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Topological structures induced by chromatic partitioning of vertex set of graphs

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Abstract: This paper presents a method of constructing topologies on vertex set of a graph G induced by chromatic partition of vertex set of the graph. It introduces colour lower approximation and colour upper approximation of vertex induced subgraphs and acquaints the open and closed sets of the topology generated by chromatic partition on the vertex set of graphs. It explores some of the properties of colour lower approximation and colour upper approximation of vertex induced subgraphs. It also establishes some new subgraphs based on the colour lower approximation and colour upper approximation and some of their properties have been studied.
Keywords: Vertex colouring, Chromatic partition, Colour lower approximation, Colour upper approximation, Graph chromatic topological space.
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1 Introduction

Theoretical topology is a bit dry but has got tremendous applications over many other fields. Although topology is a part of mathematics, it has influenced the whole world with strong effects and incredible applications. Most of the real life problems can be modeled and solved by graph theoretical concepts. The most important concept in set theory, namely, the relation on a set, is very useful not only in theoretical studies, but also, in practical applications on a wide scale. It acts as a key for bridging real life data with mathematical models such as graph theory and topological structures. Further, these notions affect the process of representing topologies or topological concepts via relations. When a practical problem has been modelled abstractly as a relation, its properties may be studied without referring to the original problem domain.

Vertex colouring and edge colouring in graph theory is used in various research areas of computer Science like data mining, clustering, image segmentation, image capturing, networking and so on. A data structure can be modeled as a tree in graph theory; network topologies in computer science can be modeled as standard and special graphs, and the new features of the original problem can be studied. Graph colouring is used in allocation of resource, scheduling and so on. Walks, paths and cycles in graph theory are used in many applications such as database design concepts, travelling salesman problem, resource networking, etc.

Allam et al. [1] studied topological structures induced by binary and preorder relations and obtained a quasi-discrete topology using a symmetric relation. Salama [11] used dominance relations and general binary relations to generate topological structures using the lower and upper approximations and also used the approach of closure and interior operator to induce topological structures. Lalithambigai and Gnanachandra in [6] described the method of generating topologies using the adjacency, incidence. non adjacency and non incidence relations on the vertex set of graphs. Lalithambigai and Gnanachandra in [8] described the method of generating topologies using the relations: in-valence, out-valence, reachability on the vertex set of digraphs. Gamorez et al. [3] determined the subbasis for the topologies on the vertex set of graphs that are obtained by the graph operations viz. the corona, the edge corona, disjunction, symmetric difference, tensor product and strong tensor product. Nianga and Canoy in [9] described the method of generating topologies on the vertex set of graphs that are obtained by the graph operations the complement, the join, the corona, the composition and the cartesian product. Nianga and Canoy in [10] used the hop neighbourhoods to generate topology on vertex set of graphs and studied topologies induced by some unary, binary operations on graphs. Lalithambigai et al. [7] explored the methods of generating topologies on vertex set of graphs using the graph mertics: distance, detour distance, circular distance and circular detour distance and related the topologies induced by different metrics for some special graphs. Many authors [4, 5, 12, 13] used various relators to generalize various topologies. This paper aims to generalize topologies induced by chromatic partitioning of vertex set of a graph which forms an equivalence relation on the vertex set of graphs.

2 Preliminaries

Fundamental definitions and preliminaries of graph theory can be found in the source [2].

An assignment of colours to the vertices of a graph in a way so that no two adjacent vertices get the same colour is called colouring of the graph. For each colour, the set of all points which get that colour is independent and is called a colour class. A colouring of a graph G using at most n colours is called an n-colouring. The chromatic number $\chi(G)$ of a graph G is the minimum number of colours needed to colour G. A graph G is called n-colourable if $\chi(G) \leq n$. Each n-colouring of G partitions $V(G)$ into n independent sets called colour classes. Such a partitioning induced by a $\chi(G)$ colouring of G is called a chromatic partitioning. A partition of $V(G)$ into the smallest possible number of independent sets is called a chromatic partitioning of G.

Two graphs G_1 and G_2 are isomorphic if there exists a bijection $f : V(G_1) \longrightarrow V(G_2)$ such that u, v are adjacent in G_1 if and only if $f(u), f(v)$ are adjacent in G_2. Two topological spaces (X, τ_1) and (Y, τ_2) are homeomorphic if there exists a bijection $f : (X, \tau_1) \longrightarrow (Y, \tau_2)$ such that both f and f^{-1} are continuous.

3 Topology induced by chromatic partition of a vertex set of graphs

This section, presents the method of generating topologies induced by chromatic partition of vertex set of graphs.

Definition 3.1. Let $G = (V(G), E(G))$ be a graph with chromatic number k and let $\mathcal{P} = \{[S_1], [S_2], \ldots, [S_k]\}$ be a chromatic partition of $V(G)$. The color neighbourhood of a vertex v of G with respect to $\mathcal{P}$, denoted by $CN(v; \mathcal{P})$, is equal to the colour class in $\mathcal{P}$ containing v, i.e., $CN(v; \mathcal{P}) = [S_i]$, where $v \in [S_i]$.

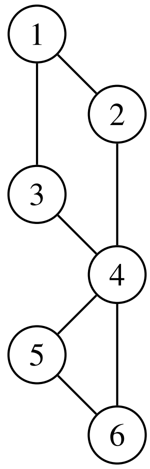
Define operators on the set of all vertex induced subgraphs of G to itself, as follows: For a vertex induced subgraph H of G, define $CL(V(H); \mathcal{P}) = \{v \in V(G) : CN(v; \mathcal{P}) \subseteq V(H)\}$ and $CU(V(H); \mathcal{P}) = \{v \in V(G) : CN(v; \mathcal{P}) \cap V(H) \neq \varnothing\}$.

$CL(V(H); \mathcal{P})$ is called colour lower approximation of $V(H)$ with respect to the chromatic partition $\mathcal{P}$ and $CU(V(H); \mathcal{P})$ is called colour upper approximation of $V(H)$ with respect to the chromatic partition $\mathcal{P}$. $CU(V(H); \mathcal{P}) - CL(V(H); \mathcal{P})$ is called colour boundary of $V(H)$ and is denoted by $CB(V(H); \mathcal{P})$.

Observation 3.2. (i) For every $v \in CL(V(H); \mathcal{P}), CN(v; \mathcal{P}) \subseteq V(H)$, and so $CN(v; \mathcal{P}) \cap V(H) \neq \varnothing$. Hence $v \in CU(V(H); \mathcal{P})$ and so $CL(V(H); \mathcal{P}) \subseteq CU(V(H); \mathcal{P})$.
(ii) Since $v \in CN(v; \mathcal{P})$, it follows that $v \in V(H)$ implies $v \in V(H) \cap CN(v; \mathcal{P})$. Hence $V(H) \subseteq CU(V(H); \mathcal{P})$.

Definition 3.3. Let G be a graph and $\mathcal{P}$ be a chromatic partition of $V(G)$. A vertex induced subgraph H of G is said to be $\mathcal{P}$ determinable if $CL(V(H); \mathcal{P}) = CU(V(H); \mathcal{P})$.

Example 3.4. Consider the following graph.



