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On properties of generalized Tridovan numbers

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Abstract: In this paper, we examine generalized Tridovan sequences and treat in detail two cases called Tridovan sequences and Tridovan–Lucas sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. In addition, we give some identities and matrices related to these sequences.
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1 Introduction

Padovan (Cordonnier) numbers are defined by the third-order recurrence relation

$$P_{n+3} = P_{n+1} + P_n, \quad P_0 = 1, \quad P_1 = 1, \quad P_2 = 1.$$

For more information on Padovan sequence, see [4, 7, 15, 19–21] and references therein. Note that any element of the Padovan sequence is calculated based on the sum of the last two terms ignoring the one immediately before. Following the same idea, Vieira and Alves [16] defined the Tridovan sequence as the sum of the three last terms ignoring the one immediately before. The addition of this new element makes the Tridovan sequence a fourth order recurrent linear sequence. Formally, they defined Tridovan numbers by the fourth-order recurrence relation

$$T_{n+4} = T_{n+2} + T_{n+1} + T_n, \quad T_0 = 0, T_1 = 1, T_2 = 0, T_3 = 1.$$

In this paper, we investigate the generalized Tridovan sequence. First we recall some information on the generalized Tetranacci sequence.

The generalized Tetranacci sequence (or generalized (r, s, t, u) sequence or generalized 4-step Fibonacci sequence) $\{W_n(W_0, W_1, W_2, W_3; r, s, t, u)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}, \quad W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3, \ \ n \geq 4 \quad (1)$$

where W_0, W_1, W_2, W_3 are arbitrary complex (or real) numbers and r, s, t, u are real numbers.

This sequence has been studied by many authors and is detailed in the extensive literature on these sequences, see for example [1, 5, 6, 8, 14, 17, 18]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{t}{u}W_{-(n-1)} - \frac{s}{u}W_{-(n-2)} - \frac{r}{u}W_{-(n-3)} + \frac{1}{u}W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1) holds for all integers n.

As $\{W_n\}$ is a fourth order recurrence sequence (difference equation), it's characteristic equation is

$$x^4 - rx^3 - sx^2 - tx - u = 0 \quad (2)$$

whose roots are $\alpha, \beta, \gamma, \delta$. Note that we have the following identities

$$\begin{aligned} \alpha + \beta + \gamma + \delta &= r, \\ \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= -s, \\ \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= t, \\ \alpha\beta\gamma\delta &= -u. \end{aligned}$$

Generalized Tetranacci numbers can be expressed, for all integers n, using Binet's formula

$$\begin{aligned} W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} \\ &+ \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \end{aligned} \quad (3)$$

where

$$\begin{aligned} p_1 &= W_3 - (\beta+\gamma+\delta)W_2 + (\beta\gamma+\beta\delta+\gamma\delta)W_1 - \beta\gamma\delta W_0, \\ p_2 &= W_3 - (\alpha+\gamma+\delta)W_2 + (\alpha\gamma+\alpha\delta+\gamma\delta)W_1 - \alpha\gamma\delta W_0, \\ p_3 &= W_3 - (\alpha+\beta+\delta)W_2 + (\alpha\beta+\alpha\delta+\beta\delta)W_1 - \alpha\beta\delta W_0, \\ p_4 &= W_3 - (\alpha+\beta+\gamma)W_2 + (\alpha\beta+\alpha\gamma+\beta\gamma)W_1 - \alpha\beta\gamma W_0. \end{aligned}$$

It is customary to choose r, s, t and u such that Eq. (2) has at least two real roots. Note that the Binet form of a sequence satisfying (2) for non-negative integers is valid for all integers n (see [3]).

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

Lemma 1.1. [14] *Suppose that* $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ *is the ordinary generating function of the generalized* (r, s, t, u) *sequence* $\{W_n\}_{n\geq 0}$. *Then,* $\sum_{n=0}^{\infty} W_n x^n$ *is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2 + (W_3 - rW_2 - sW_1 - tW_0)x^3}{1 - rx - sx^2 - tx^3 - ux^4}. \quad (4)$$

We next find Binet's formula of generalized (r, s, t, u) numbers $\{W_n\}$ by the use of generating function for W_n.

Theorem 1.1. [14] (Binet's formula of generalized (r, s, t, u) numbers)

$$\begin{aligned} W_n = {} & \frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} \\ & + \frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{q_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \end{aligned} \quad (5)$$

where

$$\begin{aligned} q_1 &= W_0\alpha^3 + (W_1 - rW_0)\alpha^2 + (W_2 - rW_1 - sW_0)\alpha + (W_3 - rW_2 - sW_1 - tW_0), \\ q_2 &= W_0\beta^3 + (W_1 - rW_0)\beta^2 + (W_2 - rW_1 - sW_0)\beta + (W_3 - rW_2 - sW_1 - tW_0), \\ q_3 &= W_0\gamma^3 + (W_1 - rW_0)\gamma^2 + (W_2 - rW_1 - sW_0)\gamma + (W_3 - rW_2 - sW_1 - tW_0), \\ q_4 &= W_0\delta^3 + (W_1 - rW_0)\delta^2 + (W_2 - rW_1 - sW_0)\delta + (W_3 - rW_2 - sW_1 - tW_0). \end{aligned}$$

Note that from (3) and (5) we have

$$\begin{aligned} & W_3 - (\beta+\gamma+\delta)W_2 + (\beta\gamma+\beta\delta+\gamma\delta)W_1 - \beta\gamma\delta W_0 \\ = {} & W_0\alpha^3 + (W_1 - rW_0)\alpha^2 + (W_2 - rW_1 - sW_0)\alpha + (W_3 - rW_2 - sW_1 - tW_0), \\ & W_3 - (\alpha+\gamma+\delta)W_2 + (\alpha\gamma+\alpha\delta+\gamma\delta)W_1 - \alpha\gamma\delta W_0 \\ = {} & W_0\beta^3 + (W_1 - rW_0)\beta^2 + (W_2 - rW_1 - sW_0)\beta + (W_3 - rW_2 - sW_1 - tW_0), \\ & W_3 - (\alpha+\beta+\delta)W_2 + (\alpha\beta+\alpha\delta+\beta\delta)W_1 - \alpha\beta\delta W_0 \\ = {} & W_0\gamma^3 + (W_1 - rW_0)\gamma^2 + (W_2 - rW_1 - sW_0)\gamma + (W_3 - rW_2 - sW_1 - tW_0), \\ & W_3 - (\alpha+\beta+\gamma)W_2 + (\alpha\beta+\alpha\gamma+\beta\gamma)W_1 - \alpha\beta\gamma W_0 \\ = {} & W_0\delta^3 + (W_1 - rW_0)\delta^2 + (W_2 - rW_1 - sW_0)\delta + (W_3 - rW_2 - sW_1 - tW_0). \end{aligned}$$

Next, we consider two special cases of the generalized (r, s, t, u) sequence $\{W_n\}$ which we call the (r, s, t, u)-Fibonacci and the (r, s, t, u)-Lucas sequences. The (r, s, t, u)-Fibonacci sequence $\{G_n\}_{n\geq 0}$ and the (r, s, t, u)-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$G_{n+4} = rG_{n+3} + sG_{n+2} + tG_{n+1} + uG_n, \quad n \geq 0, \quad (6)$$

with $G_0 = 0, G_1 = 1, G_2 = r, G_3 = r^2 + s$; and

$$H_{n+4} = rH_{n+3} + sH_{n+2} + tH_{n+1} + uH_n, \quad n \geq 0, \quad (7)$$

with $H_0 = 4, H_1 = r, H_2 = 2s + r^2, H_3 = r^3 + 3sr + 3t$.

