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$$\begin{vmatrix} 2a & c(d+d') & b(d-d') \\ c(d-d') & 2b & a(d+d') \\ b(d+d') & a(d-d') & 2c \end{vmatrix} = 0,$$

i.e., if $abc\Delta = 0$.

In the same way we find that each of the 8 points is the vertex of a cone in the system ψ if $\Delta = 0$. Two of these cones meet, apart from the line joining their vertices, in a twisted cubic which passes through all 8 points, and which therefore lies on all quadrics of ψ.

A Permutation Problem

4755 [1957, 596]. *Proposed by Chandler Davis, Institute for Advanced Study*

In how many ways can the first n positive integers be arranged in alternately increasing and decreasing order? That is, how many permutations π: $\pi(1), \cdots, \pi(n)$ are there such that the quantities $(-1)^k\{\pi(k+1)-\pi(k)\}$, for $k=1, \cdots, n-1$ have all the same sign?

Solution by W. J. Blundon, Memorial University of Newfoundland. Let P_n be the required number of arrangements of the first n positive integers, under the restriction that the common sign of the stated quantities is negative. The integer n, being the largest in the set, is necessarily of the form $\pi(2i)$, $i=1, 2, \cdots, [n/2]$. The integers to the left of n can be chosen in $\binom{n-1}{2i-1}$ ways, and each such selection can be arranged in P_{2i-1} ways. The integers to the right of n can be arranged in P_{n-2i} ways. Hence

$$P_n = \sum_{i=1}^{[n/2]} \binom{n-1}{2i-1} P_{2i-1}P_{n-2i}, \qquad n = 1, 2, \cdots,$$

where, for convenience, we define $P_0 = 1$. Putting $P_n = n!Q_n$, we have

$$Q_0 = 1; \qquad nQ_n = \sum_{i=1}^{[n/2]} Q_{2i-1}Q_{n-2i}, \qquad n = 1, 2, \cdots.$$

Define $f(x) = \sum_{n=0}^{\infty} Q_n x^n$. Then it is easily verified that

$$\{f(x)\}^2 = -1 + 2f'(x).$$

The solution of this differential equation gives

$$\sec x + \tan x = f(x) = \sum_{n=0}^{\infty} P_n x^n/n!.$$

From the well-known expansions of sec x and tan x, we have

$$P_n = \begin{cases} B_n & n \text{ even}, \\ \dfrac{2^{n+1}(2^{n+1}-1)}{n+1} B_n & n \text{ odd}, \end{cases}$$

where the B's are alternately Bernoulli numbers and Euler numbers (1/6, 1, 1/30, 5, $\cdots$,).

We now remove the restriction of the first sentence of this solution. Then, by symmetry, the required number of arrangements is $2P_n$, (except when $n=1$, when the restriction has no meaning). The number of arrangements for small n is given by

n	1	2	3	4	5	6	7	8	9	10
$2P_n$	1	2	4	10	32	122	544	2770	15872	101042

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Also solved by W. H. Furry, E. C. Milner, L. E. Clarke, A. van Heemert, and the proposer.

Editorial Note. Using known expansions for the tangent and secant, Furry gets the result in the form

$$P_n = 2(2/\pi)^{n+1}n!\{1 + (-3)^{-n-1} + 5^{-n-1} + (-7)^{-n-1} + \cdots\}.$$

For large n the convergence is rapid, and the first term provides a good asymptotic expression for P_n.

The proposer notes an obvious connection with his problem no. 4714 [1957, 679–680]. He notes also that, because of the uniqueness of the Taylor expansion, P_n is the nth derivative of $\sec x + \tan x$, evaluated at $x=0$.

Real Functions

4756 [1956, 596]. *Proposed by J. L. Massera, Institute of Mathematics and Statistics, Montevideo, Uruguay*

Let $p(x_1, \cdots, x_n)$, $q(x_1, \cdots, x_n)$ be two real functions of n real variables x_i, defined and continuous in a parallelotope R: $0 \leqq x_i \leqq a_i < \infty$. Assume that $p(x_1, \cdots) = q(x_1, \cdots) = 0$ whenever $x_1x_2 \cdots x_n = 0$, and that $p(x_1, \cdots) > 0$, $q(x_1, \cdots) \geqq 0$, when $x_1x_2 \cdots x_n \neq 0$. Prove that there exists a real function $h(u)$ of a real variable u, defined, continuous and strictly increasing for $u \geqq 0$, $h(0)=0$, such that throughout R

$$h\{q(x_1, \cdots)\} < p(x_1, \cdots).$$

Solution by Neill McShane, Yale University. Let $\mathbf{x} = (x_1, \cdots, x_n)$. Since R is a compact domain, the continuous function $r(\mathbf{x})$ defined implicitly by $q(\mathbf{x}) = r(\mathbf{x})p(\mathbf{x})$ is bounded above uniformly by some number r_0. Assume $q(\mathbf{x})$ not identically 0, since otherwise the problem is trivial. Then $r_0 > 0$. $h(u) = u/2r_0$ satisfies the demands of the problem, for

$$h(q(\mathbf{x})) = q(\mathbf{x})/2r_0 = r(\mathbf{x})p(\mathbf{x})/2r_0 < p(\mathbf{x}).$$

Also solved by J. Horváth and by the proposer.

Identity Related to the Beta- Function

4757 [1957, 596]. *Proposed by O. P. Aggarwal, University of Washington*

Prove for every integer $n \geqq 0$, and for any positive c,