Let $\mathcal{P} = \{\{1,4\},\{2,3,5\},\{6\}\}$ be a chromatic partition of $V(G)$.
If H is a vertex induced subgraph of G with $V(H) = \{1,2,3,4\}$, then $CL(V(H);\mathcal{P}) = \{1,4\}$ and $CU(V(H);\mathcal{P}) = \{1,2,3,4,5\}$. So, H is not $\mathcal{P}$-determinable.
If W is a vertex induced subgraph of G with $V(W) = \{1,4\}$, then $CL(V(W);\mathcal{P}) = \{1,4\}$ and $CU(V(W);\mathcal{P}) = \{1,4\}$. So W is $\mathcal{P}$-determinable.
But W is not $\mathcal{P}'$-determinable where $\mathcal{P}' = \{\{1,5\},\{2,3,6\},\{4\}\}$, since $CL(V(W);\mathcal{P}') = \{4\}$ and $CU(V(W);\mathcal{P}') = \{1,5,4\}$.

The definition of the colour lower approximation and colour upper approximation of $V(H)$ ensures the following observations:

Observation 3.5.

(i) For a null graph G on n vertices, $CN(v;\mathcal{P}) = V(G)$ for all $v \in V(G)$, where $\mathcal{P}$ is the unique chromatic partition of the null graph G. So, for all vertex induced subgraphs H with $|V(H)| < n, CL(V(H);\mathcal{P}) = \varnothing$ and $CU(V(H);\mathcal{P}) = V(G)$.

(ii) In a complete graph, K_n, on n vertices, $CN(v;\mathcal{P}) = \{v\}$ for all $v \in V(K_n)$, where $\mathcal{P}$ is the unique chromatic partition of K_n. So, $CL(V(H);\mathcal{P}) = CU(V(H);\mathcal{P}) = V(H)$ for every vertex induced subgraph H of K_n. Hence every vertex induced subgraph of K_n is $\mathcal{P}$- determinable.

(iii) Consider the cycle graph, C_n, on n vertices where n is even. The chromatic number of C_n, when n is even, is 2. Let $\mathcal{P} = \{[S_1],[S_2]\}$ be the chromatic partition of $V(C_n)$. Let H be the vertex induced subgraph of C_n.
If $|V(H)| = 1$, then $CL(V(H);\mathcal{P}) = \varnothing$ and $CU(V(H);\mathcal{P}) = [S_i]$, where $[S_i]$, $i = 1$ or 2, is the colour class containing $V(H)$.
If $|V(H)| < \frac{n}{2}$, then $CL(V(H);\mathcal{P}) = \varnothing$.
Let $|V(H)| = \frac{n}{2}$. If $V(H) = [S_i]$, $i = 1$ or 2, then $CL(V(H);\mathcal{P}) = V(H)$; otherwise $CL(V(H);\mathcal{P}) = \varnothing$.
If $|V(H)| = n-1$, then $V(H) = [S_i] \cup A$, where $A \subseteq V(G) - [S_i], i = 1$ or 2. Hence $CL(V(H);\mathcal{P}) = [S_i], i = 1$ or 2.

If $|V(H)| > 1$ and $V(H) \subseteq [S_i], i = 1$ or 2, then $CU(V(H); \mathcal{P}) = [S_i], i = 1$ or 2.
If $|V(H)| > 1$ and $V(H) \cap [S_i] \neq \varnothing$, for $i = 1, 2$, then $CU(V(H); \mathcal{P}) = V(C_n)$.

(iv) Consider the cycle graph, C_n, on n vertices where n is odd. The chromatic number of C_n, when n is odd, is 3. Let $\mathcal{P} = \{[S_1], [S_2], [S_3]\}$ be the chromatic partition of $V(C_n)$ where $[S_3]$ is a singleton. For C_3, if H is a vertex induced subgraph of C_3 with $|V(H)| = 1$, then $CL(V(H); \mathcal{P}) = [S_i]$, where $[S_i]$ is the color class containing $V(H)$.
For $n \geq 5$, let H be the vertex induced subgraph of C_n. If $|V(H)| = 1$ and $V(H) \subseteq [S_i]$, $i = 1$ or 2, then $CL(V(H); \mathcal{P}) = \varnothing$.
If $|V(H)| = 1$ and $V(H) = [S_3]$, then $CL(V(H); \mathcal{P}) = [S_3]$.
If $|V(H)| = 1$, then $CU(V(H); \mathcal{P}) = [S_i]$, where $[S_i]$, $i = 1, 2$ or 3, is the colour class containing $V(H)$.
If $|V(H)| < [\frac{n}{2}]$, then $CL(V(H); \mathcal{P}) = \varnothing$.
Let $|V(H)| = [\frac{n}{2}]$. If $V(H) = [S_i], i = 1$ or 2, then $CL(V(H); \mathcal{P}) = V(H)$; otherwise $CL(V(H); \mathcal{P}) = \varnothing$.
If $|V(H)| = n - 1$, then $V(H) = [S_i] \cup A$, where $A \subseteq V(G) - [S_i]$, $i = 1, 2$ or 3. Hence $CL(V(H); \mathcal{P}) = [S_i]$, $i = 1, 2$ or 3.
If $|V(H)| > 1$ and $V(H) \subseteq [S_i]$, $i = 1, 2$ or 3, then $CU(V(H); \mathcal{P}) = [S_i]$, $i = 1, 2$ or 3.
If $|V(H)| > 1$ and $V(H) \cap [S_i] \neq \varnothing$ for i = 1,2,3, then $CU(V(H); \mathcal{P}) = V(C_n)$.

(v) For a path graph P_n the chromatic index is 2. Let $\mathcal{P} = \{[S_1], [S_2]\}$, where $[S_1] = \{v_1, v_3, \ldots\}$ and $[S_2] = \{v_2, v_4, \ldots\}$ be a chromatic partition of $V(P_n)$.
If H is a vertex induced subgraph such that either $V(H) = [S_i], i = 1$ or 2, or $V(H) = [S_i] \cup A$, where $A \subseteq V(G) - [S_i], i = 1$ or 2, then $CL(V(H); \mathcal{P}) = [S_i], i = 1$ or 2. In all other cases, $CL(V(H); \mathcal{P}) = \varnothing$.
If $|V(H)| < n$ and $V(H) \subseteq [S_i], i = 1$ or 2, or $V(H) \cap [S_i] \neq \varnothing$ for $i = 1, 2$, then $CU(V(H); \mathcal{P}) = [S_i], i = 1$ or 2.

(vi) Let G be a complete bipartite graph with bipartition (V_1, V_2) and $V_1 = \{v_1, v_2, \ldots, v_n\}$, $V_2 = \{u_1, u_2, \ldots, u_m\}$.
Let $\mathcal{P} = \{[S_1], [S_2]\}$ be the chromatic partition of $V(G)$. If H is a vertex induced subgraph of G with $V(H) \subseteq V_1; |V(H)| < n$, then $CL(V(H); \mathcal{P}) = \varnothing$ and $CU(V(H); \mathcal{P}) = V_1$ and if $V(H) \subseteq V_2; |V(H)| < m$, then $CL(V(H); \mathcal{P}) = \varnothing$ and $CU(V(H); \mathcal{P}) = V_2$.
If $V(H) = V_1$ or V_2, then $CL(V(H); \mathcal{P}) = CU(V(H); \mathcal{P}) = V_1$ or V_2. Hence the vertex induced subgraphs whose vertex set is either V_1 or V_2 are $\mathcal{P}$-determinable.