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{t}{u}G_{-(n-1)} - \frac{s}{u}G_{-(n-2)} - \frac{r}{u}G_{-(n-3)} + \frac{1}{u}G_{-(n-4)}, \\ H_{-n} &= -\frac{t}{u}H_{-(n-1)} - \frac{s}{u}H_{-(n-2)} - \frac{r}{u}H_{-(n-3)} + \frac{1}{u}H_{-(n-4)}, \end{aligned}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrences (6) and (7) hold for all integers n.

The following theorem shows that the generalized Tetranacci sequence W_n at negative indices can be expressed by the sequence itself at positive indices.

Theorem 1.2. *For $n \in \mathbb{Z}$, for the generalized Tetranacci sequence (or generalized (r, s, t, u)-sequence or 4-step Fibonacci sequence) we have the following:*

$$\begin{aligned} W_{-n} &= \frac{1}{6}(-u)^{-n}(-6W_{3n} + 6H_nW_{2n} - 3H_n^2W_n + 3H_{2n}W_n + W_0H_n^3 + 2W_0H_{3n} - 3W_0H_nH_{2n}) \\ &= (-1)^{-n-1}u^{-n}(W_{3n} - H_nW_{2n} + \frac{1}{2}(H_n^2 - H_{2n})W_n - \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)W_0). \end{aligned}$$

Proof. For the proof, see Soykan [13, Theorem 1]. □

Using Theorem 1.2, we have the following corollary, see Soykan [13, Corollary 4].

Corollary 1.1. *For $n \in \mathbb{Z}$, we have*

(a) $2(-u)^{n+4}G_{-n} = -(3ru^2+t^3-3stu)^2G_n^3-(2su-t^2)^2G_{n+3}^2G_n-(-rt^2-tu+2rsu)^2G_{n+2}^2G_n - (-st^2+2s^2u+4u^2+rtu)^2G_{n+1}^2G_n + 2(3ru^2+t^3-3stu)((-2su+t^2)G_{n+3} + (-rt^2 - tu + 2rsu)G_{n+2} + (-st^2+2s^2u+4u^2+rtu)G_{n+1})G_n^2 + 2(2su-t^2)(-rt^2-tu+2rsu) G_{n+3}G_{n+2}G_n + 2(2su-t^2)(-st^2+2s^2u+4u^2+rtu)G_{n+3}G_{n+1}G_n - 2(-st^2+2s^2u+4u^2+ rtu)(-rt^2-tu+2rsu)G_{n+2}G_{n+1}G_n - 2G_{3n}u^4 + u^2(-2su+t^2)G_{2n+3}G_n + u^2(-rt^2 - tu+2rsu)G_{2n+2}G_n + u^2(-st^2+2s^2u+4u^2+rtu)G_{2n+1}G_n - 2u^2(2su-t^2)G_{2n}G_{n+3} + 2u^2(-rt^2-tu+2rsu)G_{2n}G_{n+2} + 2u^2(-st^2+2s^2u+4u^2+rtu)G_{2n}G_{n+1} - 3u^2(3ru^2 + t^3-3stu)G_{2n}G_n.$

(b) $H_{-n} = \frac{1}{6}(-u)^{-n}(H_n^3 + 2H_{3n} - 3H_{2n}H_n).$

Note that G_{-n} and H_{-n} can be given as follows by using $G_0 = 0$ and $H_0 = 4$ in Theorem 1.2,

$$G_{-n} = \frac{1}{6}(-u)^{-n}(-6G_{3n} + 6H_nG_{2n} - 3H_n^2G_n + 3H_{2n}G_n), \quad (8)$$

$$H_{-n} = \frac{1}{6}(-u)^{-n}\left(H_n^3 + 2H_{3n} - 3H_{2n}H_n\right), \quad (9)$$

respectively.

2 Generalized Tridovan sequence

In this paper, we consider the case $r = 0$, $s = 1$, $t = 1$, $u = 1$ and in this case we write $V_n = W_n$. A generalized Tridovan sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2, V_3)\}_{n\geq 0}$ is defined by the fourth order recurrence relation

$$V_n = V_{n-2} + V_{n-3} + V_{n-4} \tag{10}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2, V_3 = c_3$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - V_{-(n-2)} + V_{-(n-4)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (10) holds for all integers n.

As $\{V_n\}$ is a fourth order recurrence sequence (difference equation), its characteristic equation is

$$x^4 - x^2 - x - 1 = (x^3 - x^2 - 1)(x + 1) = 0. \tag{11}$$

The roots α, β, γ and δ of Equation (11) are given by:

$$\begin{aligned}
\alpha &= \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}, \\
\beta &= \frac{1}{3} + \omega\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega^2\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}, \\
\gamma &= \frac{1}{3} + \omega^2\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}, \\
\delta &= -1,
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Please note that there is the following:

$$\begin{aligned}
\alpha + \beta + \gamma + \delta &= 0, \\
\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= -1, \\
\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= 1, \\
\alpha\beta\gamma\delta &= -1.
\end{aligned}$$

The first generalized Tridovan numbers with positive and negative subscripts are shown in Table 1 below.

Now define two special cases for the sequence $\{V_n\}$. The Tridovan sequence $\{T_n\}_{n\geq 0}$ and the Tridovan–Lucas sequence $\{H_n\}_{n\geq 0}$ are respectively his fourth order Iterate the defining relations

$$\begin{aligned}
T_n &= T_{n-2} + T_{n-3} + T_{n-4}, \quad T_0 = 0, T_1 = 1, T_2 = 0, T_3 = 1, && (12) \\
H_n &= H_{n-2} + H_{n-3} + H_{n-4}, \quad H_0 = 4, H_1 = 0, H_2 = 2, H_3 = 3. && (13)
\end{aligned}$$

The sequences $\{T_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
T_{-n} &= -T_{-(n-1)} - T_{-(n-2)} + T_{-(n-4)}, \\
H_{-n} &= -H_{-(n-1)} - H_{-(n-2)} + H_{-(n-4)},
\end{aligned}$$

for $n = 1, 2, 3, \ldots,$ respectively. So, recurrences (12)–(13) hold for all integer n.

Table 1. A few generalized Tridovan numbers

n	V_n	V_{-n}
0	V_0	
1	V_1	$V_3 - V_1 - V_0$
2	V_2	$V_1 + V_2 - V_3$
3	V_3	$V_0 + V_1 - V_2$
4	$V_0 + V_1 + V_2$	$V_3 - 2V_1$
5	$V_1 + V_2 + V_3$	$V_2 - 2V_0$
6	$V_0 + V_1 + 2V_2 + V_3$	$2V_0 + 3V_1 - 2V_3$
7	$V_0 + 2V_1 + 2V_2 + 2V_3$	$V_0 - 2V_1 - 2V_2 + 2V_3$
8	$2V_0 + 3V_1 + 4V_2 + 2V_3$	$2V_2 - 3V_1 - 3V_0 + V_3$
9	$2V_0 + 4V_1 + 5V_2 + 4V_3$	$5V_1 + V_2 - 3V_3$
10	$4V_0 + 6V_1 + 8V_2 + 5V_3$	$5V_0 + V_1 - 3V_2$
11	$5V_0 + 9V_1 + 11V_2 + 8V_3$	$5V_3 - 8V_1 - 4V_0$
12	$8V_0 + 13V_1 + 17V_2 + 11V_3$	$4V_1 - 4V_0 + 5V_2 - 4V_3$
13	$11V_0 + 19V_1 + 24V_2 + 17V_3$	$8V_0 + 9V_1 - 4V_2 - 4V_3$

Note that T_n and H_n are the sequences A013979, A001634 in [9], respectively. Next, we present the first few values of the Tridovan and Tridovan–Lucas numbers with positive and negative subscripts (Table 2).

