Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2023, Volume 29, Number 3, 598–602
DOI: 10.7546/nntdm.2023.29.3.598-602

On special exponential Diophantine equations

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Received: 22 January 2023 **Revised:** 8 June 2023
Accepted: 17 August 2023 **Online First:** 25 August 2023

Abstract: In this paper, we will focus on the study of a special type of exponential Diophantine equations, including a proof. The main contribution of this article is the mentioned type of equations, which can only be solved by the methods of elementary mathematics.
Keywords: Diophantine, Exponential, Equation, Elementary mathematics.
2020 Mathematics Subject Classification: 11D61.

1 Introduction

In this contribution, we focus on the specific type of exponential Diophantine equation and solve it using only elementary mathematics. A special exponential equation, we mean Diophantine equation in the following form

$$x^m = y^n + a,$$

where x and y are two distinct positive integers greater than one and also bases of the exponential functions, a is a given positive integer, and m, n are unknown integers. The main contribution of this paper is a generalization of this type of the exponential Diophantine equation, including the proof in which we use methods of elementary mathematics. An investigated type of exponential equations is similar to generalized Ramanujan–Nagell Diophantine equations. Ramanujan–Nagell Diophantine equations are studied by many authors and there exist a lot of works concerning this topic. There are still open problems in this topic, which are not fully resolved yet.

This paper offers an elementary method of solving a special type of Diophantine equations, which could be of interest to a large audience.

This article follows on from the articles [1–3] and at the same time summarizes this issue of exponential Diophantine equations in a general form. In the following section, we first formulate the type of investigated exponential Diophantine equations, and then present the theorem including the proof.

2 The exponential Diophantine equations

Let $x^m - 1 = 2^n + 2^{4k+2}$, where x is an odd positive integer and $x - 1 = 2^{2k+1}$, k is given nonnegative integer and exponents m, n are integers appearing in the given equation as unknown.

In such a generalized problem, let us first realize that m, n are always nonnegative integers (even positive integers). So it probably applies

$$x^m - 1 \geq 2^{4k+2},$$

or

$$x^m \geq 2^{4k+2} + 1.$$

We get it with a simple adjustment

$$m \geq \log_x (2^{4k+2} + 1).$$

Since $m \in \mathbb{N}_0$ and holds $x = 2^{2k+1} + 1$, then

$$1 < \log_x (2^{4k+2} + 1) < 2.$$

So $m \geq 2$.

Based on the estimate for m, inequality applies $x^2 - 1 \leq 2^n + 2^{4k+2}$, which we adjust to the form $x^2 - 1 - 2^{4k+2} \leq 2^n$. Since $n \in \mathbb{N}_0$, then

$$n \geq \log_2 (x^2 - 1 - 2^{4k+2}).$$

Valid $x = 2^{2k+1} + 1$, therefore

$$\begin{aligned} &n \geq \log_2 (2^{4k+2} + 2^{2k+2} + 1 - 1 - 2^{4k+2}), \\ &n \geq \log_2 2^{2k+2}. \end{aligned}$$

So $n \geq 2k + 2$.

In article [3], solved examples of the examined type of exponential Diophantine equations for $k = 0, 1$ and 2 are presented, from which the formulations of the theorem below can be deduced.

Theorem 2.1. *Generalized problem $x^m - 1 = 2^n + 2^{4k+2}$, where x is an odd positive integer and $x - 1 = 2^{2k+1}$, k is given by nonnegative integer and exponents m, n are integers appearing in the given equation as unknown, it has exactly one solution in the form $(m, n) = (2, 2k + 2)$.*

Proof. The proof of confirmation can be divided into two parts. In the first part we prove that the ordered pair $(m, n) = (2, 2k+2)$ is a solution to a generalized problem. In the second part we prove that this solution is the only possible one.

First part of the proof.

Substituting m and n into the task assignment, we verify whether the ordered pair $(2, 2k+2)$ is a solution of the generalized problem for any nonnegative integer k. We get the equation

$$x^2 - 1 = 2^{2k+2} + 2^{4k+2}. \tag{1}$$

Using a simple adjustment ($x^2 - 1 = (x-1)(x+1)$) we obtain the equation (1) in the form

$$(x-1)(x+1) = 2^{2k+1}(2 + 2^{2k+1}).$$

Since $x - 1 = 2^{2k+1}$, then the equation (1) can be modified to the final form

$$2^{2k+1}(2 + 2^{2k+1}) = 2^{2k+1}(2 + 2^{2k+1}).$$

We have proved the first part of the proof that there is always a solution to a given equation. In the following section, we prove that this solution is the only possible one.