The following proposition presents the properties of the colour lower approximation and the colour upper approximation of $V(H)$.

Proposition 3.6. Let G be a graph with $\mathcal{P} = \{[S_1], [S_2], \ldots, [S_k]\}$ as the chromatic partition of $V(G)$. Let H and W be two vertex induced subgraphs of G. Then the following holds:

i. $CU(V(H \cup W); \mathcal{P}) = CU(V(H); \mathcal{P}) \cup CU(V(W); \mathcal{P})$ and
$CU(V(H \cap W); \mathcal{P}) \subseteq CU(V(H); \mathcal{P}) \cup CU(V(W); \mathcal{P})$.

ii. $CL(V(H \cup W);\mathcal{P}) \supseteq CL(V(H);\mathcal{P}) \cup CL(V(W);\mathcal{P})$ and
$CL(V(H \cap W);\mathcal{P}) \subseteq CL(V(H);\mathcal{P}) \cap CL(V(W);\mathcal{P})$.

iii. $CL(CU(V(H);\mathcal{P});\mathcal{P}) \subseteq CU(V(H);\mathcal{P})$; and
$CU(CL(V(H);\mathcal{P});\mathcal{P}) \subseteq CL(V(H);\mathcal{P})$.

Proof. i. Let $v \in CU(V(H \cup W);\mathcal{P})$. Then $CN(v;\mathcal{P}) \cap V(H \cup W) \neq \varnothing$. So $CN(v;\mathcal{P}) \cap (V(H) \cup V(W)) \neq \varnothing$. Hence $\{CN(v;\mathcal{P}) \cap V(H)\} \cup \{CN(v;\mathcal{P}) \cap V(W)\} \neq \varnothing$. So $CN(v;\mathcal{P}) \cap V(H) \neq \varnothing$ or $CN(v;\mathcal{P}) \cap V(W) \neq \varnothing$.
In either of the cases, $v \in CU(V(H);\mathcal{P}) \cup CU(V(W);\mathcal{P})$. Thus $CU(V(H \cup W);\mathcal{P}) \subseteq CU(V(H);\mathcal{P}) \cup CU(V(W);\mathcal{P})$.
Let $v \in CU(V(H);\mathcal{P}) \cup CU(V(W);\mathcal{P})$. Then $v \in CU(V(H);\mathcal{P})$ or $v \in CU(V(W);\mathcal{P})$. So $CN(v;\mathcal{P}) \cap V(H) \neq \varnothing$ or $CN(v;\mathcal{P}) \cap V(W) \neq \varnothing$. Hence $(CN(v;\mathcal{P}) \cap V(H)) \cup (CN(v;\mathcal{P}) \cap V(W)) \neq \varnothing$. So $CN(v;\mathcal{P}) \cap (V(H) \cup V(W)) \neq \varnothing$ and $v \in CU(V(H \cup W);\mathcal{P})$. Thus $CU(V(H);\mathcal{P}) \cup CU(V(W);\mathcal{P}) \subseteq CU(V(H \cup W);\mathcal{P})$. Hence $CU(V(H \cup W);\mathcal{P}) = CU(V(H);\mathcal{P}) \cup CU(V(W);\mathcal{P})$.

ii. Let $v \in CU(V(H \cap W);\mathcal{P})$. Then $CN(v;\mathcal{P}) \cap V(H \cap W) \neq \varnothing$. So $CN(v;\mathcal{P}) \cap V(H) \neq \varnothing$ or $CN(v;\mathcal{P}) \cap V(W) \neq \varnothing$. Hence $v \in CU(V(H);\mathcal{P})$ or $v \in CU(V(W);\mathcal{P})$. Hence $v \in CU(V(H);\mathcal{P}) \cup CU(V(W);\mathcal{P})$. Thus $CU(V(H \cap W);\mathcal{P}) \subseteq CU(V(H);\mathcal{P}) \cup CU(V(W);\mathcal{P})$.
Let $v \in CL(V(H);\mathcal{P}) \cup CL(V(W);\mathcal{P})$. Then $CN(v;\mathcal{P}) \subseteq V(H)$ or $CN(v;\mathcal{P}) \subseteq V(W)$. So $CN(v;\mathcal{P}) \subseteq V(H) \cup V(W)$. Thus $CN(v;\mathcal{P}) \subseteq V(H \cup W)$. So $v \in CL(V(H \cup W))$. Hence $CL(V(H \cup W);\mathcal{P}) \supseteq CL(V(H);\mathcal{P}) \cup CL(V(W);\mathcal{P})$.
Let $v \in CL(V(H \cap W);\mathcal{P})$. Then $CN(v;\mathcal{P}) \subseteq V(H \cap W)$. So $CN(v;\mathcal{P}) \subseteq V(H) \cap V(W)$. Thus $CN(v;\mathcal{P}) \subseteq V(H)$ and $CN(v;\mathcal{P}) \subseteq V(W)$. So $v \in CL(V(H);\mathcal{P}) \cap CL(V(W);\mathcal{P})$. Hence $CL(V(H \cap W);\mathcal{P}) \subseteq CL(V(H);\mathcal{P}) \cap CL(V(W);\mathcal{P})$.

iii. Let $v \in CL(CU(V(H);\mathcal{P});\mathcal{P})$. Then $CN(v;\mathcal{P}) \subseteq CU(V(H));\mathcal{P})$. Since $v \in CN(v;\mathcal{P})$, $v \in CU(V(H));\mathcal{P})$. Hence $CL(CU(V(H);\mathcal{P});\mathcal{P}) \subseteq CU(V(H));\mathcal{P})$.
Let $v \in CU(CL(V(H);\mathcal{P});\mathcal{P})$. So $CN(v;\mathcal{P}) \cap CL(V(H);\mathcal{P}) \neq \varnothing$. Let $u \in CN(v;\mathcal{P}) \cap CL(V(H);\mathcal{P})$. So $u \in CN(v;\mathcal{P})$ and $u \in CL(V(H);\mathcal{P})$. Now, $u \in CL(V(H);\mathcal{P}) \Rightarrow CN(u;\mathcal{P}) \subseteq V(H)$. Since $u \in CN(v;\mathcal{P}), CN(u;\mathcal{P}) = CN(v;\mathcal{P})$. So $CN(v;\mathcal{P}) \subseteq V(H)$. Thus $v \in CL(V(H);\mathcal{P})$. Hence $CU(CL(V(H);\mathcal{P});\mathcal{P}) \subseteq CL(V(H);\mathcal{P})$. □

Theorem 3.7. Let G be a graph with $\mathcal{P} = \{[S_1], [S_2], \ldots, [S_k]\}$ as a chromatic partition of $V(G)$. Then the following statements hold:

(i) $CL(V(H);\mathcal{P}) \subseteq V(H)$ for any vertex induced subgraphs H of G.

(ii) If H and W are vertex induced subgraphs of G such that $V(H) \subseteq V(W)$, then $CL(V(H);\mathcal{P}) \subseteq CL(V(W);\mathcal{P})$.

(iii) $CL(CL(V(H);\mathcal{P});\mathcal{P}) = CL(V(H);\mathcal{P})$.