Table 2. The first few values of the quintic special number with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
T_n	0	1	0	1	1	2	2	4	5	8	11	17	24	36
T_{-n}	...	0	0	1	-1	0	1	0	-2	2	1	-3	0	5
H_n	4	0	2	3	6	5	11	14	22	30	47	66	99	143
H_{-n}	...	-1	-1	2	3	-6	2	6	-5	-7	14	-1	-18	12

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

Lemma 2.1. *Suppose that $f_{V_n}(x) = \sum\limits_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized Tridovan sequence $\{V_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} V_n x^n$ is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + V_1 x + (V_2 - V_0)x^2 + (V_3 - V_1 - V_0)x^3}{1 - x^2 - x^3 - x^4}.$$

Proof. Take $r = 0$, $s = 1$, $t = 1$, $u = 1$ in Lemma 1.1. □

The previous lemma gives the following result as a specific example.

Corollary 2.1. *The generated functions of Tridovan and Tridovan–Lucas numbers are respectively*

$$\begin{aligned}\sum_{n=0}^{\infty} T_n x^n &= \frac{x}{1-x^2-x^3-x^4},\\ \sum_{n=0}^{\infty} H_n x^n &= \frac{4-2x^2-x^3}{1-x^2-x^3-x^4}.\end{aligned}$$

Using the initial conditions of (3) or (5), the generalized Tridovan number for all integers n can be expressed using Binet's formula

$$\begin{aligned}V_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\\ &\quad + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\end{aligned}$$

where

$$\begin{aligned}p_1 &= V_3-(\beta+\gamma+\delta)V_2+(\beta\gamma+\beta\delta+\gamma\delta)V_1-\beta\gamma\delta V_0\\ &= V_0\alpha^3+V_1\alpha^2+(V_2-V_0)\alpha+(V_3-V_1-V_0),\\ p_2 &= V_3-(\alpha+\gamma+\delta)V_2+(\alpha\gamma+\alpha\delta+\gamma\delta)V_1-\alpha\gamma\delta V_0\\ &= V_0\beta^3+V_1\beta^2+(V_2-V_0)\beta+(V_3-V_1-V_0),\\ p_3 &= V_3-(\alpha+\beta+\delta)V_2+(\alpha\beta+\alpha\delta+\beta\delta)V_1-\alpha\beta\delta V_0\\ &= V_0\gamma^3+V_1\gamma^2+(V_2-V_0)\gamma+(V_3-V_1-V_0),\\ p_4 &= V_3-(\alpha+\beta+\gamma)V_2+(\alpha\beta+\alpha\gamma+\beta\gamma)V_1-\alpha\beta\gamma V_0\\ &= V_0\delta^3+V_1\delta^2+(V_2-V_0)\delta+(V_3-V_1-V_0).\end{aligned}$$

For every integer n, the Tridovan and Tridovan–Lucas numbers (with initial conditions of (12) and (13)) can be expressed using Binet's formulas respectively as

$$\begin{aligned}T_n &= \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\\ &\quad + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)},\\ H_n &= \alpha^n+\beta^n+\gamma^n+\delta^n.\end{aligned}$$

3 On the recurrence properties of generalized Tridovan sequence

In this section, we express V_{-j} explicitly in terms of V_j, H_j and T_j as was done by Horadam [2] and by later researchers. Taking $r = 0$, $s = 1$, $t = 1$, $u = 1$ in Theorem 1.2, we obtain the following Proposition.

Proposition 3.1. *For $n \in \mathbb{Z}$, generalized Tridovan numbers (the case $r = 0$, $s = 1$, $t = 1$, $u = 1$) have the following identity:*

$$V_{-n} = \frac{1}{6}(-6V_{3n} + 6H_nV_{2n} - 3H_n^2V_n + 3H_{2n}V_n + V_0H_n^3 + 2V_0H_{3n} - 3V_0H_nH_{2n}).$$

From the above Proposition 3.1 (or by taking $G_n := T_n$ and $H_n := H_n$ in (8) and (9), respectively), we have the following corollary which gives the connection between the special cases of generalized Tridovan sequence at the positive index and the negative index: for Tridovan and Tridovan–Lucas numbers: take $V_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 0, T_3 = 1$ and take $V_n = H_n$ with $H_0 = 4, H_1 = 0, H_2 = 2, H_3 = 3$, respectively. Note that in this case $H_n := H_n$.

Corollary 3.1. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a) *Tridovan sequence:*

$$T_{-n} = \frac{1}{6}(-6T_{3n} + 6H_nT_{2n} - 3H_n^2T_n + 3H_{2n}T_n).$$

(b) *Tridovan–Lucas sequence:*

$$H_{-n} = \frac{1}{6}\left(H_n^3 + 2H_{3n} - 3H_{2n}H_n\right).$$

4 Simson formulas and some identities

The following theorem generalizes this result to the generalized (r, s, t, u) sequence $\{W_n\}_{n\geq 0}$.

Theorem 4.1 (Simson Formula of Generalized Tridovan Numbers)**.** *For all integers n, we have*

$$\begin{vmatrix} V_{n+3} & V_{n+2} & V_{n+1} & V_n \\ V_{n+2} & V_{n+1} & V_n & V_{n-1} \\ V_{n+1} & V_n & V_{n-1} & V_{n-2} \\ V_n & V_{n-1} & V_{n-2} & V_{n-3} \end{vmatrix} = (-1)^n \begin{vmatrix} V_3 & V_2 & V_1 & V_0 \\ V_2 & V_1 & V_0 & V_{-1} \\ V_1 & V_0 & V_{-1} & V_{-2} \\ V_0 & V_{-1} & V_{-2} & V_{-3} \end{vmatrix}. \tag{14}$$

Proof. Take $r = 0$, $s = 1$, $t = 1$, $u = 1$ in Soykan [10, Theorem 2.3]. □

The previous theorem gives the following result as a concrete example.

Corollary 4.1. *For all integers n, Simson formula of Tridovan and, Tridovan–Lucas are given as, respectively,*

$$\begin{vmatrix} T_{n+3} & T_{n+2} & T_{n+1} & T_n \\ T_{n+2} & T_{n+1} & T_n & T_{n-1} \\ T_{n+1} & T_n & T_{n-1} & T_{n-2} \\ T_n & T_{n-1} & T_{n-2} & T_{n-3} \end{vmatrix} = (-1)^{n+1} \times 2^n,$$

$$\begin{vmatrix} H_{n+3} & H_{n+2} & H_{n+1} & H_n \\ H_{n+2} & H_{n+1} & H_n & H_{n-1} \\ H_{n+1} & H_n & H_{n-1} & H_{n-2} \\ H_n & H_{n-1} & H_{n-2} & H_{n-3} \end{vmatrix} = 279 \times (-1)^n \times 2^n.$$

We can give a few basic relations between $\{T_n\}$ and $\{H_n\}$.

