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Berkeley, California

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Generation of Triangulations of the Sphere

By Robert Bowen and Stephen Fisk

It is easily seen that there is only one triangulation of the sphere with four vertices and one with five. This paper concerns an algorithm for finding all (nonisomorphic) triangulations of the 2-sphere with N vertices from those with $N - 1$. "Triangulation" shall always refer to a triangulation of the 2-sphere. First we develop a method for generating all triangulations with N vertices which may yield several triangulations of the same isomorphism type, and then we describe an isomorphism routine for eliminating these duplications.

Let T be a triangulation with $N \geqq 5$ vertices, E edges, and F faces. Let X_k denote the number of vertices of T of valency k. Then $3F = 2E$ as each face is a triangle and each edge is on two faces, and $2E = \sum kX_k$ as each edge is incident to two vertices. Hence $6F - 6E = -2E = -\sum kX_k$ and by Euler's formula we have

$$(1) \qquad 12 = 6N + 6F - 6E = 6N - \sum kX_k = \sum X_k(6 - k).$$

Since $\sum X_k(6 - k)$ is positive, T must have a vertex of valency less than six. Because every edge of T must lie on two distinct triangular faces, each vertex must have valency greater than two. Letting Q be a vertex of minimal valency, Q must have valency three, four, or five.

Case 1. Suppose Q has valency three. Then, about Q, T has the form shown in Fig. 1. Removing Q and the edges QP_k, we obtain a triangulation T' with $N - 1$ vertices. Thus we obtain T if we add the point Q to the center of the face $P_1P_2P_3$ and add the edges QP_k $(k = 1, 2, 3)$.

Case 2. Suppose Q has valency four. Then, about Q, T has the form shown in Fig. 2. By the Jordan curve theorem either P_1 is not adjacent to P_3 or P_2 is not adjacent to P_4; say P_1 is not adjacent to P_3. Then, removing Q and edges QP_k $(1 \leqq k \leqq 4)$ and adding edge P_1P_3 inside the quadrilateral $P_1P_2P_3P_4$, we obtain a triangulation T' with $N - 1$ vertices. The slight complication here is needed to insure that T' is a triangulation; for if P_1 were adjacent to P_3 in T, then T' would have multiple edges and would not be a triangulation. We now obtain T from T' by reversing the process.

Case 3. Assume Q has valency five. We claim some P_k is adjacent to no P_i other than the two shown (Fig. 3). Otherwise P_1 would be adjacent to P_3 or P_4, say P_3. Then by the Jordan curve Theorem, P_2 could be adjacent to neither P_4 nor

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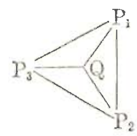

FIGURE 1

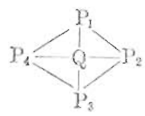

FIGURE 2

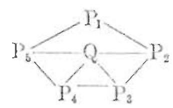

FIGURE 3

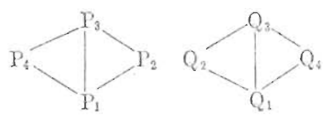

FIGURE 4

P_5. Hence we may assume (perhaps renumbering the P's) that P_1 is not adjacent to P_3 or P_4. Now removing Q and edges QP_k and adding edges P_1P_3 and P_1P_4 we obtain a triangulation T'. Reversing the process we again obtain T.

We thus have three operations which when applied in all possible ways to all triangulations of $N - 1$ vertices ($N \geqq 5$) will yield all triangulations of N vertices.

Now suppose T_1 and T_2 are two triangulations of N vertices. Ordering the faces of T_k according to the largest valency of a vertex of the face, then the second largest, and finally the third valency, let M_k be the set of maximal faces of T_k. Let $P_1P_2P_3$ be in M_2. Then an isomorphism F of T_1 onto T_2 must have $F(Q_i) = P_i$ for some $Q_1Q_2Q_3$ in M_1. Suppose F^* maps Q_i into P_i and we wish to extend F^* to a map of the whole vertex set of T_1 which induces an isomorphism of T_1 onto T_2. Moving clockwise from Q_3, let Q_4 be the next point adjacent to Q_1. As all faces of T_1 are triangles, Q_3 is adjacent to Q_4. Let P_4 be the vertex other than P_2 of T_2 which lies on a face with P_1 and P_3. If F is an isomorphism extending F^*, then as

$$F(Q_4) \neq F(Q_2) = P_2$$

and $F(Q_4)$ is adjacent to $F(Q_1) = P_1$ and $F(Q_3) = P_3$ we must have $F(Q_4) = P_4$. Continuing in this manner we see that F is determined at every vertex adjacent to Q_1. Each of these vertices then lies on a face at which F is determined; hence repeating the argument at these vertices, F is determined at all vertices at a (graph theoretical) distance two from Q_1. By induction this method determines the isomorphism F extending F^* if it exists. Applying this algorithm to all possible F^*'s and testing each F for isomorphism, we decide whether T_1 and T_2 are isomorphic.

In applying the generation algorithm to T' with $N - 1$ vertices, we do not use

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the third operation described above if its yields a triangulation with a valency three or four, or the second if it gives a triangulation with a valency three; for such triangulations will be obtained by applying earlier operations to another triangulation of $N - 1$ vertices. From formula (1) we see that if T has no valency three or four, then it has at least twelve vertices of valency five. A triangulation with twelve vertices of valency five is a regular polyhedron; there is only one such, the well-known icosahedron. Thus in obtaining all triangulations with $N \leqq 12$ vertices one need never apply the third operation, which in practice is by far the most time consuming. The generation of all such triangulations was carried out on the IBM 7094 in approximately $1\frac{1}{2}$ hours of computing time. As a check, the computation was carried out for $N \leqq 11$ with a general graph isomorphism routine (see [1] for a brief description of this routine). D. W. Grace [2] generated all trihedral polyhedra (the dual of triangulations) with 11 or fewer faces and our numbers check with his to that point. In the table below $L(N)$ is the number of triangulations and $M(N)$ the number with no valency three with N vertices.

N	$L(N)$	$M(N)$	N	$L(N)$	$M(N)$
6	2	1	10	233	12
7	5	1	11	1249	34
8	14	2	12	7595	130
9	50	5			

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University of California
Berkeley, California 94533

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