Proof. (i) Let $v \in CL(V(H);\mathcal{P})$. So $CN(v;\mathcal{P}) \subseteq V(H)$. Since $v \in CN(v;\mathcal{P})$, it follows that $v \in V(H)$. Thus $CL(V(H);\mathcal{P}) \subseteq V(H)$.

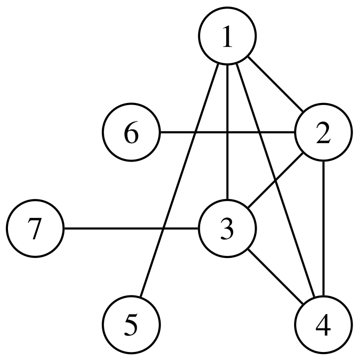
(ii) Let $v \in CL(V(H);\mathcal{P})$. So $CN(v;\mathcal{P}) \subseteq V(H) \subseteq V(W)$, which implies $v \in CL(V(W);\mathcal{P})$. Thus $CL(V(H);\mathcal{P}) \subseteq CL(V(W);\mathcal{P})$.

(iii) By the definition of $CL(V(H);\mathcal{P})$, it can be observed that $CL(V(H);\mathcal{P}) = \cup\{[S_i] : [S_i] \subseteq V(H)\}$ where $[S_i]$ are the colour classes. Hence it follows that $CL(CL(V(H);\mathcal{P});\mathcal{P}) = CL(V(H);\mathcal{P})$. □

The above theorem proves that $CL(V(H);\mathcal{P})$ is an interior operator on $V(G)$ for every vertex induced subgraph H and every chromatic partition $\mathcal{P}$ of $V(G)$. The operator $CL(V(H))$ induces a topology on $V(G)$ as given in the following definition.

Definition 3.8. Let G be a graph with chromatic partition $\mathcal{P} = [S_1], [S_2], \ldots, [S_k]\}$. Let $\tau_G^C(\mathcal{P})$ denote the topology on $V(G)$ induced by $CL(V(H);\mathcal{P})$. For any vertex induced subgraph H of G, $V(H) \in \tau_G^C(\mathcal{P})$ if and only if $CL(V(H);\mathcal{P}) = V(H)$. The topology $\tau_G^C(\mathcal{P})$ is called the graph chromatic topology and the pair $(V(G), \tau_G^C(\mathcal{P}))$ is called the graph chromatic topological space. The vertex induced subgraphs H of G with $V(H) \in \tau_G^C(\mathcal{P})$ are said to be open. The vertex induced subgraphs W of G are said to be closed if $V(G) - V(W) \in \tau_G^C(\mathcal{P})$.

Example 3.9. Consider the following graph.



Let $\mathcal{P} = \{\{1,7\},\{2,5\},\{3,6\},\{4\}\}$.
$\tau_G^C(\mathcal{P}) = \{\varnothing, \{1,7\}, \{2,5\}, \{3,6\}, \{4\}, \{1,7,2,5\}, \{1,7,3,6\}, \{1,7,4\}, \{2,5,3,6\}, \{2,5,4\}, \{3,6,4\}, \{2,5,3,6,4\}, \{1,7,3,6,4\}, \{1,7,2,5,4\}, \{1,2,3,5,6,7\}, \{1,2,3,4,5,6,7\}\}$.

Remark 3.10. 1. Let G be a graph with chromatic partition $\mathcal{P} = \{[S_1], [S_2], \ldots, [S_k]\}$. Then $\{[S_i] : 1 \le i \le \chi(G)\}$ is a basis for the topology $\tau_G^C(\mathcal{P})$.

2. $\tau_G^C(\mathcal{P})$ on vertex set of complete graphs are discrete topologies and $\tau_G^C(\mathcal{P})$ on vertex set of null graphs are indiscrete topologies.

3. A vertex induced subgraph H with $V(H) \in \tau_G^C(\mathcal{P})$ is both open and closed.

4. Let $\mathcal{P}'$ and $\mathcal{P}''$ be the chromatic partitions of the graphs G_1 and G_2 respectively. Let $\tau_{G_1}^C(\mathcal{P}')$ and $\tau_{G_2}^C(\mathcal{P}'')$ be the graph chromatic topologies on $V(G_1)$ and $V(G_2)$, respectively. Then $\tau_{G_1+G_2}^C(\mathcal{P})$, where $\mathcal{P} = \mathcal{P}' \cup \mathcal{P}''$ is a graph chromatic topology on $V(G_1 + G_2)$ which is finer than $\tau_{G_1}^C(\mathcal{P}')$ and $\tau_{G_2}^C(\mathcal{P}'')$.

5. Let $\mathcal{P} = \{[S_1], [S_2], \ldots, [S_k]\}$ be a chromatic partition of vertex set of a graph G. Let $\tau_G^C(\mathcal{P}) = \{U_i\}$. Let G' be a graph obtained by subdividing an edge uv of G. Let w be the new vertex obtained by subdividing the edge uv. If w is assigned a color i, $1 \leq i \leq \chi(G)$, then, on $V(G')$, $\tau_{G'}^C(\mathcal{P}) = \{F_i\}$, where $F_i = U_i \cup \{w\}$ when $[S_i] \subseteq U_i$; otherwise, $F_i = U_i$. If w is assigned a color j, where $j > \chi(G)$, then $\tau_G^C(\mathcal{P})$ on $V(G)$ is coarser than $\tau_{G'}^C(\mathcal{P})$ on $V(G')$.

Theorem 3.11. Let G be a graph with chromatic partition $\mathcal{P} = \{[S_1], [S_2], \ldots, [S_k]\}$. Let $\tau_G^C(\mathcal{P})$ be the topology on $V(G)$ induced by $CL(V(H); \mathcal{P})$. Then the following holds:

(i) $CU(V(H); \mathcal{P}) = V(G) - (CL(V(G) - V(H)); \mathcal{P})$, that is, $CU(V(H); \mathcal{P})$ is a closure operator on the topological space $(V(G), \tau_G^C(\mathcal{P}))$.

(ii) $CL(V(H); \mathcal{P}) = V(H)$ if and only if $CU(V(H); \mathcal{P}) = V(H)$.

Proof. (i). $v \in CU(V(H); \mathcal{P}) \Leftrightarrow CN(v; \mathcal{P}) \cap V(H) \neq \varnothing$
$\Leftrightarrow$ there exists $u \in CN(v; \mathcal{P})$ and $u \in V(H)$
$\Leftrightarrow$ there exists $u \in CN(v; \mathcal{P})$ and $u \notin V(G) - V(H)$
$\Leftrightarrow CN(v; \mathcal{P}) \nsubseteq V(G) - V(H)$
$\Leftrightarrow v \notin CL(V(G) - V(H); \mathcal{P})$
$\Leftrightarrow v \in V(G) - (CL(V(G) - V(H)); \mathcal{P})$
Hence $CU(V(H); \mathcal{P}) = V(G) - (CL(V(G) - V(H)); \mathcal{P})$.
(ii). Assume that $CL(V(H); \mathcal{P}) = V(H)$.
By (i), $CU(V(H); \mathcal{P}) = V(G) - (CL(V(G) - V(H)); \mathcal{P}) = V(G) - (V(G) - V(H)) = V(H)$.
Conversely, assume that $CU(V(H); \mathcal{P}) = V(H)$.
$V(H) = V(G) - (V(G) - V(H)) = V(G) - (CU(V(G) - V(H)); \mathcal{P}) = V(G) - (V(G) - (CL(V(H)); \mathcal{P})) = CL(V(H); \mathcal{P})$. $\square$

The following theorem characterizes the determinable subgraph in terms of topology $\tau_G^C(\mathcal{P})$.