Lemma 4.1. *The following equalities are true:*

$$\begin{aligned}279T_n &= -4H_{n+5}-44H_{n+4}+78H_{n+3}+25H_{n+2}, \quad (15)\\ 279T_n &= -44H_{n+4}+74H_{n+3}+21H_{n+2}-4H_{n+1},\\ 279T_n &= 74H_{n+3}-23H_{n+2}-48H_{n+1}-44H_n,\\ 279T_n &= -23H_{n+2}+26H_{n+1}+30H_n+74H_{n-1},\\ 279T_n &= 26H_{n+1}+7H_n+51H_{n-1}-23H_{n-2},\end{aligned}$$

and

$$\begin{aligned}H_n &= 3T_{n+5}+2T_{n+4}-4T_{n+3}-6T_{n+2},\\ H_n &= 2T_{n+4}-T_{n+3}-3T_{n+2}+3T_{n+1},\\ H_n &= -T_{n+3}-T_{n+2}+5T_{n+1}+2T_n,\\ H_n &= -T_{n+2}+4T_{n+1}+T_n-T_{n-1},\\ H_n &= 4T_{n+1}-2T_{n-1}-T_{n-2}.\end{aligned}$$

Proof. Note that all IDs apply to all integer n. We prove (15). To show (15), writing

$$T_n = a\times H_{n+5}+b\times H_{n+4}+c\times H_{n+3}+d\times H_{n+2}$$

and solving the system of equations

$$\begin{aligned}T_0 &= a\times H_5+b\times H_4+c\times H_3+d\times H_2\\ T_1 &= a\times H_6+b\times H_5+c\times H_4+d\times H_3\\ T_2 &= a\times H_7+b\times H_6+c\times H_5+d\times H_4\\ T_3 &= a\times H_8+b\times H_7+c\times H_6+d\times H_5\end{aligned}$$

we find that $a=-\frac{4}{279}, b=-\frac{44}{279}, c=\frac{26}{93}, d=\frac{25}{279}$. Other equalities can be proved similarly. □

5 Sum formulas

5.1 Sums of terms with positive subscripts

The following proposition give some formulas for generalized Tridovan numbers with positive indices.

Proposition 5.1. *If* $r=0,\ s=1,\ t=1,\ u=1$ *then for* $n\geq 0$ *we have the following formula:*

$$\sum_{k=0}^{n} V_k = \frac{1}{2}(V_{n+4}+V_{n+3}-V_{n+1}-V_3-V_2+V_0).$$

Proof. Take $r=0$, $s=1$, $t=1$, $u=1$ in Theorem 2.1 in [11]. □

Note that using

$$V_{n+4}=V_{n+2}+V_{n+1}+V_n$$

we have the formula

$$\sum_{k=0}^{n} V_k = \frac{1}{2}(V_{n+3} + V_{n+2} + V_n - V_3 - V_2 + V_0)$$

and so

$$\sum_{k=0}^{n-1} V_k = -V_n + \sum_{k=0}^{n} V_k = \frac{1}{2}(V_{n+3} + V_{n+2} - V_n - V_3 - V_2 + V_0). \quad (16)$$

From the last proposition, we have the following corollary giving the sum of Tridovan numbers (take $V_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 0, T_3 = 1$).

Corollary 5.1. *For* $n \geq 0$ *we have the following formula:*

$$\sum_{k=0}^{n} T_k = \frac{1}{2}(T_{n+4} + T_{n+3} - T_{n+1} - 1).$$

Taking $V_n = H_n$ with $H_0 = 4, H_1 = 0, H_2 = 2, H_3 = 3$ in the last proposition, we have the following system of molecular formulas for Tridovan–Lucas numbers.

Corollary 5.2. *For* $n \geq 0$ *we have the following formula:*

$$\sum_{k=0}^{n} H_k = \frac{1}{2}(H_{n+4} + H_{n+3} - H_{n+1} - 1).$$

5.2 Sums of squares of terms with positive subscripts

The following proposition give some formulas for generalized Tridovan numbers with positive indices.

Theorem 5.1. *If* $r = 0,\ s = 1,\ t = 1,\ u = 1$ *then for* $n \geq 0$ *we have the following formulas:*

(a) $\sum_{k=0}^{n} V_k^2 = \frac{1}{9}((n+3)V_{n+4}^2 + (n+2)V_{n+3}^2 - 12V_{n+2}^2 + (n-5)V_{n+1}^2 - 2(n+4)V_{n+4}V_{n+3} + 10 V_{n+4}V_{n+2} - 2(n+5)V_{n+4}V_{n+1} + 2V_{n+2}V_{n+3} + 2(n+9)V_{n+3}V_{n+1} - 4V_{n+2}V_{n+1} - 2V_3^2 - V_2^2 + 12V_1^2 + 6V_0^2 + 6V_3V_2 - 10V_3V_1 - 2V_2V_1 + 8V_3V_0 - 16V_2V_0 + 4V_1V_0)$.

(b) $\sum_{k=0}^{n} V_{k+1}V_k = \frac{1}{9}(-(n+4)V_{n+4}^2 - (n+3)V_{n+3}^2 + 3V_{n+2}^2 - (n+5)V_{n+1}^2 + 2(n+5)V_{n+4}V_{n+3} - V_{n+4}V_{n+2} + 2(n+6)V_{n+4}V_{n+1} - 2V_{n+2}V_{n+3} - (2n+11)V_{n+3}V_{n+1} - 5V_{n+2}V_{n+1} + 3V_3^2 + 2V_2^2 - 3V_1^2 + 4V_0^2 - 8V_3V_2 + V_3V_1 + 2V_2V_1 - 10V_3V_0 + 9V_2V_0 + 5V_1V_0)$.

(c) $\sum_{k=0}^{n} V_{k+2}V_k = \frac{1}{9}((n+8)V_{n+4}^2 + (n+7)V_{n+3}^2 + 6V_{n+2}^2 + (n+9)V_{n+1}^2 - (2n+9)V_{n+4}V_{n+3} - 8V_{n+4}V_{n+2} - (2n+11)V_{n+4}V_{n+1} + 2V_{n+3}V_{n+2} + (2n+1)V_{n+3}V_{n+1} + 5V_{n+1}V_{n+2} - 7V_3^2 - 6V_2^2 - 6V_1^2 - 8V_0^2 + 7V_3V_2 + 8V_3V_1 - 2V_2V_1 + 9V_3V_0 + V_2V_0 - 5V_1V_0)$.

(d) $\sum_{k=0}^{n} V_{k+3}V_k = \frac{1}{9}(-(n+6)V_{n+4}^2 - (n+5)V_{n+3}^2 - 6V_{n+2}^2 - (n+7)V_{n+1}^2 + 2(n+7)V_{n+4}V_{n+3} + 8V_{n+2}V_{n+4} + (2n+7)V_{n+4}V_{n+1} - 2V_{n+2}V_{n+3} - 2(n+3)V_{n+3}V_{n+1} - 5V_{n+2}V_{n+1} + 5V_3^2 + 4V_2^2 + 6V_1^2 + 6V_0^2 - 12V_3V_2 - 8V_3V_1 + 2V_2V_1 - 5V_3V_0 + 4V_2V_0 + 5V_1V_0)$.

Proof. We take $x = 1, r = 0,\ s = 1,\ t = 1,\ u = 1$ in Theorem 3.1 in [12]. Note that setting $x = 1, r = 0,\ s = 1,\ t = 1,\ u = 1$ in Theorem 3.1 (a), (b), (c) and (d) in [12] makes the right hand side of the sum formulas to be an indeterminate form. However, applying the L'Hospital rule gives an evaluation of the molecular formula.

