

By equating the coefficients of the terms containing q^{2n+3} one finds then at last

$$(5.4) \qquad \left\{ c(n) \equiv \sum_{k=0}^{[2n+3/71]} p_{do}(k) \, p_{do}(2n+3-71k) - \sum_{k=0}^{[n-3/71]} p_d(k) \, p_d(n-3-71k) \right.$$
$$(\mathrm{mod}\ 71), \quad n \neq 0.$$

The two partition functions $p_d(n)$ and $p_{do}(n)$ have been tabulated by G. N. WATSON [8], up to $n = 400$. They are, moreover, easily computable and relatively small numbers. The right hand side of (5.4) contains only very few terms, even less than it appears at first sight since $p_{do}(2) = 0$. For instance, for $n \leqslant 109$, (5.4) runs in full

$$c(n) \equiv p_{do}(2n+3) + p_{do}(2n+3-71) - p_d(n-3) - p_d(n-3-71)$$
$$(\mathrm{mod}\ 71), \quad 0 \neq n \leqslant 109.$$

For $n = 100$, for instance, one has

$$c(100) \equiv 354357 + 14157 - 345856 - 165 = 22493 \equiv 57 \pmod{71},$$

which involves really small numbers only in comparison to $c(100)$ itself which is a number of 53 decimal digits.

In the derivation of (5.4) M_9 was introduced. If instead one introduces M_6 and proceeds along the same lines one gets the analogous congruence

$$(5.5) \quad \left\{ c(n) \equiv \sum_{k=0}^{[2n+2/47]} p_{do}(k)\, p_{do}(2n+2-47k) + \sum_{k=0}^{[n-2/47]} p_d(k)\, p_d(n-2-47k) \right. \\ \pmod{47}, \quad n \neq 0.$$

The analogous congruence modulo 23 is more complicated and will be omitted. From the numerical data it appears that similar congruences must exist modulo 41 and modulo 59. Actually the existence of (5.4) and (5.5) was also first found empirically.

6. The computation of $c(n)$

The coefficients $c(n)$ were computed for $n = -1\,(1)\,100$ from VAN DER POL's formula [7, (38)]

$$(6.1) \qquad J(\tau) = {}^2\!/_{27}\, (\theta_2^8 + \theta_3^8 + \theta_4^8)\, (\theta_2^{-8} + \theta_3^{-8} + \theta_4^{-8}),$$

What is the explicit form of (4.3), using the methods described in section 4.

The function $s(n)$ was computed for $n = 0\,(1)\,200$ and the function $t(n)$ for $n = -1\,(1)\,100$ by means of the selfchecking relations (4.10). They were checked moreover by means of (4.12). From (4.8) and (4.11) followed $d(n)$ and $c(n)$ for $n = -1\,(1)\,100$. This whole part was checked by duplication. Next $c(n)$ was computed from the second selfchecking equation (4.2) for $n = 1\,(1)\,50$ as an independent check. At last all values of $c(n)$ were submitted to the congruence-checks (5.4) and (5.5). For special values of n the congruences (5.2) and (5.3) were used as checks. The congruence (5.1), although valid for all n, was of little use since in both ways of computing the $c(n)$ the factor 24 plays a rôle.

With regard to the remark following (5.3), it is interesting to see whether Lehner's statement also applies to the new powers of primes (< 13) that are available now (viz. $n = 27, 32, 49, 64, 81$). This appears to be the case indeed. This phenomenon is rather interesting. In order to have a further check, $c(128)$ was calculated modulo 2^{30} from Lehmer's duplication formula [4]. It was found that $c(128) \equiv 2^{29} \pmod{2^{30}}$ so that also in this case Lehner's statement holds.

The values of $c(n)$ are given in the following table.

A521

$c(n)$	n
1	−1
744	0
1 96884	1
214 93760	2
8642 99970	3
2 02458 56256	4
33 32026 40600	5
425 20233 00096	6
4465 69940 71935	7
40149 08866 56000	8
3 17644 02297 84420	9
22 56739 33095 93600	10
146 21191 14995 19294	11
874 31371 96857 75360	12
4872 01011 17981 42520	13
25497 82738 94105 25184	14
1 26142 91646 57818 43075	15
5 93121 77242 14450 58560	16
26 62842 41315 07752 45160	17
114 59912 78844 47865 13920	18
474 38786 80123 41688 13250	19
1894 49976 24889 33900 28800	20
7318 11377 31813 75192 45696	21
27406 30712 51362 46549 29920	22
99710 41659 93718 26935 33820	23
3 53074 53186 56142 70998 77376	24
12 18832 84330 42251 04333 51500	25
41 07899 60190 30790 91576 38144	26
135 35635 41518 64687 86750 77500	27
436 56892 24858 87663 46104 01280	28
1379 83758 34642 99992 55422 88376	29
4278 07822 44213 26256 70582 27200	30
13023 36938 25770 29512 80448 73221	31
38960 80061 70995 91189 43000 98560	32
1 14632 93989 00810 63777 96110 90240	33
3 31962 77091 39267 16726 36796 06784	34
9 46816 61357 02260 43164 62634 38600	35
26 61436 58257 53796 26887 21518 75584	36
73 77316 99697 25069 76080 17928 54360	37
201 76878 99472 28738 64858 00437 76000	38
544 76388 17516 16630 12316 54104 77688	39
1452 68925 44393 62169 79435 54293 76000	40
3827 76775 17393 63485 06559 83311 30120	41
9970 41660 02174 43268 73940 99688 24320	42
25683 33470 63954 06994 77401 18663 19670	43
65452 36773 14992 68312 17028 36951 44960	44
1 65078 82156 81861 74782 49628 31551 42200	45
4 12189 63080 52167 73489 54445 72343 33696	46
10 19253 51589 15767 91938 65201 10914 37835	47
24 96774 10595 07166 92603 31512 31996 72320	48
60 60574 41541 37209 99542 37822 28126 50932	49
145 81598 45321 50199 97540 39132 61539 84000	50

