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THE PROBLÈME DES MÉNAGES

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1. Introduction. The *problème des ménages* asks for the number of ways of seating at a circular table n married couples, husbands and wives alternating, so that no husband is next his own wife.

We may begin by fixing the positions of husbands or wives, say wives for courtesy's sake. The number of ways of seating the wives is $2\,n!$, for they may occupy either the "odd" or "even" seats and may then be permuted in $n!$ ways. Let the seats next the first wife be numbered 1 and 2, those next the second wife 2 and 3, etc. Then the *problème des ménages* may be restated thus: in how many ways can the numbers $1, 2, \ldots, n$ be permuted so that 1 is not in positions 1 or 2, 2 not in 2 or 3, ..., n not in n or 1. We shall denote the number of such permutations by u_n, the solution of the *problème des ménages* then being given by $2\,n!\,u_n$. For $n = 3, 4, 5$ we have $u_n = 1, 2, 13$, respectively, the permissible permutations being:

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$u_0 = 1$, $u_1 = -1$ (Knuth), $u_2 = 0$.

312	23451	34152	35421	45123.
	24153	34512	43152	
2341	24513	34521	43512	
3412	25413	35412	43521	

Thus stated, the *problème des ménages* is seen to be a natural extension of the older *problème des rencontres*, which asks for the number of permutations of $1, 2, \ldots, n$ in which every integer is out of place. The well-known answer to this latter problem is the so-called sub-factorial of n:

$$h_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-)^n \frac{1}{n!}\right] \qquad (1)$$

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1, 0, 1, 2, 9, ...

which, in the notation of finite differences, may be written compactly as $h_n = \Delta^n 0!$.

The statement and reduction of the *problème des ménages* as above are due to Lucas[5]; note that the date is 1891. He gives the recurrence formula

$$(n-2)u_n = (n^2 - 2n)u_{n-1} + nu_{n-2} + (-)^{n-1}4, \qquad (2)$$

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attributing it to Laisant, and independently to Moreau. Evidently these were communications directly to Lucas, for there appears to be no record of publication by Laisant or Moreau themselves. Using (2), Lucas tabulated the values of u_n up to $n = 20$.

Apparently unknown to Lucas (and many others after him) was the fact that (2) had been given thirteen years earlier by Cayley and Muir. The problem, in its reduced version, had been suggested to Cayley by Tait, who believed he required it in his study of knots. Subsequently[12] it appeared that what he needed was rather the number of ménages permutations, where those which are cyclic permutations of one another are not regarded as distinct. For example, for $n = 4$ we have one solution instead of two, since 2341 and 3412 are identified; and for $n = 5$ we get 11 instead of 13 on identifying 23451, 34512, and 45123. This is evidently a somewhat harder problem, and no solution appears to have been published.

Cayley's first paper[1] gave a direct application of the method of "inclusion and exclusion" (cf. § 2); the resulting formula for u_n, though explicit, is cumbersome. Shortly after, Muir[7] obtained the recurrence

$$\begin{aligned} u_n = (n-2)u_{n-1} + (2n-4)u_{n-2} + (3n-6)u_{n-3} + (4n-10)u_{n-4} \\ + (5n-14)u_{n-5} + (6n-20)u_{n-6} + (7n-26)u_{n-7} \\ + (8n-34)u_{n-8} + \cdots \end{aligned} \tag{3}$$

In an addendum[2] to Muir's paper, Cayley used (3) to derive a formal generating function. In the course of this work he obtained as a by-product a recursion formula in terms of an auxiliary quantity q_n:

$$u_n = q_n - q_{n-2} \tag{4}$$

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$$q_n = nq_{n-1} + q_{n-2} + (-)^{n-1}(n-2). \tag{5}$$

He omitted the trivial step of eliminating q from (4) and (5) which would have yielded precisely (2). The first discovery of (2) must thus be credited to neither Muir nor Laisant, but to Cayley.

In a second note four years later, Muir[8] gave an independent deduction of (2) from (3) without noting Cayley's priority. As an intermediate step he obtained the homogeneous recurrence

$$u_n = nu_{n-1} + 2u_{n-2} - (n-4)u_{n-3} - u_{n-4} \tag{6}$$

which can in turn be derived by iteration of (2).

Netto[9] recapitulated Cayley's work[1] and quoted (2) and (6), ascribing them to Muir. He made no mention of Lucas, Laisant, or ménages.

