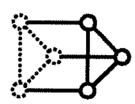
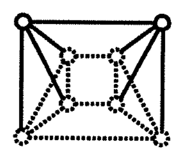
5

Regular Graphs ($v \leq 10$)

There is a trivial but important observation about natural numbers known as *The Frivolous Theorem of Arithmetic.* It states: Most natural numbers are very very very large. The same kind of observation cannot be avoided for simple graphs, especially when one is faced with trying to fathom the depths of large regular graphs, some of which are shown here.

Regular graphs, as already defined, have uniform vertex degrees, and compared to non-regulars there are surprisingly few of them. Because of their high orders of symmetry they are the gems of graph theory, and for the same reason they are the most difficult to capture in a drawing. Oddly enough it is this gem-like symmetry, this nearly total lack of distinguishing marks, which makes regular graphs difficult to study, difficult to generate, even difficult to count.





A regular graph is identified partly by its common degree. For example in a 3-regular graph the degree of every vertex is 3. It is well known that every 3-regular graph has a 2-regular subgraph — namely a cycle or a bouquet of disconnected cycles — and it was recently proved that every 4-regular graph has a 3-regular subgraph. Though it seems this is the beginning of a trend, it is not true that every 5-regular graph has a 4-regular subgraph. Similarly, the idea fails for all larger degrees of regularity.

Theorem 11 For $r \leq 4$, every r-regular simple graph contains an (r−1)-regular subgraph.

As in Chapter 2 the graphs are arranged in complementary pairs. Specifications are given in three lines describing topological and graph-theoretic names, outstanding characteristics, and symmetry group name and/or order. See Chapter 2 for explanations of symbols. And there are two new properties: T and S. For each pair of vertices in a transitive graph (T), there is an automorphism that maps one to the other. For each pair of edges in an edge-transitive graph, there is an automorphism that maps one to the other. A symmetric graph (S) is both transitive and edge-transitive. Note that the complement of a symmetric graph is not necessarily symmetric.

v = 2

0-regular — **1-regular**

E_2

K_2, P_2, $K_{1,1}$
S
$Z_2 = S_2$

v = 3

0-regular — **2-regular**

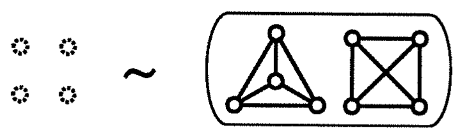
E_3

K_3, C_3, 3-cycle, triangle
g:f = 3:1, S
$S_3 = D_3$ (ord 6)

v = 4

0-regular — **3-regular**

E_4



K_4, W_4, tetrahedron
g:f = 3:4, S
S_4 (ord 24)

1-reg — **2-reg**

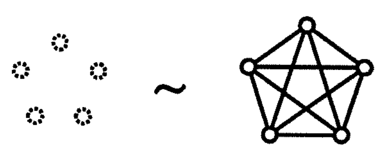
$2K_2$

C_4, $K_{2,2}$, 4-cycle, quadrilateral
g:f = 4:1, B S
$D_4 = Z_2(Z_2)$ (ord 8)

v = 5

0-regular — **4-regular**

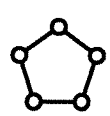
E_5



K_5, 4-simplex, Schläfli {3,3,3}
g:f = 3:10, S non-P
S_5 (ord 120)

2-reg

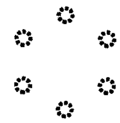
C_5, 5-cycle, pentagon
g:f = 5:1, S, self-complementary
D_5 (ord 10)



v = 6

0-regular — **5-regular**

E_6



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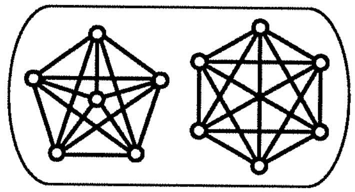
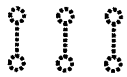


K_6
g:f = 3:20, S
S_6 (ord 720)

1-reg — **4-reg**

$3K_2$



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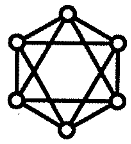
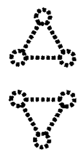


$K_{2,2,2}$, octahedron, 3-antiprism, 4-bipyramid
g:f = 3:8, P S
$S_4 \times Z_2 = S_3(Z_2)$ (ord 48)

2-reg — **3-reg**

$2K_3$

1



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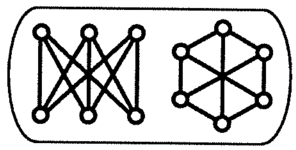
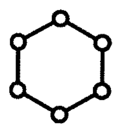