Theorem 3.12. Let G be a graph with chromatic partition $\mathcal{P} = \{[S_1], [S_2], \ldots, [S_k]\}$. For every vertex induced subgraphs H of G, the following statements are equivalent:
(i) H is $\mathcal{P}$-determinable.
(ii) $V(H) \in \tau_G^C(\mathcal{P})$ i.e., $CL(V(H); \mathcal{P}) = V(H)$
(iii) $V(G) - V(H) \in \tau_G^C(\mathcal{P})$. i.e., $CU(V(H); \mathcal{P}) = V(H)$

Proof. $\underline{(i) \Rightarrow (ii)}$ By Theorem 3.7, $CL(V(H); \mathcal{P}) \subseteq V(H)$. Now, $v \notin CL(V(H); \mathcal{P})$ and H is $\mathcal{P}$-determinable implies $v \notin CU(V(H); \mathcal{P})$. So, $CN(v; \mathcal{P}) \cap V(H) = \varnothing$. Since $v \in CN(v; \mathcal{P})$, it follows that $v \notin V(H)$. Hence $V(H) \subseteq CL(V(H); \mathcal{P})$.
$\underline{(ii) \Rightarrow (iii)}$ By Theorem 3.7, $CL(V(G) - V(H); \mathcal{P}) \subseteq V(G) - V(H)$. Now, $v \notin CL(V(G) - V(H); \mathcal{P})$ implies $v \in V(G) - (CL(V(G) - V(H); \mathcal{P}))$, which implies $v \in CU(V(H); \mathcal{P})$, which implies $CN(v; \mathcal{P}) \cap V(H) \neq \varnothing$. Hence there exists $u \in CN(v; \mathcal{P})$ and $u \in V(H)$. So $u \in CN(v; \mathcal{P})$ and $u \in CL(V(H); \mathcal{P})$. Hence $CN(v; \mathcal{P}) = CN(u; \mathcal{P})$ and $CN(u; \mathcal{P}) \subseteq V(H)$. Thus $CN(v; \mathcal{P}) \subseteq V(H)$, which implies $v \in V(H)$. So $v \notin V(G) - V(H)$. Hence $V(G) - V(H) \subseteq CL(V(G) - V(H); \mathcal{P})$.

$\underline{(iii) \Rightarrow (i)}$ By hypothesis, $CL(V(G)-V(H);\mathcal{P}) = V(G)-V(H) \Rightarrow V(G)-(CU(V(H);\mathcal{P})) = V(G)-V(H) \Rightarrow CU(V(H);\mathcal{P}) = V(H)$. By Theorem 3.11, $CL(V(H);\mathcal{P}) = V(H)$. So $CL(V(H);\mathcal{P}) = CU(V(H))$. Hence H is $\mathcal{P}$-determinable. □

Theorem 3.13. In a graph chromatic topological space $(V(G), \tau_G^C(\mathcal{P}))$, the following holds for every vertex induced subgraph H:

(i) $CU(CL(V(H);\mathcal{P});\mathcal{P}) = (CB(V(H);\mathcal{P}))^c$ when $(CB(V(H);\mathcal{P})^c \subseteq CU(V(H);\mathcal{P})$.

(ii) $CU(CB(V(H);\mathcal{P});\mathcal{P}) = (CL(V(H);\mathcal{P}))^c$ when $(CL(V(H);\mathcal{P}))^c \subseteq CU(V(H);\mathcal{P})$.

Proof. (i) Let $v \in CU(CL(V(H);\mathcal{P});\mathcal{P})$. So, $CN(v;\mathcal{P}) \cap V(H) \neq \varnothing$. Hence there exists $u \in CN(v;\mathcal{P})$ and $CN(u;\mathcal{P}) \subseteq V(H)$. Since $u \in CN(v;\mathcal{P}), CN(u;\mathcal{P}) = CN(v;\mathcal{P})$. So, $v \in CL(V(H);\mathcal{P})$. Since $(CB(V(H);\mathcal{P}))^c \subseteq CU(V(H);\mathcal{P})$ and $CL(V(H);\mathcal{P}) \subseteq CU(V(H);\mathcal{P}), v \in (CB(V(H);\mathcal{P}))^c$, which implies $CU(CL(V(H);\mathcal{P});\mathcal{P}) \subseteq (CB(V(H);\mathcal{P}))^c$.

Let $v \in (CB(V(H);\mathcal{P}))^c$. Since $(CB(V(H);\mathcal{P}))^c \subseteq CU(V(H);\mathcal{P})$ and $CL(V(H);\mathcal{P}) \subseteq CU(V(H);\mathcal{P}), v \in CL(V(H);\mathcal{P})$. Also $v \in CN(v;\mathcal{P})$. So $CN(v;\mathcal{P}) \cap CL(V(H);\mathcal{P}) \neq \varnothing$. Hence $v \in CU(CL(V(H);\mathcal{P});\mathcal{P})$.

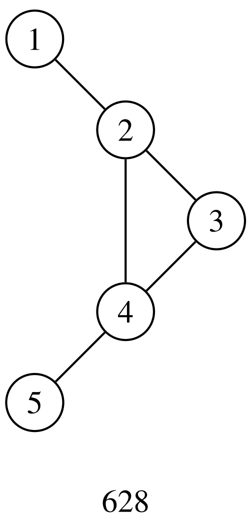
Thus $CU(CL(V(H);\mathcal{P});\mathcal{P}) = (CB(V(H);\mathcal{P}))^c$.

(ii) Let $v \in CU(CB(V(H);\mathcal{P});\mathcal{P})$. So, $CN(v;\mathcal{P}) \cap CB(V(H);\mathcal{P}) \neq \varnothing$. Hence there exists $u \in CN(v;\mathcal{P})$ and $u \in CB(V(H);\mathcal{P})$. So, $u \in CN(v;\mathcal{P})$ and $u \notin CL(V(H);\mathcal{P})$. Since $u \in CN(v;\mathcal{P}), CN(u;\mathcal{P}) = CN(v;\mathcal{P})$. So, $CN(v;\mathcal{P}) \nsubseteq V(H)$. Hence $v \notin CL(V(H);\mathcal{P})$ which gives $v \in (CL(V(H);\mathcal{P}))^c$. Hence $CU(CB(V(H);\mathcal{P});\mathcal{P}) \subseteq (CL(V(H);\mathcal{P}))^c$.

