4

Trees ($v \leq 12$)

No subclass of simple graphs has been studied more carefully than the trees — the connected *acyclic* simple graphs — mainly because trees have been used for modeling problems ranging freely from the real to the numerical, and because computer design and use rely so heavily on abstract branching structures.

Every connected graph contains at least one tree that spans all the vertices of the graph, just as a skeleton spans a body. This is called a spanning tree. It is not obvious that fleshing out all trees with additional edges can generate all connected graphs. But it is easy to see that the deletion of edges from a connected graph can eliminate its cycles without disconnecting it. To prune a graph down to a tree, choose a cycle and erase one of its edges. This cannot disconnect the graph, since a pair of vertices **A** and **B** cannot have depended on that missing edge to be connected by a path. There is still an **A**-to-**B** path the other way around that cycle. Continue the process until no cycles remain. The result will not be a *forest* of disconnected trees but a single connected tree.

There are many interesting and useful facts about trees. Here are five of the best.

Theorem 10 In any tree:

(1) $v = e + 1$ (the number of vertices is one more than the number of edges).
(2) any two vertices are connected by a unique path.
(3) the removal of any edge disconnects the tree, and the removal of any vertex of degree at least 2 disconnects the tree.
(4) the addition of any one edge creates one cycle.
(5) there are one or two centers and one or two centroids (a CENTER of a tree is a vertex that minimizes the maximum distance to the other vertices; a CENTROID or *barycenter* of a tree is a vertex that minimizes the maximum weight of the branches that emanate from that vertex).

The following trees are ordered roughly according to the number of vertices of degree at least three. As in Chapter 1, the drawings have been left unmarked so that readers can assign diverse notations to them, but an additional set of drawings — beginning on page 135 — shows centers (filled with black) and centroids (circled).