Taylor,[13] giving no references, derived the recurrence

$$nu_{n+2} = (n^2 + n + 1)(u_{n+1} + u_n) + (n+1)u_{n-1} \tag{7}$$

q_n starts 0, 1, 2, 14, 82, ...
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which is easy to deduce from (2). By elimination from a set of equations like (7) he found an expression for u_n as a determinant, which has probably only formal interest. Alone of all authors in this respect, he chose to seat the men first.

MacMahon[6] gave an operational solution equivalent to the observation that u_n is the coefficient of $x_1 x_2 \dots x_n$ in

$$(y - x_1 - x_2)(y - x_2 - x_3) \dots (y - x_n - x_1)$$

where $y = x_1 + x_2 + \dots + x_n$. He quoted (2) without proof, ascribing it to Laisant.

Schöbe,[11] quoting Lucas, gave a systematic derivation of the various recurrences. He used an auxiliary quantity b_n related to Cayley's q by

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$$b_{n+1} = q_n - (-)^n.$$

He found the interesting new expression

h_n^2 =A335691

$$n!\, b_{n+1} = \sum_{i=0}^{n} (-)^i \binom{n}{i} h^2{}_{n-i} = \Delta^n h_0{}^2$$

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where h_n is given by (1), and proved that $u_n/n! \to e^{-2}$ as $n \to \infty$.

A new chapter in the subject opened with the publication in 1934 of a brilliant communication from J. Touchard.[14] In effect he revived Cayley's search for an explicit formula and stated without proof a simple one which Cayley had missed:

$$u_n = n! - \frac{2n}{2n-1}\binom{2n-1}{1}(n-1)! + \frac{2n}{2n-2}\binom{2n-2}{2}(n-2)! - \dots \quad (8)$$

Proofs of (8) and other related results were supplied later by Kaplansky[3] and Riordan.[10]

What we wish to do here is to derive these old, and some new, results by a systematic procedure: the symbolic method.[4] We hope to show its power both in getting results and in uniting related problems.

2. The Symbolic Method. The basis of the method about to be explained has been known for a long time and is a vital tool in many investigations. It has been variously called the "method of inclusion and exclusion," "principle of cross-classification," "sieve method," etc. Let there be N objects and a set of properties, say for definiteness three: a, b, and c (the extension to any number of properties will be evident). Suppose $N(a)$ objects have property a, $N(b)$ have b, $N(ab)$ have both a

and b, etc. Then the number of objects having none of the properties is

$$N - N(a) - N(b) - N(c) + N(ab) + N(bc) + N(ca) - N(abc).$$

For our purpose it is technically more convenient to use the equivalent formulation in terms of probability; here the method goes by the name of *Poincaré's formula*. Let A, B, C be events, $p(A)$ the probability of A, $p(AB)$ the joint probability of A and B, etc. Then the probability that none of A, B, C happen is

$$1 - p(A) - p(B) - p(C) + p(AB) + p(BC) + p(CA) - p(ABC). \quad (9)$$

The form of (9) suggests immediately a product of factors:

$$[1 - p(A)][1 - p(B)][1 - p(C)], \quad (10)$$

and in fact if A, B, C are independent, (10) is correct. However, even if the events are dependent, (10) will remain valid provided we agree that products like $p(A)p(B)$ are to be construed *symbolically* as meaning $p(AB)$. With this convention, the door is opened for the algebraic manipulations to follow.

3. Ménages Polynomials. In the *problème des ménages* (and in a host of similar problems) the events under study are of the form "i is in the jth place." Let p_{ij} denote the probability of this event. Then our task is to compute

$$(1 - p_{11})(1 - p_{12})(1 - p_{22})(1 - p_{23}) \ldots (1 - p_{nn})(1 - p_{n1}). \quad (11)$$

Let us pause to observe how a product of p's is to be computed. It is clear that $p_{ij} = (n-1)!/n!$, there being $(n-1)!$ favorable cases out of the total of $n!$. For a product $p_{ij}p_{kl}$ (= joint probability that i is jth and k is lth) there are two possibilities. Firstly it may be zero if the events are incompatible. For example $p_{23}p_{24} = 0$ since 2 cannot be both 3rd and 4th, and $p_{31}p_{51} = 0$ since 3 and 5 cannot both be 1st. Otherwise $p_{ij}p_{kl} = (n-2)!/n!$. Similarly the product of k of the p's will be $(n-k)!/n!$ unless they are incompatible; and the latter will occur whenever there is any duplication in the first subscripts or in the second subscripts of the p's.