(a) We use in Theorem 3.1 (a) in [12]. If we set $r = 0$, $s = 1$, $t = 1$, $u = 1$ in Theorem 3.1 (a) in [12], then we have

$$\sum_{k=0}^{n} x^k V_k^2 = \frac{g_1(x)}{(x-1)(x+2x^2+x^3-1)(-x^2+x^3-1)(x+x^3+1)}$$

where

$g_1(x) = -x^{n+4}(-x^6 - x^5 + x^4 + x^3 + x^2 + x - 1)V_{n+4}^2 - x^{n+3}(-x^6 - x^5 + x^4 + x^3 + x^2 + x - 1)V_{n+3}^2 + x^{n+2}(x^8 + x^7 + 2x^5 - 2x^4 - 2x^2 - x + 1)V_{n+2}^2 - x^{n+1}(-x^9 + x^6 - 3x^5 + x^4 + x^3 + 2x^2 + x - 1)V_{n+1}^2 + x^3(-x^6 - x^5 + x^4 + x^3 + x^2 + x - 1)V_3^2 + x^2(-x^6 - x^5 + x^4 + x^3 + x^2 + x - 1)V_2^2 - x(x^8 + x^7 + 2x^5 - 2x^4 - 2x^2 - x + 1)V_1^2 + (-x^9 + x^6 - 3x^5 + x^4 + x^3 + 2x^2 + x - 1)V_0^2 - 2x^4(-x^4 + x^2 + x)V_2V_3 - 2x^4(-x^5 - x^4 + x^3 + x)V_1V_3 + 2x^4(-x^4 + x^3 + x^2 - 1)V_1V_2 - 2x^4(-x^5 + x^3 + x^2)V_0V_3 + 2x^4(x^3 + x - 1)V_2V_0 + 2x^4(-x^5 + x^2 + x - 1)V_0V_1 + 2x^{n+5}(-x^4 + x^2 + x)V_{n+3}V_{n+4} + 2x^{n+5}(-x^5 - x^4 + x^3 + x)V_{n+2}V_{n+4} + 2x^{n+5}(-x^5 + x^3 + x^2)V_{n+1}V_{n+4} - 2x^{n+5}(-x^4 + x^3 + x^2 - 1)V_{n+2}V_{n+3} - 2x^{n+5}(x^3 + x - 1)V_{n+3}V_{n+1} - 2x^{n+5}(-x^5 + x^2 + x - 1)V_{n+1}V_{n+2}.$

For $x = 1$, the right-hand side of the above summation formula is indeterminate. Here we can apply the L'Hospital rule. Then we get (a) using

$$\sum_{k=0}^{n} V_k^2 = \left. \frac{\frac{d}{dx}(g_1(x))}{\frac{d}{dx}((x-1)(x+2x^2+x^3-1)(-x^2+x^3-1)(x+x^3+1))} \right|_{x=1}.$$

(b) We use in Theorem 3.1 (b) in [12]. If we set $r = 0$, $s = 1$, $t = 1$, $u = 1$ in Theorem 3.1 (b) in [12], then we have

$$\sum_{k=0}^{n} x^k V_{k+1} V_k = \frac{g_2(x)}{(x-1)(x+2x^2+x^3-1)(-x^2+x^3-1)(x+x^3+1)}$$

where

$g_2(x) = x^{n+4}(-x^5 + x^3 + x^2)V_{n+4}^2 + x^{n+5}(-x^3 + x + 1)V_{n+3}^2 - x^{n+5}(x^4 - x^3 + x^2 - 1)V_{n+2}^2 + x^{n+5}(-x^5 + x^3 + x^2)V_{n+1}^2 - x^{n+3}(-x^6 - x^5 + 2x^4 + x^3 + 2x^2 - 1)V_{n+3}V_{n+4} - x^{n+4}(-2x^5 + x^4 + x^3 + x^2 - 1)V_{n+2}V_{n+4} - x^{n+4}(-x^6 - x^5 + 2x^4 + x^3 + 2x^2 - 1)V_{n+1}V_{n+4} - x^{n+2}(x^7 - 3x^5 + x^4 - x^3 + 2x^2 + x - 1)V_{n+2}V_{n+3} + x^{n+5}(-x^4 + x^3 + x^2 + 1)V_{n+1}V_{n+3} - x^{n+1}(x^9 + x^8 - 2x^7 - 4x^5 + x^4 + x^3 + 2x^2 + x - 1)V_{n+1}V_{n+2} - x^3(-x^5 + x^3 + x^2)V_3^2 - x^4(-x^3 + x + 1)V_2^2 + x^4(x^4 - x^3 + x^2 - 1)V_1^2 - x^4(-x^5 + x^3 + x^2)V_0^2 + x^2(-x^6 - x^5 + 2x^4 + x^3 + 2x^2 - 1)V_2V_3 + x^3(-2x^5 + x^4 + x^3 + x^2 - 1)V_1V_3 + x(x^7 - 3x^5 + x^4 - x^3 + 2x^2 + x - 1)V_1V_2 + x^3(-x^6 - x^5 + 2x^4 + x^3 + 2x^2 - 1)V_0V_3 - x^4(-x^4 + x^3 + x^2 + 1)V_0V_2 + (x^9 + x^8 - 2x^7 - 4x^5 + x^4 + x^3 + 2x^2 + x - 1)V_0V_1.$

For $x = 1$, the right-hand side of the above summation formula is indeterminate. Here we can apply the L'Hospital rule. Then we get (b) using

$$\sum_{k=0}^{n} V_{k+1} V_k = \left. \frac{\frac{d}{dx}(g_2(x))}{\frac{d}{dx}((x-1)(x+2x^2+x^3-1)(-x^2+x^3-1)(x+x^3+1))} \right|_{x=1}.$$

(c) We use in Theorem 3.1 (c) in [12]. If we set $r = 0$, $s = 1$, $t = 1$, $u = 1$ in Theorem 3.1 (c) in [12], then we have

$$\sum_{k=0}^{n} x^k V_{k+2} V_k = \frac{g_3(x)}{(x-1)(x+2x^2+x^3-1)(-x^2+x^3-1)(x+x^3+1)}$$

where

$g_3(x) = -x^{n+4}(x^3+x-1)V_{n+4}^2 - x^{n+3}(x^3+x-1)V_{n+3}^2 + x^{n+5}(-x^5-x^2+x+1)V_{n+2}^2 - x^{n+5}(x^3+x-1)V_{n+1}^2 + x^{n+3}V_{n+3}V_{n+4}(-x^5+x^4+x^3+x) - x^{n+2}(-x^8-2x^5+x^4+x^3+x^2+x-1)V_{n+2}V_{n+4} + x^{n+4}(-x^5+x^4+x^3+x)V_{n+1}V_{n+4} + x^{n+3}(x^5-2x^4+x^2-x+1)V_{n+2}V_{n+3} - x^{n+1}(-x^8-x^7+x^6-x^5+x^4+2x^3+x^2+x-1)V_{n+1}V_{n+3} - x^{n+5}(x^5-x^4+x^3+x-2)V_{n+1}V_{n+2} + x^3(x^3+x-1)V_3^2 + x^2(x^3+x-1)V_2^2 - x^4(-x^5-x^2+x+1)V_1^2 + x^4(x^3+x-1)V_0^2 - x^2(-x^5+x^4+x^3+x)V_2V_3 + x(-x^8-2x^5+x^4+x^3+x^2+x-1)V_1V_3 - x^2(x^5-2x^4+x^2-x+1)V_1V_2 - x^3(-x^5+x^4+x^3+x)V_0V_3 + (-x^8-x^7+x^6-x^5+x^4+2x^3+x^2+x-1)V_0V_2 + x^4(x^5-x^4+x^3+x-2)V_0V_1.$

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get (c) using

$$\sum_{k=0}^{n} V_{k+2} V_k = \left. \frac{\frac{d}{dx}(g_3(x))}{\frac{d}{dx}((x-1)(x+2x^2+x^3-1)(-x^2+x^3-1)(x+x^3+1))} \right|_{x=1}.$$