$K_{3,3}$, 9-rhomb, 3-Möbius-ladder,
Utilities graph
g:f = 4:9, B S non-P
$Z_2(S_3)$ (ord 72)

C_6, 6-cycle, hexagon
g:f = 6:1, B S
D_6 (ord 12)

2



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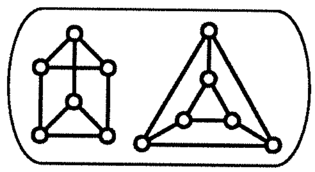
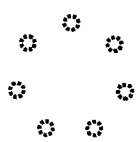


3-prism
g:f = 3:2, P T
$D_6 = S_3 \times Z_2$ (ord 12)

v = 7

0-regular — **6-regular**

E_7



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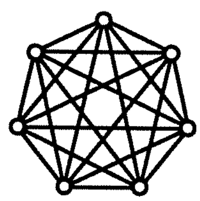
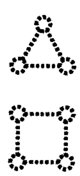


K_7
g:f = 3:35, S, torus embeddable
S_7 (ord 5040)

2-reg — **4-reg**

$C_3 + C_4$

1



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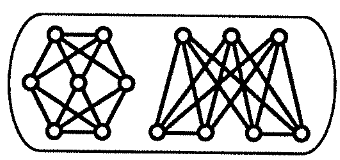
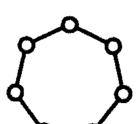


g:f = 3:6
$S_3 \times D_4$ (ord 48)

C_7, 7-cycle, heptagon
g:f = 7:1, S
D_7 (ord 14)

2



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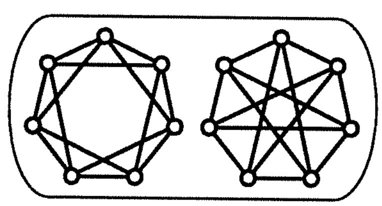
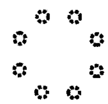


7-Möbius-truss
g:f = 3:7, T
D_7 (ord 14)

v = 8

0-regular — **7-regular**

E$_8$



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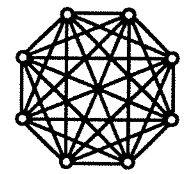
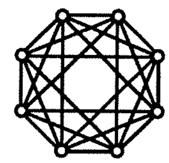


K$_8$
g:f = 3:56, S
S$_8$ (ord 40,320)

1-reg — **6-reg**

4K$_2$

~

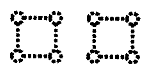


K$_{2,2,2,2}$, Schläfli {3,3,4}
g:f = 3:32, S
S$_4$(Z$_2$) (ord 384)

2-reg — **5-reg**

2C$_4$

1



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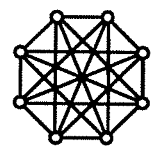
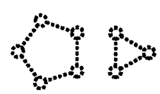


g:f = 3:16, T
Z$_2$(D$_4$) (ord 128)

C$_3$ + C$_5$

2



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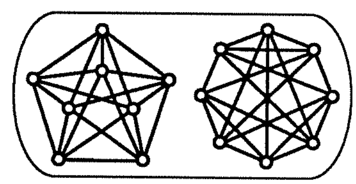
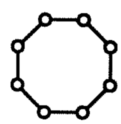


g:f = 3:15
D$_5$ × S$_3$ (ord 60)

C$_8$, 8-cycle, octagon
g:f = 8:1, B S
D$_8$ (ord 16)

3



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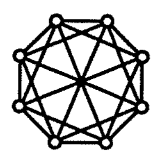
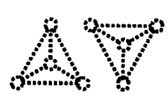


g:f = 3:16, T
D$_8$ (ord 16)

3-reg — **4-reg**

2K$_4$

1



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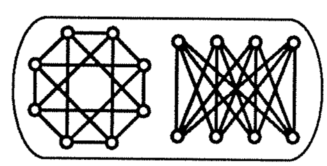
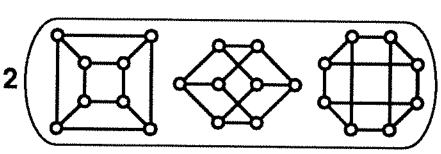


K$_{4,4}$
g:f = 4:36, B S
Z$_2$(S$_4$) (ord 1152)

3-cube, 4-prism
g:f = 4:6, B P S
S$_4$ × Z$_2$ = S$_3$(Z$_2$)
(ord 48)