Following the notation of finite differences we may write $(n-k)!/n! = E^k(n-0)!/n!$ or simply E^k. Then the preceding paragraph can be summarized as follows: to evaluate

$$(1 - p_{ij})(1 - p_{kl}) \ldots$$

first eliminate all products that vanish, and then replace each surviving p by E.

The simplest example of this procedure is the *problème des rencontres* which calls for the evaluation of

$$(1 - p_{11})(1 - p_{22}) \ldots (1 - p_{nn}).$$

Here no products vanish so the answer is $(1 - E)^n$, in agreement with (1). For a less trivial example consider the following set of restrictions:

1 not 1st or 2nd
2 not 2nd
3 not 3rd or 4th
4 not 4th

and so on in groups of two (it being supposed that n is even). We compute as follows

$$(1 - p_{11})(1 - p_{12})(1 - p_{22}) = 1 - p_{11} - p_{12} - p_{22} + p_{11}p_{22}$$

$$(1 - p_{33})(1 - p_{34})(1 - p_{44}) = 1 - p_{33} - p_{34} - p_{44} + p_{33}p_{44}, \text{ etc.}$$

Since we have now eliminated all vanishing products the answer $(1 - 3E + E^2)^{n/2}$ is apparent. Further examples can be found in reference 4.

In evaluating (11) we unfortunately do not find any such happy resolution into factors as in the above examples. However, an approach which suggests itself is to compute the result, say L_k, of detaching the first k factors of (11).

$$\begin{aligned} L_1 &= 1 - p_{11} = 1 - E \\ L_2 &= (1 - p_{11})(1 - p_{12}) = 1 - 2E \\ L_3 &= (1 - p_{11})(1 - p_{12})(1 - p_{22}) = 1 - 3E + E^2 \\ L_4 &= 1 - 4E + 3E^2 \\ L_5 &= 1 - 5E + 6E^2 - E^3 \\ L_6 &= 1 - 6E + 10E^2 - 4E^3. \end{aligned}$$

Coefficients are A011973 (or equally A115139)

One may without difficulty guess

$$L_k = 1 - kE + \binom{k-1}{2} E^2 - \binom{k-2}{3} E^3 \ldots. \qquad (12)$$

We can get an inductive proof of (12) by deriving a suitable recursion formula. For example

$$L_7 = L_6(1 - p_{44}) = L_6 - L_6 p_{44}.$$

Now the effect of p_{44} on $L_6 = (1 - p_{11}) \ldots (1 - p_{34})$ is to knock out p_{34} and leave L_5 with which it no longer conflicts. Hence

$$L_7 = L_6 - EL_5,$$

and in general

$$L_k = L_{k-1} - EL_{k-2}$$

from which (12) follows readily. It may be remarked that the preceding algebraic argument parallels the combinatorial version in reference 3.

When we reach L_{2n-1} we have imposed all restrictions except one: that n not be 1st. The analogous problem might be called "non-circular ménages" and in fact it corresponds precisely to a straight instead of circular table. If M_n is the polynomial for (11), then

$$\begin{aligned} M_n &= L_{2n-1}(1 - p_{n1}) \\ &= L_{2n-1} - L_{2n-3}(1 - p_{11})(1 - p_{nn})p_{n1} \\ &= L_{2n-1} - EL_{2n-3}, \end{aligned}$$

and, using (12), we can write M_n as

$$M_n = 1 - \frac{2n}{2n-1}\binom{2n-1}{1}E + \frac{2n}{2n-2}\binom{2n-2}{2}E^2 - \dots \quad (13)$$

On replacing E^k by $(n-k)!/n!$ we get precisely (8).

It is perhaps somewhat more elegant to have E operate directly on $0!$. This is accomplished by passing to the polynomial

$$U_n(E) = E^n M_n(1/E), \text{ i.e.,}$$

$$U_n = E^n - \frac{2n}{2n-1}\binom{2n-1}{1}E^{n-1} + \frac{2n}{2n-2}\binom{2n-2}{2}E^{n-2} - \dots \quad (14)$$

Following Touchard[14] we may also write U_n compactly as a Tchebycheff polynomial:

$$U_n = 2\cos\left[2n\cos^{-1}\left(\sqrt{E}/2\right)\right]. \quad (15)$$

We list the first few of these polynomials:

