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A formal operator involving Fermatian numbers

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Abstract: In this note, old and new properties of Fermatian numbers $\underline{z}_n = \frac{1-z^n}{1-z}$ are recalled. A new formal operator is defined and some identities and extensions are discussed.
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1 Introduction

Fermatian numbers can be defined by

$$\underline{z}_n = \frac{1-z^n}{1-z}$$

and they get their name from the French mathematician and lawyer, Pierre de Fermat (1607–1665). These numbers constitute the ordered set of integer solutions of the congruence in Fermat's Little Theorem:

$$z^{p-1} \equiv 1 \pmod{p}, \quad \text{for a prime } p. \tag{1}$$

For an integer $z > 1$, if a composite integer x divides $z^{x-1} - 1$, then x is called a *Fermat pseudoprime to base* z. Shanks called any integer solution of (1), including even numbers and other composites, a Fermatian number [12].

Various Fermatian properties were developed by Sylvester [19] and have since been extended by many mathematicians [11], especially for computational problems because (1) was the basis for early primality tests. We shall mainly consider them here as generalized integers in order to show their wide range of connections in number theory. For notational convenience, Carlitz [4] used $\underline{z}_n = [n]$, though it is less suggestive for some of the properties. Carlitz himself used $[n]$ with other meanings [3]. It is obvious that

$$\underline{z}_n! = \underline{z}_n\,\underline{z}_{n-1}\cdots\underline{z}_1$$

and

$$\underline{z}_n = \begin{cases} -z^n\,\underline{z}_{|n|}\,, & n < 0\,, \\ 1\,, & n = 0\,, \\ 1 + z + z^2 + \cdots + z^{n-1}\,, & n > 0\,, \end{cases}$$

and, therefore,

$$\underline{1}_n = n \quad \text{and} \quad \underline{1}_n! = n!\,.$$

To give the reader a feel for these generalized integers, Table 1 lists their first few values, and Tables 2 and 3 relate the partial sequences of Table 1 to some of the integer sequences in Sloane [18]. These can illuminate many more connections than those we will attempt here.

Table 1. First nine Fermatian numbers of the first nine indices

$z\backslash n$	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	1	3	7	15	31	63	127	255	511
3	1	4	13	40	121	364	1093	3280	9841
4	1	5	21	85	341	1365	5461	21845	87381
5	1	6	31	156	781	3906	19531	97656	488281
6	1	7	43	259	1555	9331	55987	335923	2015539
7	1	8	57	400	2801	19608	137257	960800	6725601
8	1	9	73	585	4681	37449	299593	2396745	19173961
9	1	10	91	820	7381	66430	597871	5380840	48427561

The sums of the rising diagonals are part of the sequence $\{1, 3, 7, 16, 39, 105, 315, 1048, \ldots\}$ which is the sequence A103439 [18], so it is worth checking other such connections. For instance, the first falling diagonal $\{1, 3, 13, 85, 781, 9331, \ldots\}$ in Table 1 is sequence A023037 in [18].

Table 2. $\underline{z}_n$ for subscripts $n = 1, 2, \ldots, 9$ in parts of Sloane sequences [18]

$\underline{z}_n$	$\underline{z}_1$	$\underline{z}_2$	$\underline{z}_3$	$\underline{z}_4$	$\underline{z}_5$	$\underline{z}_6$	$\underline{z}_7$	$\underline{z}_8$	$\underline{z}_9$
A	000012	000027	002061	053689	053699	053700	053716	053717	102909

Table 3. $\underline{z}_n$ for indices $z = 1, 2, \ldots, 9$ in parts of Sloane sequences [18]

$\underline{z}_n$	$\underline{1}_n$	$\underline{2}_n$	$\underline{3}_n$	$\underline{4}_n$	$\underline{5}_n$	$\underline{6}_n$	$\underline{7}_n$	$\underline{8}_n$	$\underline{9}_n$
A	000027	000225	003462	002450	003463	003464	023000	023001	002452

These are not new, but their presentation in this manner is [14, 15]. They suggest the use of difference operators, but we use analogies of differential operators in Section 3.