2



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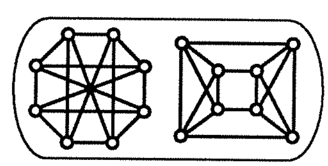
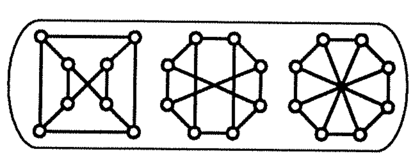


g:f = 3:8, T
S$_4$ × Z$_2$ (ord 48)

4-Möbius-ladder
g:f = 4:4, T
D$_8$ (ord 16)

3



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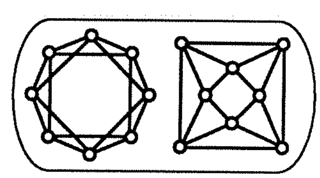
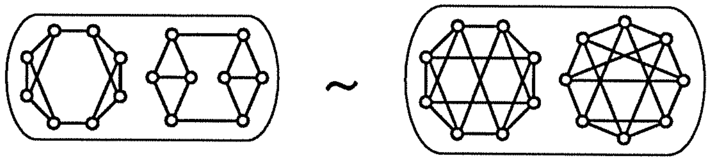


4-antiprism
g:f = 3:8, P T
D$_8$ (ord 16)

g:f = 3:4, P
$D_4 \times Z_2$ (ord 16)

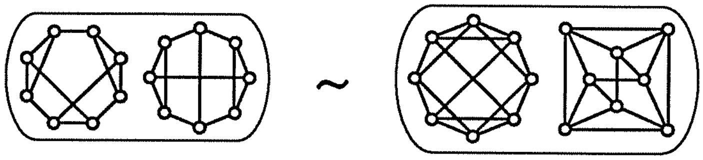
4



g:f = 3:4
$D_4 \times Z_2$ (ord 16)

bi-truncated tetrahedron
g:f = 3:2, P
$Z_2 \times Z_2$ (ord 4)

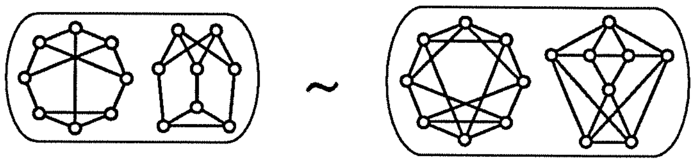
5



g:f = 3:6
$Z_2 \times Z_2$ (ord 4)

g:f = 3:1
$D_6 = S_3 \times Z_2$ (ord 12)

6

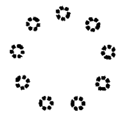


g:f = 3:7
D_6 (ord 12)

v = 9

0-regular — **8-regular**

E_9



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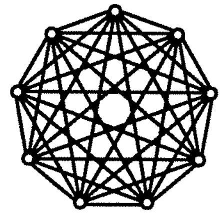
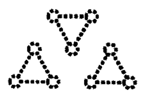


K_9
g:f = 3:84, S
S_9 (ord 362,880)

2-reg — **6-reg**

$3C_3$

1



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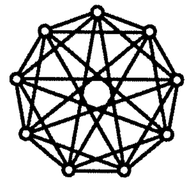
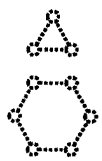


$K_{3,3,3}$
g:f = 3:27, S
$S_3(S_3)$ (ord 1296)

$C_3 + C_6$

2



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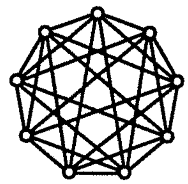
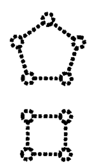


g:f = 3:29
$D_6 \times S_3$ (ord 72)

$C_4 + C_5$

3



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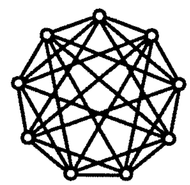
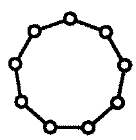


g:f = 3:30
$D_5 \times D_4$ (ord 80)

C_9, 9-cycle, nonagon
g:f = 8:1, S
D_9 (ord 18)

4



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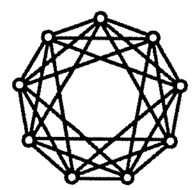
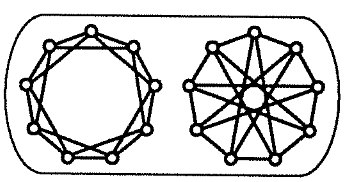


g:f = 3:30, T
D_9 (ord 18)

4-reg

9-Möbius-truss
g:f = 3:9, T
D_9 (ord 18)