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$$\begin{aligned} U_2 &= E^2 - 4E + 2 \\ U_3 &= E^3 - 6E^2 + 9E - 2 \\ U_4 &= E^4 - 8E^3 + 20E^2 - 16E + 2 \\ U_5 &= E^5 - 10E^4 + 35E^3 - 50E^2 + 25E - 2 \\ U_6 &= E^6 - 12E^5 + 54E^4 - 112E^3 + 105E^2 - 36E + 2. \end{aligned}$$

Thus for $n = 5$,

$$\begin{aligned} u_5 &= U_5\, 0! \\ &= 5! - 10(4!) + 35(3!) - 50(2!) + 25(1!) - 2(0!) \\ &= 13. \end{aligned}$$

4. Polynomial Relations. The polynomials $M_n(E)$ are useful in more problems than the simple ménages problem whose solution has just been given. We derive here some of Touchard's results having this wider extent.

Writing

$$M_n(E) = \sum_{i=0}^{n} a_{n,\,i}\,(-E)^i,$$

as in (13), we may readily show that

$$a_{n,\,i} = a_{n-1,\,i} + 2a_{n-1,\,i-1} - a_{n-2,\,i-2}. \tag{16}$$

Indeed this follows at once from

$$M_n(E) = (1 - 2E)M_{n-1}(E) - E^2 M_{n-2}(E), \tag{17}$$

which is a consequence of the two relations of § 3:

$$\begin{aligned} L_k &= L_{k-1} - EL_{k-2}, \\ M_n &= L_{2n-1} - EL_{2n-3}. \end{aligned}$$

Also:

$$U_n = (E - 2)U_{n-1} - U_{n-2}. \tag{18}$$

It may be observed that (18) is a recurrence relation for Tchebycheff polynomials, in agreement with (15).

Writing $f_n(E) = (E - 2)^n$, it follows from (18) and mathematical induction that

$$U_n = \sum_{i=0} (-)^i \frac{n}{i} \binom{n - i - 1}{i - 1} f_{n-2i}. \tag{19}$$

As $f_n\, 0!$ approaches $n!\, e^{-2}$, (19) corresponds to the asymptotic formula

$$u_n \sim n!\, e^{-2} \left[1 - \frac{1}{n - 1} + \frac{1}{2!(n - 1)_2} + \ldots + \frac{(-1)^i}{i!(n - 1)_i} + \ldots \right] \tag{20}$$

The notation $(n - 1)_i$ is the C. Jordan factorial notation:

$$(n - 1)_i \equiv (n - 1)(n - 2) \ldots (n - 1 - i + 1).$$

Again, by (18)

$$\begin{aligned}
(E-1)U_{n-1} &= U_n + U_{n-1} + U_{n-2} \\
(E-1)^2U_{n-2} &= (E-1)U_{n-1} + (E-1)U_{n-2} + (E-1)U_{n-3} \\
&= U_n + 2U_{n-1} + 3U_{n-2} + 2U_{n-3} + U_{n-4} \\
&= U^{n-4}(1 + U + U^2)^2,
\end{aligned}$$

where the multiplication in the last is symbolic; by induction

$$(E-1)^m U_{n-m} = U^{n-2m}(1 + U + U^2)^m. \tag{21}$$

The polynomial $\varphi(n, m) = (E-1)^m U_{n-m}$ enumerates permutations discordant with the identity permutation and a permutation of cycle structure $1^m(n-m)$; e. g., $\varphi(5, 2)$ enumerates permutations discordant (having no elements alike in any position) with the two permutations

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 3 \end{matrix}$$

Finally, by (15):

$$\begin{aligned}
U_iU_j &= 4\cos(2i\theta)\cos(2j\theta), \quad \cos\theta = \frac{1}{2}\sqrt{E} \\
&= 2[\cos 2(i+j)\theta + \cos 2(i-j)\theta] \\
&= U_{i+j} + U_{i-j}
\end{aligned}$$

if the convention $U_{-n} \equiv U_n$ is made, and by iteration of this the general result due to Touchard is reached:

$$U_{i_1}U_{i_2}\ldots U_{i_s} = \Sigma\, U_{i_1 \pm i_2 \cdots \pm i_s}, \tag{22}$$

with the sum on the right over the 2^{s-1} possible assignments of $+$ and $-$ signs, and $U_n \equiv U_{-n}$, $U_0 = 2$, $U_1 = E - 2$.

