Integer Partitions

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Let L denote the positive octant of the regular d-dimensional cubic lattice. Each vertex $(j_1, j_2, \ldots, j_d)$ of L is adjacent to all vertices of the form $(j_1, j_2, \ldots, j_k + 1, \ldots, j_d)$, $1 \leq k \leq d$. A **d-partition** of a positive integer n is an assignment of nonnegative integers $n_{j_1,j_2,\ldots,j_d}$ to the vertices of L, subject to both an ordering condition

$$n_{j_1,j_2,\ldots,j_d} \geq \max_{1\leq k\leq d} n_{j_1,j_2,\ldots,j_k+1,\ldots,j_d}$$

and a summation condition $\sum n_{j_1,j_2,\ldots,j_d} = n$. The summands in the d-partition are thus nonincreasing in each of the d lattice directions. We agree to suppress all zero labels. A 1-partition is the same as an ordinary partition; a 2-partition is often called a **plane partition** and a 3-partition is often called a **solid partition**. Three sample plane partitions of $n = 26$ are

$$\begin{pmatrix} 8 \\ 9 \\ 9 \end{pmatrix}, \qquad \begin{pmatrix} 1 & & & \\ 1 & & & \\ 2 & 2 & 1 & \\ 4 & 2 & 1 & 1 \\ 5 & 3 & 2 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 7 & 6 & 4 & 4 & 3 & 1 & 1 \end{pmatrix}.$$

Let $p_d(n)$ denote the number of d-partitions of n. The generating functions [1]

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} p_1(n)x^n &= 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + \cdots \\ &= \prod_{m=1}^{\infty} (1 - x^m)^{-1}, \end{aligned}$$

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} p_2(n)x^n &= 1 + x + 3x^2 + 6x^3 + 13x^4 + 24x^5 + 48x^6 + 86x^7 + 160x^8 + \cdots \\ &= \prod_{m=1}^{\infty} (1 - x^m)^{-m} \end{aligned}$$

give rise to well-known asymptotics [2, 3, 4, 5]:

$$\begin{aligned} p_1(n) &\sim \frac{1}{4\sqrt{3}n}\exp\left(\pi\sqrt{\frac{2n}{3}}\right) \\ &\sim (0.1443375672...)n^{-1}\exp\left((2.5650996603...)n^{1/2}\right), \end{aligned}$$

$$\begin{aligned} p_2(n) &\sim \frac{\zeta(3)^{7/36}e^{\zeta'(-1)}}{2^{11/36}\sqrt{3\pi}n^{25/36}}\exp\left(3\zeta(3)^{1/3}\left(\frac{n}{2}\right)^{2/3}\right) \\ &\sim (0.2315168134...)n^{-25/36}\exp\left((2.0094456608...)n^{2/3}\right) \end{aligned}$$

as $n\to\infty$, where $\zeta(3)=1.2020569031...$ is Apéry's constant [6] and $\zeta'(-1)=-0.1654211437...=2(-0.0827105718...)=\ln(0.8475366941...)$ is closely related to the Glaisher-Kinkelin constant [7]. Although an infinite product expression for the generating function [1]

$$1+\sum_{n=1}^{\infty}p_3(n)x^n = 1+x+4x^2+10x^3+26x^4+59x^5+140x^6+307x^7+684x^8+\cdots$$

remains unknown, it is conjectured that [8, 9]

$$\begin{aligned} p_3(n) &\sim \frac{C}{n^{61/96}}\exp\left(\frac{2^{7/4}\pi}{3^{5/4}5^{1/4}}n^{3/4}+\frac{\sqrt{15}\zeta(3)}{\sqrt{2}\pi^2}n^{1/2}-\frac{15^{5/4}\zeta(3)^2}{2^{7/4}\pi^5}n^{1/4}\right) \\ &\sim C\,n^{-61/96}\exp\left((1.7898156270...)n^{3/4}+(0.3335461354...)n^{1/2}-(0.0414392867...)n^{1/4}\right) \end{aligned}$$

for some constant $C>0$. The evidence for this asymptotic formula includes exact enumerations (for $n\le 68$) and Monte Carlo simulation. See [10, 11, 12, 13] for more about planar partitions and [14, 15, 16, 17] for more about solid partitions.