(d) We use in Theorem 3.1 (d) in [12]. If we set $r = 0$, $s = 1$, $t = 1$, $u = 1$ in Theorem 3.1 (d) in [12], then we have

$$\sum_{k=0}^{n} x^k V_{k+3} V_k = \frac{g_4(x)}{(x-1)(x+2x^2+x^3-1)(-x^2+x^3-1)(x+x^3+1)}$$

where

$g_4(x) = x^{n+4}(x^3-x+1)V_{n+4}^2 + x^{n+3}(x^3-x+1)V_{n+3}^2 + x^{n+2}(x^7+x^5-x^4+x^3-2x^2-x+1)V_{n+2}^2 + x^{n+5}(x^3-x+1)V_{n+1}^2 - x^{n+3}(-x^7+x^5+x^4+x^3+2x^2-x-1)V_{n+3}V_{n+4} + x^{n+2}(-x^7-2x^5+x^4+x^3+x^2)V_{n+2}V_{n+4} - x^{n+1}(x^6-2x^5+x^4+2x^2+x-1)V_{n+1}V_{n+4} + x^{n+2}(-x^8+x^7+x^6+x^3-3x^2+1)V_{n+2}V_{n+3} + x^{n+3}(-x^7+x^4-x^3+3x^2)V_{n+1}V_{n+3} + x^{n+2}(x^6+2x^5-3x^4+2x^3-2x^2-x+1)V_{n+1}V_{n+2} - x^3(x^3-x+1)V_3^2 - x^2(x^3-x+1)V_2^2 - x(x^7+x^5-x^4+x^3-2x^2-x+1)V_1^2 - x^4(x^3-x+1)V_0^2 + x^2(-x^7+x^5+x^4+x^3+2x^2-x-1)V_2V_3 - x(-x^7-2x^5+x^4+x^3+x^2)V_1V_3 + (x^6-2x^5+x^4+2x^2+x-1)V_0V_3 - x(-x^8+x^7+x^6+x^3-3x^2+1)V_1V_2 - x^2(-x^7+x^4-x^3+3x^2)V_0 V_2 - x(x^6+2x^5-3x^4+2x^3-2x^2-x+1)V_0V_1.$

For $x = 1$, the right-hand side of the above summation formula is indeterminate. Here we can apply the L'Hospital rule. Then we get (d) using

$$\sum_{k=0}^{n} V_{k+3} V_k = \left. \frac{\frac{d}{dx}(g_4(x))}{\frac{d}{dx}((x-1)(x+2x^2+x^3-1)(-x^2+x^3-1)(x+x^3+1))} \right|_{x=1}. \quad \square$$

From the last theorem, we have the following corollary which gives sum formulas of Tridovan numbers (take $V_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 0, T_3 = 1$).

Corollary 5.3. *For $n \geq 0$, Tridovan numbers have the following properties:*

(a) $\sum_{k=0}^{n} T_k^2 = \frac{1}{9}((n+3)T_{n+4}^2 + (n+2)T_{n+3}^2 - 12T_{n+2}^2 + (n-5)T_{n+1}^2 - 2(n+4)T_{n+4}T_{n+3} + 10 T_{n+4}T_{n+2} - 2(n+5)T_{n+4}T_{n+1} + 2T_{n+2}T_{n+3} + 2(n+9)T_{n+3}T_{n+1} - 4T_{n+2}T_{n+1})$.

(b) $\sum_{k=0}^{n} T_{k+1}T_k = \frac{1}{9}(-(n+4)T_{n+4}^2 - (n+3)T_{n+3}^2 + 3T_{n+2}^2 - (n+5)T_{n+1}^2 + 2(n+5)T_{n+4}T_{n+3} - T_{n+4}T_{n+2} + 2(n+6)T_{n+4}T_{n+1} - 2T_{n+2}T_{n+3} - (2n+11)T_{n+3}T_{n+1} - 5T_{n+2}T_{n+1} + 1)$.

(c) $\sum_{k=0}^{n} T_{k+2}T_k = \frac{1}{9}((n+8)T_{n+4}^2 + (n+7)T_{n+3}^2 + 6T_{n+2}^2 + (n+9)T_{n+1}^2 - (2n+9)T_{n+4}T_{n+3} - 8T_{n+4}T_{n+2} - (2n+11)T_{n+4}T_{n+1} + 2T_{n+3}T_{n+2} + (2n+1)T_{n+3}T_{n+1} + 5T_{n+1}T_{n+2} - 5)$.

(d) $\sum_{k=0}^{n} T_{k+3}T_k = \frac{1}{9}(-(n+6)T_{n+4}^2 - (n+5)T_{n+3}^2 - 6T_{n+2}^2 - (n+7)T_{n+1}^2 + 2(n+7)T_{n+4}T_{n+3} + 8T_{n+2}T_{n+4} + (2n+7)T_{n+4}T_{n+1} - 2T_{n+2}T_{n+3} - 2(n+3)T_{n+3}T_{n+1} - 5T_{n+2}T_{n+1} + 3)$.

Taking $T_n = H_n$ with $H_0 = 4, H_1 = 0, H_2 = 2, H_3 = 3$ in the last theorem, we have the following system of molecular formulas for Tridovan–Lucas numbers.

Corollary 5.4. *For $n \geq 0$, Tridovan–Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} H_k^2 = \frac{1}{9}((n+3)H_{n+4}^2 + (n+2)H_{n+3}^2 - 12H_{n+2}^2 + (n-5)H_{n+1}^2 - 2(n+4)H_{n+4}H_{n+3} + 10 H_{n+4}H_{n+2} - 2(n+5)H_{n+4}H_{n+1} + 2H_{n+2}H_{n+3} + 2(n+9)H_{n+3}H_{n+1} - 4H_{n+2}H_{n+1} + 78)$.

(b) $\sum_{k=0}^{n} H_{k+1}H_k = \frac{1}{9}(-(n+4)H_{n+4}^2 - (n+3)H_{n+3}^2 + 3H_{n+2}^2 - (n+5)H_{n+1}^2 + 2(n+5)H_{n+4}H_{n+3} - H_{n+4}H_{n+2} + 2(n+6)H_{n+4}H_{n+1} - 2H_{n+2}H_{n+3} - (2n+11)H_{n+3}H_{n+1} - 5H_{n+2}H_{n+1} + 3)$.

(c) $\sum_{k=0}^{n} H_{k+2}H_k = \frac{1}{9}((n+8)H_{n+4}^2 + (n+7)H_{n+3}^2 + 6H_{n+2}^2 + (n+9)H_{n+1}^2 - (2n+9)H_{n+4}H_{n+3} - 8H_{n+4}H_{n+2} - (2n+11)H_{n+4}H_{n+1} + 2H_{n+3}H_{n+2} + (2n+1)H_{n+3}H_{n+1} + 5H_{n+1}H_{n+2} - 57)$.

(d) $\sum_{k=0}^{n} H_{k+3}H_k = \frac{1}{9}(-(n+6)H_{n+4}^2 - (n+5)H_{n+3}^2 - 6H_{n+2}^2 - (n+7)H_{n+1}^2 + 2(n+7)H_{n+4}H_{n+3} + 8H_{n+2}H_{n+4} + (2n+7)H_{n+4}H_{n+1} - 2H_{n+2}H_{n+3} - 2(n+3)H_{n+3}H_{n+1} - 5H_{n+2}H_{n+1} + 57)$.

The following proposition give some formulas for generalized Tridovan numbers with positive indices.