2 Some old and new properties

The notational advantages of $\underline{z}_n$ follow from the paper of Hoggatt and Bicknell [7] who proved that for the general left-justified r-nomial triangle induced by the expansion $\underline{z}_n$ and taking sums over p rows and then 1 column in effect that

$$\underline{z}_n^r = \sum_{j=0}^{n(r-1)} \left\{ \begin{matrix} n \\ j \end{matrix} \right\}_r x^j ,$$

where the r-nomial coefficient is the entry in the n-th row and j-th column of the generalized Pascal triangle [2]. Furthermore, corresponding to this, if T_n is sum of the rising diagonals of the multinomial triangle generated by $\underline{z}_n$ [5], i.e., $T_n = \sum_{k=1}^{r} T_{n-k}$, then

$$T_n = \sum_{j=0}^{\lfloor n(r-1)/2 \rfloor} \left\{ \begin{matrix} n-k \\ k \end{matrix} \right\}_r .$$

It is also known that the Fermatian numbers satisfy first and second order recurrence relations, namely,

$$\underline{z}_n = z\,\underline{z}_{n-1} + 1$$

and

$$\underline{z}_n = (z+1)\,\underline{z}_{n-1} - z\,\underline{z}_{n-2} \,. \tag{2}$$

The latter, not surprisingly, gives rise to some Horadam number analogies. Indeed, from the recurrence relation (2), with initial conditions stated earlier, from [20, (3.3)] (see also [1]), we may write

$$\underline{z}_{n+1} = \left(\sqrt{z}\right)^{n-1} \left((z+1)\, U_{n-1}\left(\frac{z+1}{2\sqrt{z}}\right) - \sqrt{z}\, U_{n-2}\left(\frac{z+1}{2\sqrt{z}}\right) \right),$$

where $\{U_n(x)\}_{n\geqslant 0}$ are the Chebyshev polynomials of the second kind satisfying the three-term recurrence relations

$$U_{n-1}(x) = 2xU_n(x) - U_{n+1}(x)\,, \quad \text{for } n = 1, 2, \ldots,$$

with the initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$, or, equivalently,

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}\,, \quad \text{with } x = \cos\theta \quad (0 \leqslant \theta < \pi),$$

for all $n = 0, 1, 2 \ldots$.

There do not seem to be clear recurrence relations connecting $\underline{z}_n$ and $\underline{z-1}_n$, but there are patterns if we utilize connections among the rows and columns in Table 1, as in (3) below. For example,

$$\underline{z}_1 = \underline{z-1}_1 + 0\,, \quad \underline{z}_2 = \underline{z-1}_2 + 1, \quad \underline{z}_3 = \underline{z-1}_3 + 2z\,,$$

but thereafter the connections are not so elegant, though we can relate them all back to $\underline{z}_3$ with partial recurrence relations, as is indicated in the next section.

In the days of 'open access' to journals, rather than list previously developed properties in detail the reader is referred to some key papers. Generalizations of the polynomials of Bernoulli, Euler and Hermite have been defined in terms of generalized integers in the form of these Fermatian integers [13]. These are closely related to the q-series extensively studied by Leonard Carlitz. These various analogues of the classical special functions are inter-related with one another and also to some of the problems posed by Morgan Ward. The works of Henry Gould and Vern Hoggatt for the ordinary Bernoulli and Euler numbers are also extensively cited to produce to produce analogous results built upon what were called Fermatian exponentials [16]. More specifically, some of the arithmetic functions which Mollie Horadam developed for sequences of generalized integers have been applied to Fermatian numbers [17].

Somewhat surprisingly,

$$z^{n-1} = \underline{z}_n^2 - \underline{z}_{n-1}\underline{z}_{n-2}$$

because of the neat relationship

$$z = \frac{\underline{z}_n - 1}{\underline{z}_{n-1}}\,,$$

from which we can obtain

$$\begin{aligned} \underline{z}_n^2 - \underline{z}_{n-1}\underline{z}_{n-2} &= \underline{z}_n^2 - \frac{1}{z}(\underline{z}_n - 1)(z\,\underline{z}_n + 1) \\ &= \underline{z}_n - \underline{z}_{n-1} \\ &= z^{n-1}\,, \end{aligned}$$

as required.