1



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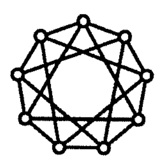
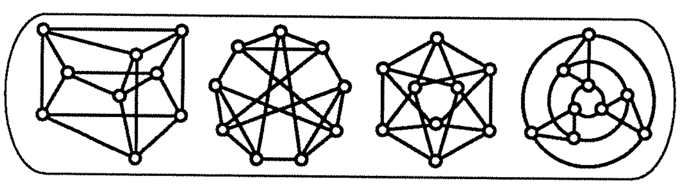


2

3,3-Möbius-torus
g:f = 3:3, T
D_9 (ord 18)

self-complementary

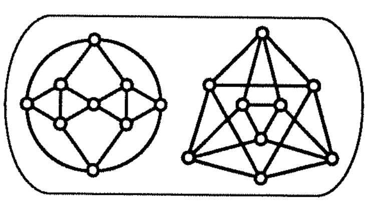
3



3,3-torus, $K_3 \times K_3$
g:f = 3:6, S
$Z_2(S_3)$ (ord 72)

g:f = 3:8, P
$D_6 = S_3 \times Z_2$ (ord 12)

4



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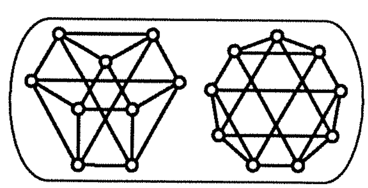
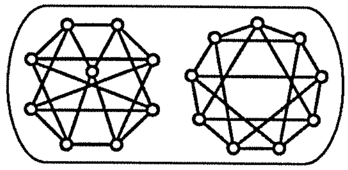


5

g:f = 3:4
D_6 (ord 12)

g:f = 3:2
$D_4 \times Z_2$ (ord 16)

6



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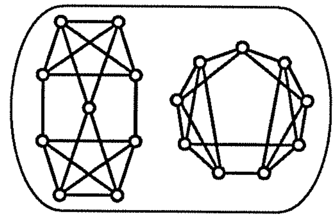
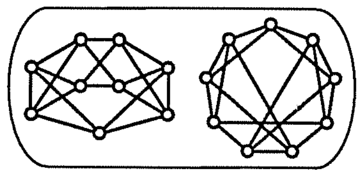


7

g:f = 3:10
$D_4 \times Z_2$ (ord 16)

self-complementary

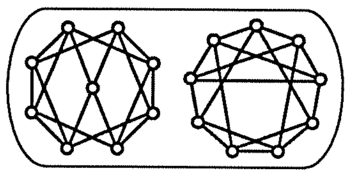
8



g:f = 3:6
(ord 32)

g:f = 3:4
D_4 (ord 8)

9



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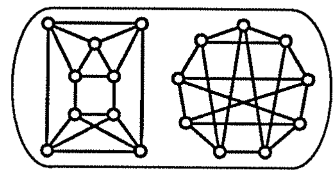
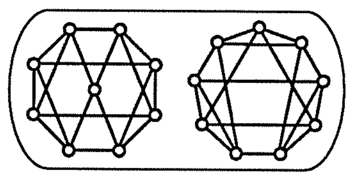


10

g:f = 3:8
D_4 (ord 8)

self-complementary

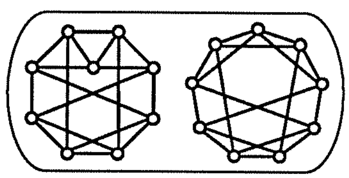
11



g:f = 3:6
$Z_2 \times Z_2 \times Z_2$ (ord 8)

g:f = 3:5
$Z_2 \times Z_2$

12



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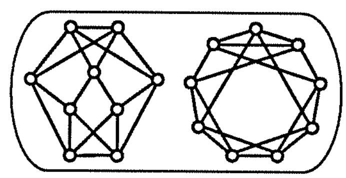
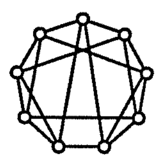


13

g:f = 3:7
$Z_2 \times Z_2$

g:f = 3:5
Z_2

14



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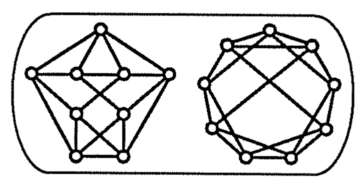
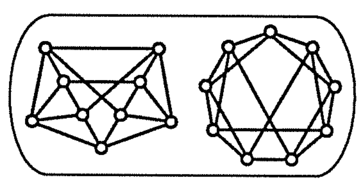


15

g:f = 3:7
Z_2

self-complementary

16



g:f = 3:6
Z_2