Proposition 5.2. *If $r = 0$, $s = 1$, $t = 1$, $u = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} (-1)^k V_k^2 = \frac{1}{6}((-1)^n (V_{n+4}^2 - V_{n+3}^2 - 4V_{n+2}^2 + 5V_{n+1}^2 + 2V_{n+4}V_{n+3} + 4V_{n+4}V_{n+2} - 2V_{n+4}V_{n+1} - 4V_{n+3}V_{n+2} - 6V_{n+3}V_{n+1}) + V_3^2 - V_2^2 - 4V_1^2 + 5V_0^2 + 2V_3V_2 + 4V_3V_1 - 4V_2V_1 - 2V_3V_0 - 6V_2V_0)$.

(b) $\sum_{k=0}^{n} (-1)^k V_{k+1}V_k = \frac{1}{6}((-1)^n (V_{n+4}^2 - V_{n+3}^2 + 2V_{n+2}^2 - V_{n+1}^2 + 2V_{n+4}V_{n+3} - 2V_{n+4}V_{n+2} - 2V_{n+4}V_{n+1} - 4V_{n+3}V_{n+2} + 6V_{n+2}V_{n+1}) + V_3^2 - V_2^2 + 2V_1^2 - V_0^2 + 2V_3V_2 - 2V_3V_1 - 2V_3V_0 - 4V_2V_1 + 6V_1V_0)$.

(c) $\sum_{k=0}^{n} (-1)^k V_{k+2}V_k = \frac{1}{6}(3(-1)^n (V_{n+4}^2 - V_{n+3}^2 - V_{n+1}^2 - 2V_{n+2}V_{n+1}) + 3V_3^2 - 3V_2^2 - 3V_0^2 - 6V_1V_0)$.

(d) $\sum_{k=0}^{n}(-1)^k V_{k+3}V_k = \frac{1}{6}((-1)^n (V_{n+4}^2 - V_{n+3}^2 - 4V_{n+2}^2 - V_{n+1}^2 + 2V_{n+4}V_{n+3} + 4V_{n+4}V_{n+2} + 4 V_{n+4}V_{n+1} - 4V_{n+3}V_{n+2} - 6V_{n+3}V_{n+1} - 6V_{n+2}V_{n+1}) + V_3^2 - V_2^2 - 4V_1^2 - V_0^2 + 2V_3V_2 + 4V_3V_1 - 4V_2V_1 + 4V_3V_0 - 6V_2V_0 - 6V_1V_0)$.

Proof. Take $x = -1, r = 0, s = 1, t = 1, u = 1$ in Theorem 3.1 in [12]. □

From the last proposition, we have the following corollary which gives sum formulas of Tridovan numbers (take $V_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 0, T_3 = 1$).

Corollary 5.5. *For $n \geq 0$, Tridovan numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k T_k^2 = \frac{1}{6}((-1)^n (T_{n+4}^2 - T_{n+3}^2 - 4T_{n+2}^2 + 5T_{n+1}^2 + 2T_{n+4}T_{n+3} + 4T_{n+4}T_{n+2} - 2T_{n+4}T_{n+1} - 4T_{n+3}T_{n+2} - 6T_{n+3}T_{n+1}) + 1)$.

(b) $\sum_{k=0}^{n}(-1)^k T_{k+1}T_k = \frac{1}{6}((-1)^n (T_{n+4}^2 - T_{n+3}^2 + 2T_{n+2}^2 - T_{n+1}^2 + 2T_{n+4}T_{n+3} - 2T_{n+4}T_{n+2} - 2T_{n+4}T_{n+1} - 4T_{n+3}T_{n+2} + 6T_{n+2}T_{n+1}) + 1)$.

(c) $\sum_{k=0}^{n}(-1)^k T_{k+2}T_k = \frac{1}{6}(3\,(-1)^n (T_{n+4}^2 - T_{n+3}^2 - T_{n+1}^2 - 2T_{n+2}T_{n+1}) + 3)$.

(d) $\sum_{k=0}^{n}(-1)^k T_{k+3}T_k = \frac{1}{6}((-1)^n (T_{n+4}^2 - T_{n+3}^2 - 4T_{n+2}^2 - T_{n+1}^2 + 2T_{n+4}T_{n+3} + 4T_{n+4}T_{n+2} + 4 T_{n+4}T_{n+1} - 4T_{n+3}T_{n+2} - 6T_{n+3}T_{n+1} - 6T_{n+2}T_{n+1}) + 1)$.

Taking $V_n = H_n$ with $H_0 = 4, H_1 = 0, H_2 = 2, H_3 = 3$ in the last proposition, we have the following corollary which presents sum formulas of Tridovan–Lucas numbers.

Corollary 5.6. *For $n \geq 0$, Tridovan–Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n}(-1)^k H_k^2 = \frac{1}{6}((-1)^n (H_{n+4}^2 - H_{n+3}^2 - 4H_{n+2}^2 + 5H_{n+1}^2 + 2H_{n+4}H_{n+3} + 4H_{n+4}H_{n+2} - 2H_{n+4}H_{n+1} - 4H_{n+3}H_{n+2} - 6H_{n+3}H_{n+1}) + 25)$.

(b) $\sum_{k=0}^{n}(-1)^k H_{k+1}H_k = \frac{1}{6}((-1)^n (H_{n+4}^2 - H_{n+3}^2 + 2H_{n+2}^2 - H_{n+1}^2 + 2H_{n+4}H_{n+3} - 2H_{n+4}H_{n+2} - 2H_{n+4}H_{n+1} - 4H_{n+3}H_{n+2} + 6H_{n+2}H_{n+1}) - 23)$.

(c) $\sum_{k=0}^{n}(-1)^k H_{k+2}H_k = \frac{1}{6}(3\,(-1)^n (H_{n+4}^2 - H_{n+3}^2 - H_{n+1}^2 - 2H_{n+2}H_{n+1}) - 33)$.

(d) $\sum_{k=0}^{n}(-1)^k H_{k+3}H_k = \frac{1}{6}((-1)^n (H_{n+4}^2 - H_{n+3}^2 - 4H_{n+2}^2 - H_{n+1}^2 + 2H_{n+4}H_{n+3} + 4H_{n+4}H_{n+2} + 4H_{n+4}H_{n+1} - 4H_{n+3}H_{n+2} - 6H_{n+3}H_{n+1} - 6H_{n+2}H_{n+1}) + 1)$.

5.3 A sum formula

We have the following proposition.

Proposition 5.3. *For all integers m and j with $H_m^2 - H_{2m} - 2H_m + 2(1 - H_{-m})(-1)^m + 2 \neq 0$, we have*

$$\sum_{k=0}^{n} V_{mk+j} = \frac{\Delta}{\frac{1}{2}(H_m^2 - H_{2m} - 2H_m + 2(1 - H_{-m})(-1)^m + 2)}$$

where $\Delta = -V_{mn+2m+j} + (H_m - 1)V_{mn+m+j} + (-1)^m (1 - H_{-m}) V_{mn+j} + (-1)^m V_{mn-m+j} - (-1)^m V_{-m+j} + V_{2m+j} - (H_m - 1)V_{m+j} - (H_m + \frac{1}{2}(H_{2m} - H_m^2) - 1)V_j$.

Proof. Take $r = 0$, $s = 1$, $t = 1$, $u = 1$ and $H_n = H_n$ in Soykan [14, Theorem 15]. $\square$

A special case of the above proposition is the following corollary.