We may also obtain an arbitrary order recurrence relation to connect the columns of Table 1 through induction on n, namely:

$$\begin{aligned} \underline{z}_n &= z\,\underline{z}_{n-1} + 1 \\ &= z\,(z\,\underline{z}_{n-2} + 1) + 1 \\ &= z^2\,\underline{z}_{n-2} + z + 1 \\ &= z^2\,(z\,\underline{z}_{n-3} + 1) + z + 1 \\ &= z^3\,\underline{z}_{n-3} + z^2 + z + 1 \\ &= z^3\,\underline{z}_{n-3} + \underline{z}_3 \\ &\vdots \\ &= z^k\,\underline{z}_{n-k} + \underline{z}_k\,. \end{aligned} \tag{3}$$

For example,

$$\underline{z}_n = z^{n-3}\,\underline{z}_3 + \underline{z}_{n-3}\,.$$

3 Formal operators

We can extend these analogies further by defining *formally* operators (cf. [23]):

$$D_{z,x}x^n = (1 - z^n)\, x^n\,,$$

and

$$D_{x,z}x^n = \underline{z}_n\, x^n\,,$$

so that

$$D_{x,1}x^n = n\, x^{n-1} = Dx^n\,,$$

and

$$(1 - z)D_{x,z}x^n = (1 - z^n)\, x^n = D_{z,x}x^n\,.$$

Other formal properties follow if we define

$$D_{z,x}z = 0 \quad \text{and} \quad D_{x,z}ax^n = aD_{x,z}x^n\,,$$

where a is a constant, and for $f(y)$, a function on y,

$$D_{x,z}f(y) = D_{y,z}f(y)D_{x,z}y\,,$$

which when z is unity reduces to

$$D_{x,1}f(y) = D_{y,1}f(y)D_{x,1}y\,.$$

This can be demonstrated to be the ordinary chain rule $D_x f(y) = D_y f(y)D_x y$. Other properties then include

$$D_{x,z}y^n = \underline{z}_n\, y^{n-1}D_{y,z}y$$

and for u and v, functions of x,

$$D_{x,z}(u^n + v^n) = D_{x,z}u^n + D_{x,z}v^n\,,$$

and

$$D^n_{x,z}uv = \sum_{r=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} D^r_{x,z}u\, D^{n-r}_{x,z}v\,,$$

where

$$\begin{bmatrix} n \\ j \end{bmatrix} = \frac{\underline{z}_n}{\underline{z}_r\, \underline{z}_{n-r}}\,.$$

This is an analogue of Leibnitz' theorem for the n-th derivative of a product of two functions, namely,

$$D^n uv = \sum_{r=0}^{n} \binom{n}{j} D^r u\, D^{n-r} v\,.$$

We can then apply these to Fermatian Bernoulli, Euler and Hermite polynomials. For instance, consider Fermatian Bernoulli numbers (cf. [21]):

$$\frac{tE_z(xt)}{E_z(t) - 1} = \sum_{n=0}^{\infty} B_{n,z}\, \frac{t^n}{\underline{z}_n!},$$

in which we have Fermatian exponential functions

$$E_z(t) = \sum_{n=0}^{\infty} \frac{t^n}{z_n}\,.$$

To round off this section, we can suggest two more related formal operators for the interested reader to explore, namely,

$$I_{z,x} f(x) = D_{x,z}^{-1} f(x) \quad \text{and} \quad I_{x,z} f(x) = (1-z) I_{z,x}\,.$$

Thus, $I_{x,1} x^n$ and $\int x^n\,dx$ differ only by a constant, which be made zero with suitable limits in a given context.

4 Concluding comments

While applied mathematics depends on pure mathematics, the latter should not have to justify its existence through applications. Nevertheless, Fermatian numbers have applications in primality testing with Poulet numbers. Criteria for divisibility never go away because of their intimate connections with security [22].

Fermat pseudoprimes $[8, 9]$ to base 2 are also called Sarrus numbers or Poulet numbers, P_n [24]. The first few Poulet numbers are $\{341, 561, 645, 1105, 1387, \ldots\}$ (see [18, A001567]). Pomerance et al [10] computed $21\ 853$ Poulet numbers less than 25×10^9, and that the number of Poulet numbers $< x$ for sufficiently large x satisfy (cf. [6])

$$\exp\left((\ln x)^{5/14}\right) < P_2(x) < \exp\left(-\frac{\ln x \ln\ln\ln x}{2\ln\ln x}\right)\,.$$

Super-Poulet numbers are those Poulet numbers whose divisors d satisfy $d \mid 2(2^{d-1}-1)$, clearly related to Fermatian numbers, since a Poulet number is a Fermat pseudoprime to base 2.

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