Let $v \in (CL(V(H);\mathcal{P}))^c$. Since $(CL(V(H);\mathcal{P}))^c \subseteq CU(V(H);\mathcal{P}), v \in CU(V(H);\mathcal{P})$. So, $v \in CB(V(H);\mathcal{P})$. So, $CN(v;\mathcal{P}) \cap CB(V(H);\mathcal{P}) \neq \varnothing$. Hence,
$v \in CU(CB(V(H);\mathcal{P});\mathcal{P})$ which gives $(CL(V(H);\mathcal{P}))^c \subseteq CU(CB(V(H);\mathcal{P});\mathcal{P})$. Thus, $CU(CB(V(H);\mathcal{P});\mathcal{P}) = (CL(V(H);\mathcal{P}))^c$. □

The following example illustrates that equality does not hold in Theorem 3.13 when $(CB(V(H);\mathcal{P}))^c \nsubseteq CU(V(H);\mathcal{P})$ or $(CL(V(H);\mathcal{P}))^c \nsubseteq CU(V(H);\mathcal{P})$.

Example 3.14. Consider the following graph.



Let $\mathcal{P} = \{\{1,3,5\},\{2\},\{4\}\}$
Let $V(H) = \{1,2,3\}$.
$CU(V(H);\mathcal{P}) = \{1,2,3,5\}; CL(V(H);\mathcal{P}) = \{2\}; CB(V(H);\mathcal{P}) = \{1,3,5\}$.
$(CL(V(H);\mathcal{P}))^c = \{1,3,4,5\}$ and $(CB(V(H);\mathcal{P}))^c = \{2,4\}$.
$CU(CL(V(H);\mathcal{P});\mathcal{P}) = \{2\} \neq \{2,4\}; CU(CB(V(H);\mathcal{P})) = \{1,3,5\} \neq \{1,3,4,5\}$.

Theorem 3.15. Let G_1 and G_2 be two graphs with the chromatic partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively. If the graphs G_1 and G_2 are isomorphic, then the chromatic topological spaces $(V(G_1), \tau^C_{G_1}(\mathcal{P}_1))$ and $(V(G_2), \tau^C_{G_2}(\mathcal{P}_2))$ are homeomorphic.

Proof. Since G_1 and G_2 are isomorphic, there exists a bijection $f : V(G_1) \longrightarrow V(G_2)$ such that u, v are adjacent in G_1 if and only if $f(u), f(v)$ are adjacent in G_2. Clearly, $\chi(G_1) = \chi(G_2)$. Let $\chi(G_1) = \chi(G_2) = \chi$. For all $i = 1,2,\ldots,\chi$, if $[S_i] = \{v_1, v_2, \ldots, v_n\}$ in G_1, then in G_2, $[S_i] = \{f(v_1), f(v_2), \ldots, f(v_n)\}$. Hence, it is easy to prove f and f^{-1} are continuous. Thus, $(V(G_1), \tau^C_{G_1}(\mathcal{P}_1))$ and $(V(G_2), \tau^C_{G_2}(\mathcal{P}_2))$ are homeomorphic. □

Corollary 3.16. If G is self complementary, then the chromatic topological spaces $(V(G), \tau^C_G(\mathcal{P}))$ and $(V(G^c), \tau^C_{G^c}(\mathcal{P}))$ are homeomorphic.

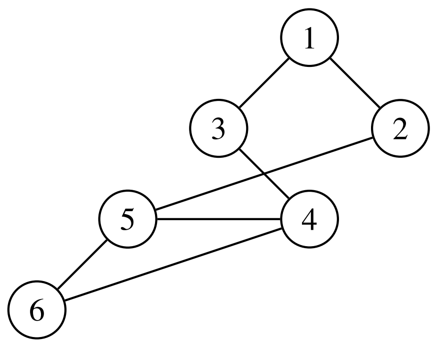
Proof. Since G is self complementary, G and G^c are isomorphic. Hence by Theorem 3.15, $(V(G), \tau^C_G(\mathcal{P}))$ and $(V(G^c), \tau^C_{G^c}(\mathcal{P}))$ are homeomorphic. □

4 Some new subgraphs with respect to graph chromatic topology

This section establishes some new subgraphs with respect to the topologies induced by chromatic partition of vertex set of graphs and explores some of their properties.

Definition 4.1. Let $(V(G), \tau^C_G(\mathcal{P}))$ be the graph chromatic topological space. Pivot of a vertex induced subgraph H, $PV(H;\mathcal{P})$, is defined as $PV(H;\mathcal{P}) = \cap\{V(W) : W$ is a vertex induced subgraph of G; V(W) $\in \tau^C_G(\mathcal{P})$ and $V(H) \subseteq V(W)\}$

Example 4.2. Consider the following graph.



Let $\mathcal{P} = \{\{1,5\},\{2,3,6\},\{4\}\}$
$\tau^C_G(\mathcal{P}) = \{\varnothing, \{1,5\}, \{2,3,6\}, \{4\}, \{1,2,3,5,6\}, \{1,4,5\}, \{2,3,4,6\}, \{1,2,3,4,5,6\}\}$
Let $V(H) = \{1,2,3\}$. $PV(H;\mathcal{P}) = \{1,2,3,5,6\}$.

The definition of Pivot of a vertex induced subgraph ensures the following observations:

Observation 4.3. Let G be a graph with chromatic partition $\mathcal{P} = \{[S_1], [S_2], \ldots, [S_k]\}$. Let H be a vertex induced subgraph of G. Then the following holds:

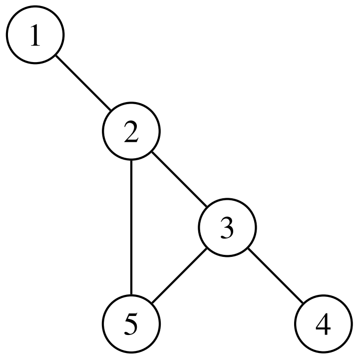
1. If $V(H) = \{v_i\}$, then $PV(H; \mathcal{P}) = [S_i]$ where $[S_i]$ is the colour class containing v_i.
2. If $V(H) \in \tau_G^C(\mathcal{P})$, then $PV(H; \mathcal{P}) = V(H)$.
3. In all the other cases, $PV(H; \mathcal{P})$ is the union of the colour classes containing the vertices of H.

Definition 4.4. Let G be a graph with chromatic partition $\mathcal{P}$. A vertex induced subgraph H of G is said to be a $\mathcal{P}-$ pivotic subgraph if $PV(H; \mathcal{P}) = V(H)$.

Remark 4.5. A vertex induced subgraph H of a graph G with $V(H) \in \tau_G^C(\mathcal{P})$ is $\mathcal{P}-$ pivotic.

Definition 4.6. Let G be a graph with chromatic partition $\mathcal{P}$. A vertex induced subgraph H of G is said to be a chromatic free subgraph with respect to $\mathcal{P}$ if $v \notin CU(V(H) - \{v\}; \mathcal{P})$ for every $v \in V(H)$.

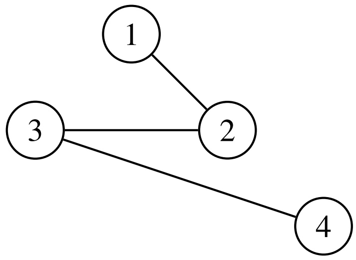
Example 4.7. Consider the following graph.