Corollary 5.7. *The following IDs apply:*

1. $m = 1, j = 0$.
   - **(a)** $\sum_{k=0}^{n} V_k = \frac{1}{2}(V_{n+2} + V_{n+1} + 2V_n + V_{n-1} - V_3 - V_2 + V_0)$.
   - **(b)** $\sum_{k=0}^{n} T_k = \frac{1}{2}(T_{n+2} + T_{n+1} + 2T_n + T_{n-1} - 1)$.
   - **(c)** $\sum_{k=0}^{n} H_k = \frac{1}{2}(H_{n+2} + H_{n+1} + 2H_n + H_{n-1} - 1)$.
2. $m = -1, j = 0$.
   - **(a)** $\sum_{k=0}^{n} V_{-k} = \frac{1}{2}(-V_{-n+1} - V_{-n} - 2V_{-n-1} - V_{-n-2} + V_3 + V_2 + V_0)$.
   - **(b)** $\sum_{k=0}^{n} T_{-k} = \frac{1}{2}(-T_{-n+1} - T_{-n} - 2T_{-n-1} - T_{-n-2} + 1)$.
   - **(c)** $\sum_{k=0}^{n} H_{-k} = \frac{1}{2}(-H_{-n+1} - H_{-n} - 2H_{-n-1} - H_{-n-2} + 9)$.
3. $m = 3, j = 1$.
   - **(a)** $\sum_{k=0}^{n} V_{3k+1} = \frac{1}{2}(V_{3n+7} - 2V_{3n+4} + V_{3n-2} - V_{3n+1} - V_3 - V_2 + 2V_1 + V_0)$.
   - **(b)** $\sum_{k=0}^{n} T_{3k+1} = \frac{1}{2}(T_{3n+7} - 2T_{3n+4} + T_{3n-2} - T_{3n+1} + 1)$.
   - **(c)** $\sum_{k=0}^{n} H_{3k+1} = \frac{1}{2}(H_{3n+7} - 2H_{3n+4} + H_{3n-2} - H_{3n+1} - 1)$.
4. $m = -3, j = -1$.
   - **(a)** $\sum_{k=0}^{n} V_{-3k-1} = \frac{1}{2}(-V_{-3n+2} + 2V_{-3n-1} + V_{-3n-4} - V_{-3n-7} + V_3 - V_2 + V_0)$.
   - **(b)** $\sum_{k=0}^{n} T_{-3k-1} = \frac{1}{2}(-T_{-3n+2} + 2T_{-3n-1} + T_{-3n-4} - T_{-3n-7} + 1)$.
   - **(c)** $\sum_{k=0}^{n} H_{-3k-1} = \frac{1}{2}(-H_{-3n+2} + 2H_{-3n-1} + H_{-3n-4} - H_{-3n-7} + 5)$.

6 Matrices related to generalized Tridovan numbers

This section presents specific matrix relationships for generalized Tridovan numbers. We define the square matrix A of order 4 as:

$$A = A_{0111} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = -1$. We also define

$$B_n = \begin{pmatrix} T_{n+1} & T_n + T_{n-1} + T_{n-2} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} + T_{n-3} & T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & T_{n-2} + T_{n-3} + T_{n-4} & T_{n-2} + T_{n-3} & T_{n-2} \\ T_{n-2} & T_{n-3} + T_{n-4} + T_{n-5} & T_{n-3} + T_{n-4} & T_{n-3} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_{n+1} & V_n + V_{n-1} + V_{n-2} & V_n + V_{n-1} & V_n \\ V_n & V_{n-1} + V_{n-2} + V_{n-3} & V_{n-1} + V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-2} + V_{n-3} + V_{n-4} & V_{n-2} + V_{n-3} & V_{n-2} \\ V_{n-2} & V_{n-3} + V_{n-4} + V_{n-5} & V_{n-3} + V_{n-4} & V_{n-3} \end{pmatrix}.$$

Theorem 6.1. *For all integers $m, n \geq 0$, we have*

(a) $B_n = A^n$.

(b) $C_1 A^n = A^n C_1$.

(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof. Take $r = 0$, $s = 1$, $t = 1$, $u = 1, G_n = T_n$ in [14, Theorem 19]. □

Note that using the identity

$$279T_n = 74H_{n+3} - 23H_{n+2} - 48H_{n+1} - 44H_n$$

and Theorem 6.1, we see that

$$A^n = \frac{1}{279} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \quad n \geq 0,$$

where

$$\begin{aligned} a_{11} &= 74H_{n+4} - 23H_{n+3} - 48H_{n+2} - 44H_{n+1} \\ a_{21} &= 74H_{n+3} - 23H_{n+2} - 48H_{n+1} - 44H_n \\ a_{31} &= 74H_{n+2} - 23H_{n+1} - 48H_n - 44H_{n-1} \\ a_{41} &= 74H_{n+1} - 23H_n - 48H_{n-1} - 44H_{n-2} \end{aligned}$$

$$\begin{aligned} a_{12} &= 74H_{n+3} + 51H_{n+2} + 3H_{n+1} - 115H_n - 92H_{n-1} - 44H_{n-2} \\ a_{22} &= 74H_{n+2} + 51H_{n+1} + 3H_n - 115H_{n-1} - 92H_{n-2} - 44H_{n-3} \\ a_{32} &= 74H_{n+1} + 51H_n + 3H_{n-1} - 115H_{n-2} - 92H_{n-3} - 44H_{n-4} \\ a_{42} &= 74H_n + 51H_{n-1} + 3H_{n-2} - 115H_{n-3} - 92H_{n-4} - 44H_{n-5} \end{aligned}$$

$$\begin{aligned} a_{13} &= 74H_{n+3} + 51H_{n+2} - 71H_{n+1} - 92H_n - 44H_{n-1} \\ a_{23} &= 74H_{n+2} + 51H_{n+1} - 71H_n - 92H_{n-1} - 44H_{n-2} \\ a_{33} &= 74H_{n+1} + 51H_n - 71H_{n-1} - 92H_{n-2} - 44H_{n-3} \\ a_{43} &= 74H_n + 51H_{n-1} - 71H_{n-2} - 92H_{n-3} - 44H_{n-4} \end{aligned}$$

$$\begin{aligned} a_{14} &= 74H_{n+3} - 23H_{n+2} - 48H_{n+1} - 44H_n \\ a_{24} &= 74H_{n+2} - 23H_{n+1} - 48H_n - 44H_{n-1} \\ a_{34} &= 74H_{n+1} - 23H_n - 48H_{n-1} - 44H_{n-2} \\ a_{44} &= 74H_n - 23H_{n-1} - 48H_{n-2} - 44H_{n-3} \end{aligned}$$

Some properties of matrix A^n can be given as

$$A^n = A^{n-2} + A^{n-3} + A^{n-4}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = (-1)^n$$

for all integers m and n.

Theorem 6.2. *For all* $m, n,$ *we have*

$$V_{n+m} = V_n T_{m+1} + V_{n-1}(T_m + T_{m-1} + T_{m-2}) + V_{n-2}(T_m + T_{m-1}) + V_{n-3} T_m.$$

Proof. Take $r = 0$, $s = 1$, $t = 1$, $u = 1, G_n = T_n$ in [14, Theorem 20]. □

Corollary 6.1. *For all integers* $m, n,$ *we have*

$$\begin{aligned} T_{n+m} &= T_n T_{m+1} + T_{n-1}(T_m + T_{m-1} + T_{m-2}) + T_{n-2}(T_m + T_{m-1}) + T_{n-3} T_m, \\ H_{n+m} &= H_n T_{m+1} + H_{n-1}(T_m + T_{m-1} + T_{m-2}) + H_{n-2}(T_m + T_{m-1}) + H_{n-3} T_m. \end{aligned}$$

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