c(n)								n
347	82974	25351	24906	52111	11193	03264	16268	51
822	82309	23604	86379	46346	57066	92508	05760	52
1930	75525	46782	25741	67329	52965	87752	61720	53
4494	97224	12333	74771	55078	53776	07541	22752	54
10384	83010	58794	97940	68925	15368	59324	35825	55
23814	07585	30992	24134	99951	81283	96335	84128	56
54214	49889	87656	47230	00378	95797	97720	88000	57
1 22553	65475	04082	06615	35516	23305	01657	60000	58
2 75134	11092	85948	64606	92553	08616	87146	59374	59
6 13542	89505	30361	36170	69338	27228	48587	77600	60
13 59250	92428	36550	38097	01809	16661	62894	74168	61
29 92109	83800	07688	36650	74958	85452	33318	70720	62
65 45530	43491	65030	30643	85476	04156	99053	65270	63
142 31976	35972	71606	23108	02114	65424	36536	81152	64
307 60954	73477	19676	30396	15540	12847	95239	17200	65
661 00917	73782	87162	74459	09215	08064	15869	54240	66
1412 35833	72861	18490	82870	80245	89187	32135	44410	67
3001 00414	97911	12962	58941	10839	46623	40095	18080	68
6341 98425	35335	41630	77601	14920	60361	94613	13664	69
13331 26252	93210	23532	85518	96736	23687	92354	81600	70
27877 50248	90624	32847	67184	93296	34876	93051	98947	71
57998 94663	06862	70977	78971	24287	02702	89346	56000	72
1 20064 76859	24154	07996	57067	63561	79539	59481	73320	73
2 47334 29811	83106	50913	62656	13239	67886	40929	91488	74
5 07071 19308	98997	08057	00789	06280	84219	65198	46750	75
10 34690 66408	50426	35622	63168	39259	82257	41159	46496	76
21 01594 58102	75143	25069	10589	02482	07991	00864	59520	77
42 49352 00246	86459	96896	93275	41404	17894	12398	69440	78
85 53998 18184	24975	89405	37694	48098	79684	98086	43878	79
171 44484 30238	56632	32305	05079	66626	55430	46332	41600	80
342 15552 55551	89176	73198	38691	23583	94201	19784	93364	81
679 98684 36672	14052	17195	40980	18582	52260	99449	65120	82
1345 82384 70689	81684	95259	62168	82155	84589	79008	27370	83
2652 88632 13847	03560	25223	21296	59440	09217	23815	85408	84
5208 62134 25202	53933	69315	34883	96012	72044	83857	83600	85
10186 63549 71409	56830	21681	12072	29975	61148	07976	01792	86
19845 94685 77153	87241	69587	80804	25504	86362	87388	82125	87
38518 94383 02834	97365	36939	13362	43138	88225	01457	92000	88
74484 51892 92890	17811	71998	98327	68142	07693	12594	10120	89
1 43507 17246 72834	53885	51522	23427	82991	19235	32076	03200	90
2 75501 04261 67891	53749	08061	78938	36796	95113	39297	83496	91
5 27036 05805 32817	64188	08922	00416	29201	19197	55057	56160	92
10 04730 45344 09390	42843	89896	53654	12981	69030	71458	27840	93
19 08864 09832 13103	02488	60473	90986	18405	93893	84773	79584	94
36 14432 17930 44626	81879	67680	91204	64684	97513	08362	05250	95
68 21306 83268 93807	76546	62982	56534	65084	00341	84769	04448	96
128 31568 45093 05662	37049	15719	10171	04861	21743	36342	89960	97
240 60143 44493 76049	97591	58609	03804	73418	08640	16968	39680	98
449 72195 69801 18067	40150	81827	51777	54986	40947	29105	49646	99
837 98831 11070 74769	12751	95038	47574	52703	80191	83390	72000	100

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