Let $\mathcal{P} = \{\{1,3\}, \{2,4\}, \{5\}\}$
$\tau_G^C(\mathcal{P}) = \{\varnothing, \{1,3\}, \{2,4\{, \{5\}, \{1,2,3,4\}, \{1,3,5\}, \{2,4,5\}, \{1,2,3,4,5\}\}$
Let $V(H) = \{2,3\}$.
$CU(\{3\}; \mathcal{P}) = \{1,3\}; CU(\{2\}; \mathcal{P}) = \{2,4\}$. So, H is a chromatic free subgraph of G.
Let $V(H) = \{1,3\}$.
$CU(\{3\}; \mathcal{P}) = \{1,3\}$ and $1 \in CU(\{3\}; \mathcal{P})$. So H is not a chromatic free subgraph of G.

Remark 4.8. A vertex induced subgraph H with $V(H) \in \tau_G^C(\mathcal{P})$ is not a chromatic free subgraph of a graph G.
The converse need not be true. i.e, The vertex set of a vertex induced subgraph which is not a chromatic free subgraph does not belong to $\tau_G^C(\mathcal{P})$

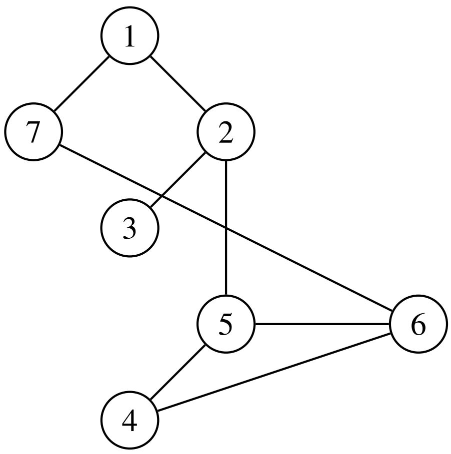
Example 4.9. Consider the following graph.



Let $\mathcal{P} = \{\{1,3\},\{2,4\}\}$
$\tau_G^C(\mathcal{P}) = \{\varnothing, \{1,3\},\{2,4\},\{1,2,3,4\}\}$
Let $V(H) = \{1,2,3\}$.
$CU(\{2,3\};\mathcal{P}) = \{1,2,3,4\}$ and $1 \in CU(\{2,3\};\mathcal{P})$.
So H is not a free subgraph of G with respect to $\mathcal{P}$. But $V(H) \notin \tau_G^C(\mathcal{P})$.

Definition 4.10. Let $(V(G), \tau_G^C(\mathcal{P}))$ be the graph chromatic topological space. Let H be a vertex induced subgraph of G and $v \in V(G)$. v is said to be a chromatic adherent vertex of $V(H)$ with respect to $\mathcal{P}$ if $M \cap (V(H) - \{v\}) \neq \varnothing$ for every $M \in \tau_G^C(\mathcal{P})$ containing v. The set of all chromatic adherent vertices of $V(H)$ is called the chromatic derived set of $V(H)$ with respect to $\mathcal{P}$ and is denoted by $CD(V(H);\mathcal{P})$.

Example 4.11. Consider the following graph.



$\mathcal{P} = \{\{1,3,6\},\{2,4,7\},\{5\}\}$
$\tau_G^C(\mathcal{P}) = \{\varnothing, \{1,3,6\},\{2,4,7\},\{5\},\{1,6,3,5\},\{1,6,3,7,2,4\},\{7,2,4,5\},\{1,2,3,4,5,6,7\}\}$
Let $V(H) = \{1,2,3\}$.
$CD(V(H);\mathcal{P}) = \{1,3,4,6,7\}$

Theorem 4.12. In a graph chromatic topological space $(V(G), \tau_G^C(\mathcal{P}), CU(V(H);\mathcal{P}) = V(H) \cup CD(V(H);\mathcal{P})$ for every vertex induced subgraph H of G.

Proof. Let $v \in V(H) \cup CD(V(H);\mathcal{P})$. Then $v \in V(H)$ or $v \in CD(V(H);\mathcal{P})$. If $v \in V(H)$, then $v \in CU(V(H);\mathcal{P})$. Let $v \notin V(H)$. Then $v \in CD(V(H);\mathcal{P})$. So, $M \cap (V(H) - \{v\}) \neq \varnothing$ for every $M \in \tau_G^C(\mathcal{P})$ containing v. Since $v \notin V(H)$, $M \cap V(H) \neq \varnothing$. Hence, $v \in CU(V(H);\mathcal{P})$. Therefore, $V(H) \cup CD(V(H);\mathcal{P}) \subseteq CU(V(H);\mathcal{P})$. Let $v \in CU(V(H);\mathcal{P})$.

If $v \in V(H)$, then $v \in V(H) \cup CD(V(H);\mathcal{P})$. Let $v \notin V(H)$. Since $v \in CU(V(H);\mathcal{P})$, $V(H) \cap M \neq \varnothing$ for every $M \in \tau_G^C(\mathcal{P})$ containing v. Hence, $M \cap (V(H) - \{v\}) \neq \varnothing$. So, $v \in CD(V(H);\mathcal{P})$. Thus $v \in V(H) \cup CD(V(H);\mathcal{P})$. Therefore, $CU(V(H);\mathcal{P}) \subseteq V(H) \cup CD(V(H);\mathcal{P})$. Hence, $CU(V(H);\mathcal{P}) = V(H) \cup CD(V(H);\mathcal{P})$. $\square$

Corollary 4.13. For a vertex induced subgraph W, $V(W) \in \tau_G^C(\mathcal{P})$ if and only if $CD(V(W);\mathcal{P}) \subseteq V(W)$.

Proof. $V(W) \in \tau_G^C(\mathcal{P})$ if and only if $CL(V(W);\mathcal{P}) = V(W)$, if and only if $CU(V(W);\mathcal{P}) = V(W)$, if and only if $V(W) \cup CD(V(W);\mathcal{P}) = V(W)$, if and only if $CD(V(W);\mathcal{P}) \subseteq V(W)$. $\square$

Theorem 4.14. Let $(V(G), \tau_G^C(\mathcal{P}))$ be a graph chromatic topological space and H, W be vertex induced subgraphs of G. Then the following holds:

i. If $V(H) \subseteq V(W)$, then $CD(V(H);\mathcal{P}) \subseteq CD(V(W);\mathcal{P})$.

ii. $CD(V(H \cup W);\mathcal{P}) = CD(V(H);\mathcal{P}) \cup CD(V(W);\mathcal{P})$.

iii. $CD(V(H \cap W);\mathcal{P}) \subseteq CD(V(H);\mathcal{P}) \cap CD(V(W);\mathcal{P})$.

iv. If $v \in CD(V(H);\mathcal{P})$, then $v \in CD(V(H) - \{v\};\mathcal{P})$.