Second part of the proof.

Suppose there are other solutions in the form:

a) Let $(m, n) = (2, 2k+z)$ be a solution, where z is a positive integer and $z \geq 3$. After substituting into the generalized equation and subsequent modifications we get

$$\begin{aligned} x^2 - 1 &= 2^{2k+z} + 2^{4k+2}, \\ (x-1)(x+1) &= 2^{2k+1}(2^{z-1} + 2^{2k+1}), \\ 2^{2k+1}(2 + 2^{2k+1}) &= 2^{2k+1}(2^{z-1} + 2^{2k+1}), \\ 2 + 2^{2k+1} &= 2^{z-1} + 2^{2k+1}, \\ 2 &= 2^{z-1}. \end{aligned}$$

And equality occurs just when $z = 2$. However, this is a dispute with the fact that $z \geq 3$.

b) Let $(m, n) = (y, 2k+2)$ be a solution where y is a positive integer and $y \geq 3$. After substituting into the generalized equation and subsequent modifications we get

$$\begin{aligned} x^y - 1 &= 2^{2k+2} + 2^{4k+2}, \\ (x-1)(x^{y-1} + x^{y-2} + \cdots + x + 1) &= 2^{2k+1}(2 + 2^{2k+1}), \\ 2^{2k+1}(x^{y-1} + x^{y-2} + \cdots + x + 1) &= 2^{2k+1}(2 + 2^{2k+1}), \\ x^{y-1} + x^{y-2} + \cdots + x + 1 &= 2 + 2^{2k+1}, \\ x^{y-1} + x^{y-2} + \cdots + x + 1 &= x + 1. \end{aligned}$$

The equality occurs just when $y = 2$. However, this is a dispute with the fact that $y \geq 3$.

c) Let the solution be an ordered pair $(m, n) = (y, 2k+z)$, where $y > 2$ and at the same time $2k + z > 2k + 2$. So a pair of equations must hold

$$x^y - 1 = 2^{2k+z} + 2^{4k+2}, \tag{2}$$
$$x^2 - 1 = 2^{2k+2} + 2^{4k+2}. \tag{3}$$

Subtracting the equation (3) from the equation (2) we get the equation

$$x^y - x^2 = 2^{2k+z} - 2^{2k+2}. \tag{4}$$

By successive modifications of the equation (4) we get

$$\begin{aligned} x^2(x^{y-2} - 1) &= 2^{2k+1}(2^{z-1} - 2), \\ x^2(x-1)(x^{y-3} + x^{y-4} + \cdots + x + 1) &= 2^{2k+1}(2^{z-1} - 2), \\ 2^{2k+1}x^2(x^{y-3} + x^{y-4} + \cdots + x + 1) &= 2^{2k+1}(2^{z-1} - 2), \\ x^2(x^{y-3} + x^{y-4} + \cdots + x + 1) &= 2^{z-1} - 2. \end{aligned}$$

By multiplying the left side of the last given equation and dividing the number 2 from the right side, we obtain the equation

$$x^{y-1} + x^{y-2} + \cdots + x^3 + x^2 = 2(2^{z-2} - 1). \tag{5}$$

So $2 \mid (x^{y-1} + x^{y-2} + \cdots + x^3 + x^2)$. Since $2 \nmid x^2$ (x is an odd positive integer), then necessarily $2 \mid (x^{y-3} + x^{y-4} + \cdots + x + 1)$. This can only be done if y is an even number (given the assumption of $y \geq 4$). We further modify the equation (5)

$$\begin{aligned} x^2(x+1)(x^{y-4} + \cdots + x^2 + 1) &= 2(2^{z-2} - 1), \\ x^2(2^{2k+1} + 2)(x^{y-4} + \cdots + x^2 + 1) &= 2(2^{z-2} - 1). \end{aligned}$$