0.1. Hardy-Ramanujan-Rademacher. The Hardy-Ramanujan-Rademacher formula for $p_1(n)$ is a spectacular exact result [18, 19, 20, 21, 22, 23, 24, 25, 26]:

$$p_1(n)=\frac{\pi}{2^{5/4}3^{3/4}}\left(n-\frac{1}{24}\right)^{-3/4}\sum_{k=1}^{\infty}\frac{A_k(n)}{k}I_{3/2}\left(\sqrt{\frac{2}{3}}\frac{\pi}{k}\sqrt{n-\frac{1}{24}}\right)$$

where

$$I_{3/2}(x)=\sqrt{\frac{2x}{\pi}}\left(\frac{\cosh(x)}{x}-\frac{\sinh(x)}{x^2}\right)$$

is the modified Bessel function of order 3/2,

$$A_k(n) = \sum_{\substack{\gcd(h,k)=1,\\ 1\le h<k}} \omega_{h,k} \exp\left(\frac{-2\pi i n h}{k}\right),$$

and $\omega_{h,k} = \exp(\pi i s(h,k))$ is the unique $24k^{\text{th}}$ root of unity with Dedekind sum

$$s(h,k) = \sum_{m=1}^{k-1} \left(\frac{m}{k} - \left\lfloor\frac{m}{k}\right\rfloor - \frac{1}{2}\right)\left(\frac{hm}{k} - \left\lfloor\frac{hm}{k}\right\rfloor - \frac{1}{2}\right).$$

For example,

$$A_1(n) = 1, \qquad A_2(n) = (-1)^n, \qquad A_3(n) = 2\cos\left(\frac{\pi(12n-1)}{18}\right),$$

$$A_4(n) = 2\cos\left(\frac{\pi(4n-1)}{8}\right), \qquad A_5(n) = 2\cos\left(\frac{\pi(2n-1)}{5}\right) + 2\cos\left(\frac{4\pi n}{5}\right).$$

Defining

$$c = \sqrt{\frac{2}{3}}\pi, \qquad \lambda(n) = \sqrt{n - \frac{1}{24}},$$

$$\mu(n) = c\lambda(n), \qquad A_k^*(n) = A_k(n)/\sqrt{k},$$

we have the following variations:

$$\begin{aligned} p_1(n) &= \frac{1}{2^{1/2}\pi} \sum_{k=1}^{\infty} A_k(n) k^{1/2} \frac{d}{dn}\left[\frac{\sinh(c\lambda(n)/k)}{\lambda(n)}\right] \\ &= 2\frac{3^{1/2}}{24n-1} \sum_{k=1}^{\infty} A_k^*(n)\left[\left(1 - \frac{k}{\mu(n)}\right)\exp\left(\frac{\mu(n)}{k}\right) + \left(1 + \frac{k}{\mu(n)}\right)\exp\left(-\frac{\mu(n)}{k}\right)\right]. \end{aligned}$$

In contrast, the original Hardy-Ramanujan formula is only an asymptotic expansion:

$$\begin{aligned} p_1(n) &\sim \frac{1}{2^{3/2}\pi} \sum_{k=1}^{\infty} A_k(n) k^{1/2} \frac{d}{dn}\left[\frac{\exp(c\lambda(n)/k)}{\lambda(n)}\right] \\ &\sim 2\frac{3^{1/2}}{24n-1} \sum_{k=1}^{\infty} A_k^*(n)\left(1 - \frac{k}{\mu(n)}\right)\exp\left(\frac{\mu(n)}{k}\right), \end{aligned}$$

which was later proved to be divergent by Lehmer [27, 28, 29]. Therefore Rademacher's contribution was the identification of a small additional term that forces the original Hardy-Ramanujan series to converge.

A third formula for $p_1(n)$:

$$p_1(n) \sim \frac{\pi}{2^{5/4}3^{3/4}}\lambda(n)^{-3/2}\sum_{k=1}^{\infty}\frac{A_k(n)}{k}I_{-3/2}\left(\frac{c\lambda(n)}{k}\right)$$

appears in Almkvist [30, 31] and is a consequence of a more general theory (to be discussed shortly). The only difference between this formula and the Hardy-Ramanujan-Rademacher formula is that $I_{-3/2}$ appears rather than $I_{3/2}$. It is believed to be divergent, but this has not yet been proved. For practical purposes, using the modified Bessel function of order $-3/2$:

$$I_{-3/2}(x) = \sqrt{\frac{2x}{\pi}}\left(\frac{\sinh(x)}{x} - \frac{\cosh(x)}{x^2}\right)$$

gives only slightly different numerical results (for large $\sqrt{n}/k$).