Formula (22) may be used in the enumeration of 3-line Latin rectangles (Riordan[10]) and results in the following formula, published in *The Amer. Math. Monthly*, v. 53, 1946, p. 18:

$$_3K_n = \sum_{i=0}^{m} \binom{n}{i} h_i h_{n-i} u_{n-2i}, \quad m = \left[\frac{1}{2}n\right]. \tag{23}$$

In this formula $_3K_n$ is the number of reduced 3-line Latin rectangles, that is, with the first row in natural order, h_n is the sub-factorial of n (given by (1)) and to avoid an exceptional case, u_0 is taken as unity.

5. Recurrence Relations. As is well known, the polynomials U_n define not only the ménages numbers u_n but also more general numbers say $u_{n,r}$ of permutations such that r elements are in forbidden positions; thus

$$u_{n,r} = M_n(E)\psi_{r,0}, \tag{24}$$

with $\psi_{r,k} = E^k\psi_{r,0} = (-)^r \binom{k}{r} (n-k)!$,

or

$$G_n(t) = \sum_r u_{n,r} t^r = \sum_i a_{n,i}(n-i)!(t-1)^i \tag{25}$$

$$= \sum_i \frac{2n}{2n-i}\binom{2n-i}{i}(n-i)!(t-1)^i.$$

To derive recurrences, it is convenient to consider two related generating functions, as follows:

$$H_n(t) = \sum_r v_{n,r} t^r = \sum_i \binom{2n-i}{i}(n-i)!(t-1)^i \tag{26}$$

$$I_n(t) = \sum_r w_{n,r} t^r = \sum_i \binom{2n-i+1}{i}(n-i)!(t-1)^i. \tag{27}$$

Note that $H_n(t)$ corresponds to L_{2n-1} in the same way that G_n corresponds to M_n; hence $v_{n,r}$ enumerates permutations for "non-circular ménages" (cf. § 3).

Then it follows from the relation (implicit in: $M_n = L_{2n-1} - EL_{2n-3}$):

$$\frac{2n}{2n-i}\binom{2n-i}{i} = \binom{2n-i}{i} + \binom{2n-i-1}{i-1}, \tag{28}$$

that

$$\begin{aligned} G_n &= H_n + (t-1)H_{n-1}, \\ &= nI_{n-1} + 2(t-1)^n, \\ &= I_n - (t-1)^2 I_{n-2}. \end{aligned} \tag{29}$$

Also the ordinary binomial recurrence shows that:

$$H_n = I_n - (t-1)I_{n-1}. \tag{30}$$

Combination of these leads to:

$$\begin{aligned} (n-1)G_{n+1} &= (n^2-1)G_n + (n+1)(t-1)^2G_{n-1} - 4(t-1)^{n+1} \\ nH_{n+1} &= (n^2+n-1+t)H_n + (n+1)(t-1)^2H_{n-1} - 2(t-1)^{n+1} \\ I_{n+1} &= (n+1)I_n + (t-1)^2I_{n-1} + 2(t-1)^{n+1}. \end{aligned} \tag{31}$$

These in turn, of course, correspond to recurrences for $u_{n,r}$, $v_{n,r}$, and $w_{n,r}$ of which we quote only those for $r = 0$ (abbreviating $u_{n,0}$ to u_n, etc.).

$$\begin{aligned} (n-1)u_{n+1} &= (n^2-1)u_n + (n+1)u_{n-1} + 4(-)^n \qquad (32) \\ nv_{n+1} &= (n^2+n-1)v_n + (n+1)v_{n-1} + 2(-)^n \\ w_{n+1} &= (n+1)w_n + w_{n-1} - 2(-)^n. \end{aligned}$$

The first of these is equation (2) of § 1.

Simpler formulas follow from differentiation of generating functions; thus, indicating derivatives by primes:

$$G'_n = 2nH_{n-1} = nI'_{n-1} + 2n(t-1)^{n-1} = 2nG_{n-1} - (t-1)\frac{n}{n-1}G'_{n-1} \qquad (33)$$
$$\begin{aligned} H'_n &= (2n-1)H_{n-1} - (t-1)H'_{n-1} \\ I'_n &= 2nI_{n-1} - (t-1)I'_{n-1}. \end{aligned}$$

Corresponding to the last three are the recurrences:

$$\begin{aligned} (n-1)ru_{n,\,r} &= nru_{n-1,\,r} + n(2n-r-1)u_{n-1,\,r-1} \\ rv_{n,\,r} &= rv_{n-1,\,r} + (2n-r)v_{n-1,\,r-1} \\ rw_{n,\,r} &= rw_{n-1,\,r} + (2n-r+1)w_{n-1,\,r-1}. \end{aligned}$$

Table 1 shows the numbers $u_{n,\,r}$ for $n \leq 10$.