We get the equation by a simple modification

$$x^2(2^{2k} + 1)(x^{y-4} + \cdots + x^2 + 1) = 2^{z-2} - 1. \tag{6}$$

Expression $(2^{2k} + 1) \mid (2^{z-2} - 1)$, i.e., a fraction of $\frac{2^{z-2}-1}{2^{2k}+1}$ must be an integer (even a positive integer). From this fraction follows the assumption that $z - 2 \geq 2k$ holds

$$\frac{2^{z-2} - 1}{2^{2k} + 1} = 2^{z-2k-2} - \frac{2^{z-2k-2} + 1}{2^{2k} + 1},$$

i.e., a given fraction is a positive integer just when the positive integer is also a fraction $\frac{2^{z-2k-2}+1}{2^{2k}+1}$. From the given fraction follows another assumption that $z - 2k - 2 \geq 2k$ and holds

$$\frac{2^{z-2k-2} + 1}{2^{2k} + 1} = 2^{z-4k-2} - \frac{2^{z-4k-2} - 1}{2^{2k} + 1},$$

i.e., a given fraction is a positive integer just when the positive integer (or 0) is also a fraction $\frac{2^{z-4k-2}-1}{2^{2k}+1}$.

We can continue this process. Suppose that the algorithm does not end after a finite number of steps, it is infinite, therefore, the conditions for z go to infinity for any $k \in \mathbb{N}$. The number z must be greater than all the limits arising from the algorithm, i.e., such $z \in \mathbb{N}$

does not exist. For $k = 0$ the condition follows $2 = (2^0 + 1) \mid (2^{z-2} - 1)$, where $2^{z-2} - 1$ is either an odd number or equal to 0 for $z = 2$. In the first case, we get a dispute because there is no odd number that is divisible by 2, in the latter case we have a dispute assuming since $z > 2$.

The algorithm must therefore end after a finite number of steps. It is also obvious that it must end after an even number of steps and therefore $z = 4kp + 2$, where $k, p \in \mathbb{N}$ (case $k = 0$ we solved in case b). We also found that y must be even, i.e., $y = 2l$, where $l \in \mathbb{N}$ a $l \geq 2$. Equation (2) using the relation $x^2 = (2^{2k+1} + 1)^2 = 2^{4k+2} + 2^{2k+2} + 1$ can be rewritten into shape

$$x^{y-2} \cdot (2^{4k+2} + 2^{2k+2} + 1) - 1 = 2^{z-2} \cdot 2^{2k+2} + 2^{4k+2}.$$

Using the relations $z = 4kp + 2$ and $y = 2l$, we obtain an equation in the form

$$x^{2l-2} \cdot (2^{4k+2} + 2^{2k+2} + 1) - 1 = 2^{4kp} \cdot 2^{2k+2} + 2^{4k+2}. \tag{7}$$

Equality in the equation (7) occurs just when $p = 0$ (implies $l = 1$), which is a dispute with the assumptions $l \geq 2, p \in \mathbb{N}$. Otherwise, the equation (7) can be modified as follows:

$$\begin{aligned} x^{2l-2} \cdot x^2 &= 2^{4k+2} + 2^{2k+2} + 1 + (2^{4kp} - 1) \cdot 2^{2k+2} \\ x^{2l-2} \cdot x^2 &= x^2 + (2^{4kp} - 1) \cdot 2^{2k+2} \\ x^2 \cdot (x^{2l-2} - 1) &= (2^{4kp} - 1) \cdot 2 \cdot (x - 1). \end{aligned}$$

So $x^2 \mid (2^{4kp} - 1) \cdot 2 \cdot (x - 1)$. Since $x^2 \nmid 2$ and $x^2 \nmid (x - 1)$, then necessarily $x^2 \mid (2^{4kp} - 1)$. The fraction $\frac{2^{4kp} - 1}{x^2} = \frac{2^{4kp} - 1}{2^{4k+2} + 2^{2k+2} + 1}$ must be integer.

Continuing with the procedure described above, we would obtain infinitely many conditions for $p \in \mathbb{N}$ that would produce an infinite decreasing sequence of positive integers, which is not possible. So p must be 0. $\square$

3 Conclusion

The investigated type of exponential Diophantine equations has just one solution, always in the form $(m, n) = (2, 2k + 2)$, where $k \in \mathbb{N}_0$.

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