Analogous series exist for plane partitions. The terms involve neither exponentials nor Bessel functions, but rather a new function

$$g(x,\gamma) = \sum_{\nu=0}^{\infty}\frac{x^{2\nu+\gamma-1}}{\nu!\Gamma(2\nu+\gamma)}$$

that satisfies the third-order differential equation

$$xg'''(x,\gamma) - (\gamma-3)g''(x,\gamma) - 2g(x,\gamma) = 0$$

(the derivatives are taken with respect to x) as well as

$$g'(x,\gamma) = g(x,\gamma-1), \qquad 2g(x,\gamma+2) + (\gamma-1)g(x,\gamma) = xg(x,\gamma-1).$$

A heuristic argument in [30, 31] gives that

$$p_2(n) \sim \varphi_1(n) + \varphi_2(n) + \varphi_3(n) + \cdots$$

as $n \to \infty$, where

$$\varphi_1(n) = \zeta(3)^{13/24}e^{\zeta'(-1)}\sum_{k=0}^{\infty}a_{2k}\zeta(3)^k g\left(n\sqrt{\zeta(3)}, -\frac{1}{12} - 2k\right)$$

and a_{2k} is the coefficient of x^{2k} in the Maclaurin series of

$$h(x) = \exp\left(-\sum_{j=1}^{\infty}\frac{2(2j+1)!\zeta(2j)\zeta(2j+2)}{j(2\pi)^{4j+2}}x^{2j}\right),$$

$$\varphi_2(n) = (-1)^n 2^{-5/3}\zeta(3)^{7/12}e^{2\zeta'(-1)}\sum_{k=0}^{\infty} b_{2k}\left(\frac{\zeta(3)}{8}\right)^k g\left(n\sqrt{\frac{\zeta(3)}{8}}, -\frac{1}{6}-2k\right)$$

and b_{2k} is the coefficient of y^{2k} in the Maclaurin series of

$$\frac{h(2y)^5}{h(y)h(4y)^2},$$

and so forth. The additional terms $\varphi_3(n)$, $\varphi_4(n)$ appear in [30] and $\varphi_5(n)$, $\varphi_6(n)$ appear in [31]. Taken together, these terms provide remarkably accurate estimates of $p_2(n)$. Govindarajan & Prabhakar [32] revisited Almkvist's results, using a modified function

$$\tilde{g}(x,\gamma) = \frac{1}{2}\sum_{\nu=0}^{\infty}\frac{x^\nu}{\nu!\Gamma((3-\gamma+\nu)/2)}$$

that seems better behaved than $g(x,\gamma)$ and evidently does for $p_2(n)$ akin to what Rademacher's modification of Hardy-Ramanujan did for $p_1(n)$.

0.2. Addendum. Recent Monte Carlo work indicates that [33]

$$\lim_{n\to\infty} n^{-3/4}\ln\left(p_3(n)\right) \approx 1.822 > 1.789... = \frac{2^{7/4}\pi}{3^{5/4}5^{1/4}},$$

contradicting [8, 9]. The asymptotics of solid partitions appear to differ sharply from those of line and plane partitions; in addition to sub-leading terms of order $n^{1/2}$, $n^{1/4}$ and $\ln(n)$, there seems to be an oscillatory function at the $n^{-1/4}$ level. Theory lags far behind numerical experimentation here. Let

$$\begin{aligned} 1+\sum_{n=1}^{\infty} q(n)x^n &= 1+x+4x^2+10x^3+26x^4+59x^5+141x^6+310x^7+692x^8+\cdots \\ &= \prod_{m=1}^{\infty}\left(1-x^m\right)^{-m(m+1)/2}. \end{aligned}$$

Although the MacMahon conjecture is incorrect ($p_3(n) \neq q(n)$ for $n > 5$), there is still a possibility that $p_3(n) \sim q(n)$ as $n \to \infty$. The conjectured asymptotics for $p_3(n)$ given earlier are validated asymptotics for $q(n)$. In a recent breakthrough, Kotesovec [34] deduced that the multiplicative constant C for $q(n)$ is

$$2^{-157/96}15^{-13/96}\exp\left(-\frac{\zeta(3)}{8\pi^2}+\frac{75\zeta(3)^3}{2\pi^8}+\frac{\zeta'(-1)}{2}\right)\pi^{1/24} = 0.2135951604...$$

and we look forward to seeing underlying details.

Let us consider one of many possible variations on 1-partitions. Define $\hat{p}_1(n)$ to be the number of partitions of n into integers, each of which may occur only an odd number of times. It can be shown that [35]

$$\hat{p}_1(n) \sim \frac{B}{2\pi\, n} \exp\left(2B\sqrt{n}\right)$$

where

$$\begin{aligned} B^2 &= \frac{\pi^2}{12} + \int_0^1 \frac{\ln(1+x-x^2)}{x} dx = \frac{\pi^2}{12} + 2\ln(\varphi)^2 \\ &= \frac{\pi^2}{12} + 0.4631296411... = (1.1338415562...)^2 \end{aligned}$$

and $\varphi = (1+\sqrt{5})/2$ is the Golden mean.

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