6. Asymptotic Formulas. To develop an asymptotic formula for $u_{n,\,r}$, the following relations which we take without proof, are required

$$\frac{u_{n,\,r}}{n!} = \frac{1}{r!}\sum_{i=0} \frac{(-)^i}{i!} M_{(r+i)}, \qquad (34)$$

$$M_{(i)} = a_{n,\,i} \Big/ \binom{n}{i}, \qquad (35)$$

where $M_{(i)}$ is the ith factorial moment of the distribution $u_{n,\,r}$ and $a_{n,\,i}$ is the coefficient of $(-E)^i$ in polynomial $M_n(E)$; note that $a_{n,\,i}$ has recurrence (16).

Equation (34) is easily evaluated if $a_{n,\,i}$ is expanded in the form:

$$a_{n,\,i} = \sum_{j=0} b_{i,\,j}\binom{n-j}{i-j}. \qquad (36)$$

By (16) this is possible if

$$b_{i,\,j} = 2b_{i-1,\,j} - b_{i-2,\,j-1} \qquad (37)$$

with boundary condition $b_{i,\,0} = 2^i$ and $a_{i,\,1} = 2$. Then

$$\begin{aligned} b_{i,\,1} &= (i-1)2^{i-2} \qquad (38) \\ b_{i,\,2} &= (i^2-5i-2)2^{i-5} \end{aligned}$$

By (35);

$$\frac{M_{(i)}}{2^i} = 1 - \frac{i-1}{4}\frac{i}{n} + \frac{i^2 - 5i - 2}{32}\frac{i(i-1)}{n(n-1)} - \dots \qquad (39)$$

so that

$$\frac{u_{n,r}}{n!} = \frac{2^r}{r!}\sum_i \frac{(-2)^i}{i!}\left[1 - \frac{(r+i)_2}{4n} + \frac{(r+i)_4 - 8(r+i)_2}{32(n)_2} - \dots\right], \qquad (40)$$

since

$$i(i-1)(i^2 - 5i - 2) = (i)_4 - 8(i)_2.$$

Using the Vandermonde relation

$$(r+i)_j = \sum_k \binom{j}{k}(r)_{j-k}(i)_k$$

equation (40) is readily evaluated with the result

$$\frac{u_{n,r}}{n!} = \frac{2^r e^{-2}}{r!}\left[1 - \frac{(r-1)(r-4)}{4n} + \frac{f_2(r)}{4(n)_2}\right] + 0(n^{-3}) \qquad (41)$$

where

$$f_2(r) = 3\binom{r}{4} - 6\binom{r}{3} + 4\binom{r}{2} - 2$$

For the range $r = 0$ to 10 the values of $(r-1)(r-4)$ and $f_2(r)$ are as follows

r	0	1	2	3	4	5	6	7	8	9	10
$(r-1)$ $(r-4)$	4	0	−2	−2	0	4	10	18	28	40	54
$f_2(r)$	−2	−2	2	4	1	−7	−17	−23	−16	16	88.

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Note that (41) is consistent with (20), though less extensive for this instance.

The close approximation of (41) to the true distribution for sufficiently large values of n is shown by the following comparison for $n = 10$:

r	0	1	2	3	4	5	6
Exact	0.12119	0.26896	0.28551	0.19173	0.09064	0.03171	0.00835
Approx.	0.12105	0.26917	0.28571	0.19174	0.09047	0.03178	0.00845

The corresponding expression for non-circular ménages numbers $v_{n,\,r}$ is

$$\frac{v_{n,\,r}}{n!} = \frac{2^r e^{-2}}{r!}\left[1 - \frac{r(r-3)}{4n} + \frac{g_2(r)}{4(n)_2}\right] + 0(n^{-3}) \qquad (42)$$

with

$$g_2(r) = 3\binom{r}{4} - 3\binom{r}{3} + 2r - 2.$$

Δ = A94314

TABLE 1. MÉNAGES NUMBERS $u_{n,\,x}$

n/x	0	1	2	3	4	5	6	7	8	9	10
2	0	0	2								
3	1	0	3	2							
4	2	8	4	8	2						
5	13	30	40	20	15	2					
6	80	192	210	152	60	24	2				
7	579	1344	1477	994	479	140	35	2			
8	4738	10800	11672	7888	3660	1232	280	48	2		
9	43387	97434	104256	32958	32958	11268	2856	504	63	2	
10	439792	976000	1036050	695760	328920	115056	30300	6000	840	80	2

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