Proof. i. Let $v \in CD(V(H);\mathcal{P})$. Then, $M \cap (V(H) - \{v\}) \neq \varnothing$ for every $M \in \tau_G^c$ containing v. Since $V(H) \subseteq V(W), M \cap (V(W) - \{v\}) \neq \varnothing$ for every $M \in \tau_G^C(\mathcal{P})$ containing v. So, $v \in CD(V(W);\mathcal{P})$. Hence $CD(V(H);\mathcal{P}) \subseteq CD(V(W);\mathcal{P})$.

ii. As, $V(H) \subseteq V(H \cup W)$ and $V(W) \subseteq V(H \cup W)$, by (i), $CD(V(H);\mathcal{P}) \cup CD(V(W);\mathcal{P}) \subseteq CD(V(H \cup W);\mathcal{P})$. Let $v \notin CD(V(H);\mathcal{P}) \cup CD(V(W);\mathcal{P})$. Then, $v \notin CD(V(H);\mathcal{P})$ and $v \notin CD(V(W);\mathcal{P})$. So, there exists M and $N \in \tau_G^C(\mathcal{P})$ containing v such that $M \cap (V(H) - \{v\}) = \varnothing$ and $N \cap (V(W) - \{v\}) = \varnothing$. Hence it follows that, $(M \cap N) \cap (V(H) - \{v\}) = \varnothing$ and $(M \cap N) \cap (V(W) - \{v\}) = \varnothing$. Also, $M \cap N \in \tau_G^C(\mathcal{P})$ and $v \in M \cap N$. Therefore, $(M \cap N) \cap ((V(H) \cup V(W)) - \{v\}) = \varnothing$. So, $v \notin CD(V(H \cup W);\mathcal{P})$. Hence $CD(V(H \cup W);\mathcal{P}) \subseteq CD(V(H);\mathcal{P}) \cup CD(V(W);\mathcal{P})$. Thus $CD(V(H \cup W);\mathcal{P}) = CD(V(H);\mathcal{P}) \cup CD(V(W);\mathcal{P})$.

iii. As, $V(H) \cap V(W) \subseteq V(H)$ and $V(H) \cap V(W) \subseteq V(W)$, by (i), $CD(V(H) \cap V(W);\mathcal{P}) \subseteq CD(V(H);\mathcal{P})$ and $CD(V(H) \cap V(W);\mathcal{P}) \subseteq CD(V(W);\mathcal{P})$. So, $CD(V(H) \cap V(W);\mathcal{P}) \subseteq CD(V(H);\mathcal{P}) \cap CD(V(W);\mathcal{P})$.

iv. Let $v \in CD(V(H);\mathcal{P})$. Then, $M \cap (V(H) - \{v\}) \neq \varnothing$ for every $M \in \tau_G^C(\mathcal{P})$ containing v. As, $(V(H) - \{v\}) - \{v\} = V(H) - \{v\}$, $M \cap ((V(H) - \{v\}) - \{v\}) \neq \varnothing$ for every $M \in \tau_G^C$ containing v. Hence, $v \in CD(V(H) - \{v\};\mathcal{P})$. $\square$

Remark 4.15. Let $(V(G), \tau_G^C(\mathcal{P}))$ be a graph chromatic topological space. If H and W are vertex disjoint vertex induced subgraphs of G, then $CD(V(H);\mathcal{P}) \cap CD(V(W);\mathcal{P}) \subseteq CD(V(H \cap W);\mathcal{P})$ does not hold.

Theorem 4.16. Let $(V(G), \tau_G^C(\mathcal{P}))$ be a graph chromatic topological space. Let H be a vertex induced subgraph of G such that $|V(H)| = 1$. Then $CD(V(H); \mathcal{P}) = CU(V(H); \mathcal{P}) - V(H)$.

Proof. Let $u \in CD(V(H); \mathcal{P})$. Then $M \cap (V(H) - \{u\}) \neq \varnothing$ for every $M \in \tau_G^C(\mathcal{P})$ containing u. Suppose that $u \in V(H)$. Then $V(H) = \{u\}$. So, $M \cap (V(H) - \{u\}) = \varnothing$, which is a contradiction. Thus, $u \notin V(H)$. Since $CD(V(H); \mathcal{P}) \subseteq CU(V(H); \mathcal{P}), u \in CU(V(H); \mathcal{P})$. So $u \in CU(V(H); \mathcal{P}) - V(H)$. Therefore, $CD(V(H); \mathcal{P}) \subseteq CU(V(H); \mathcal{P}) - V(H)$. Let $u \in CU(V(H); \mathcal{P}) - V(H)$. Then $M \cap V(H) \neq \varnothing$ for every $M \in \tau_G^C$ containing u and $u \notin V(H)$. Hence $M \cap (V(H) - \{u\}) \neq \varnothing$ for every $M \in \tau_G^C$ containing u. So, $u \in CD(V(H); \mathcal{P})$. Therefore, $CU(V(H); \mathcal{P}) - V(H) \subseteq CD(V(H); \mathcal{P})$. Thus, $CD(V(H); \mathcal{P}) = CU(V(H); \mathcal{P}) - V(H)$. □

5 Conclusion

In this paper a method of constructing topologies on vertex set of a graph G induced by chromatic partition of vertex set of the graph is presented. Colour lower approximation and colour upper approximation of vertex induced subgraphs are introduced and the open and closed sets of the topology generated by chromatic partition on the vertex set of graphs are acquainted. Some of the properties of colour lower approximation and colour upper approximation of vertex induced subgraphs are also explored. It is proved that the chromatic topological spaces associated with the isomorphic graphs are homeomorphic. Some new subgraphs based on colour upper approximation and colour lower approximation such as pivotic subgraphs, chromatic free subgraphs, chromatic derived set of a vertex induced subgraph have been established and some of their properties have been studied.

References

[1] Allam, A. A., Bakeir, M. Y., & Abo-Tabl, E. A. (2008). Some methods for generating topologies by relations. *Bulletin of the Malaysian Mathematical Sciences Society*, 31(1), 35–45.

[2] Chartrand, G., & Zhang, P. (2006). *Introduction to Graph Theory*. Tata McGraw-Hill, New Delhi.

[3] Gamorez, A. E., Nianga, C. G. S., & Canoy Jr., S. R. (2019). Topologies induced by neighborhoods of a graph under some binary operation. *European Journal of Pure and Applied Mathematics*, 12(3), 749–755.

[4] Kovacs, Z. (1991). *Properties of Relators Generated by Interiors*. Diploma Thesis, Lajos Kosuth University, Debrecen (Hungarian).

[5] Kurdics, J. (1991). *Connected and Well-chained Relator Spaces*. Doctoral Dissertation, Lajos Kosuth University, Debrecen (Hungarian).

[6] Lalithambigai, K., & Gnanachandra. P. (2023). Topologies induced on Vertex Set of Graphs. *Mapana - Journal of Sciences*, 22(1), 35–52.

[7] Lalithambigai, K., & Gnanachandra. P., & Jafari. S. (2023). Topologies induced by Graph Metrics on the Vertex Set of Graphs. *Mathematics for Applications*, 12, 73–88.

[8] Lalithambigai, K., & Gnanachandra. P. (2023). Topological Structures on Vertex Set of Digraphs. *Baghdad Science Journal*, 20(1 Special Issue), 350–358.

[9] Nianga, C. G. S., & Canoy Jr., S. R. (2019). On topologies induced by some unary and binary operations. *European Journal of Pure and Applied Mathematics*, 12(2), 499–505.

[10] Nianga, C. G. S., & Canoy Jr., S. R. (2019). On a finite topological space induced by hop neighborhoods of a graph. *Advances and Applications in Discrete Mathematics*, 21(1), 79–89.

[11] Salama, A. S. (2008). Topologies induced by relations with applications. *Journal of Computer Science*, 4(10), 877–887.

[12] Száz, Á. (1992). Inverse and symmetric relators. *Acta Mathematica Hungarica*, 60, 157–176.

[13] Száz, Á. (1992). Structures derivable from relators. *Singularité*, 3, 14–30.