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On a sequence derived from the Laplace transform of the characteristic polynomial of the Fibonacci sequence

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Abstract: Recently, based on the Laplace transform of the characteristic polynomial of the Fibonacci sequence, Deveci and Shannon established a new sequence and analysed some of its properties. They disclosed in particular the odd terms. In this short note, we provide a matricial representation for this sequence as well as one in terms of the Chebyshev polynomials of the second kind. The subsequence of the even terms is also disclosed.

Keywords: Fibonacci sequence, Recurrence, Chebyshev polynomials of the second kind, Determinant, Tridiagonal matrices.
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1 Introduction

The celebrated Fibonacci sequence whose first terms are

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots$$

satisfy the recurrence relation

$$F_n = F_{n-1} + F_{n-2}\,, \quad \text{for } n \geqslant 2, \tag{1.1}$$

with initial conditions $F_0 = 0$ and $F_1 = 1$. This sequence is known to go back at least to the 6th or 8th centuries with the works of Virahanka [22]. The extent to which Fibonacci introduced the sequence and the Hindu-Arabic numeration systems to Europe can also be disputed, as it might have been Gerbert d'Aurillac, the French Pope Sylvester II in the late 10th century, or others familiar with Arab achievements and the use of the abacus [12, Chapter 3].

In the present paper, we provide a short survey and a matricial representation for the sequence established by Deveci and Shannon, as well as one in terms of the Chebyshev polynomials of the second kind. The subsequence of the even terms is also disclosed.

2 A survey and new results

One of the ways to understand the Fibonacci sequence, and perhaps one of the least explored, is through the theory of orthogonal polynomials, namely the Chebyshev polynomials of the second kind $(U_n(x))_{n\geqslant 0}$. Notwithstanding, this connection was for example explored by R. G. Buschman [6] in the early 1960s, with the equality

$$F_n = (-\mathring{\imath})^{n-1}\, U_{n-1}\left(\frac{\mathring{\imath}}{2}\right)\,,$$

where $\mathring{\imath}$ represents the unit imaginary number. This was also subject of consideration by A. F. Horadam later in [18]. Recall that the Chebyshev polynomials of the second kind $(U_n(x))_{n\geqslant 0}$ are the orthogonal polynomials satisfying the three-term recurrence relation

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)\,, \quad \text{for } n = 1, 2, \ldots\ , \tag{2.1}$$

with initial conditions $U_{-1}(x) = 0$ and $U_0(x) = 1$. Two of the most well-known explicit formulas for these polynomials are

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}\,, \quad \text{with } x = \cos\theta \quad (0 \leqslant \theta < \pi),$$

and

$$U_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k}, \tag{2.2}$$

for all $n = 0, 1, 2\ldots$. From (2.1), we can conclude immediately that

$$F_{n+1} = \det \begin{pmatrix} 1 & 1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & 1 \end{pmatrix}_{n\times n}$$

and, from here, explore the properties of the determinants in matrix theory. For interesting extensions and applications, the reader is referred to [1–5, 9, 13, 14].

Another way to understand the Fibonacci sequence is consider it as a particular case of the sequence $\{w_n \equiv w_n(a,b;p,q)\}$ defined by the second-order homogeneous linear recurrence with constant coefficients

$$w_n = pw_{n-1} - qw_{n-2}\,, \quad \text{for } n \geqslant 2,$$

with initial conditions $w_0 = a$ and $w_1 = b$, for arbitrary integers a and b, established in 1965 by Horadam [17, Section 3].

Setting

$$T_n = \begin{pmatrix} p & 1 & & \\ q & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & q & p \end{pmatrix}_{n\times n},$$

we have (cf. e.g. [1, 14])

$$\det T_n = (\sqrt{q})^n\, U_n\left(\frac{p}{2\sqrt{q}}\right).$$

From here, we can obtain

$$w_{n+1} = (\sqrt{q})^n \left(\frac{b}{\sqrt{q}}\,U_{n-1}\left(\frac{p}{2\sqrt{q}}\right) - a\,U_{n-2}\left(\frac{p}{2\sqrt{q}}\right)\right). \tag{2.3}$$

To our knowledge, (2.3) was first established by Udrea in [24]. For an extension to periodic sequences the reader is referred to [5].

Recently, Deveci and Shannon in [8], based on the Laplace transform of the characteristic polynomial of the recurrence relation (1.1), defined a new sequence (y_n) satisfying

$$y_n = -w_{n-1} + 2w_{n-2}\,, \quad \text{for } n > 2, \tag{2.4}$$

with initial conditions $y_1 = 0$ and $y_2 = 1$. The first terms of this sequence are

$$0,\ 1,\ -1,\ 3,\ -5,\ 11,\ -21,\ 43,\ -85,\ 171,\ -341,\ 683,\ -1365\,,\ldots$$

From (2.3), setting $p = -1, q = -2, a = 0$, and $b = 1$, we have the explicit formula

$$y_n = (\mathbb{i}\sqrt{2})^{n-2} U_{n-2}\left(\frac{\mathbb{i}\sqrt{2}}{4}\right)$$

and a possible matricial representation is

$$y_n = \det \begin{pmatrix} 0 & 1 & & & \\ -1 & -1 & 1 & & \\ & -2 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & -2 & -1 \end{pmatrix}_{n \times n} \tag{2.5}$$

based on [5, (2.1)].

Among several properties, from the triangle for (y_n) in [8] it is claimed that for n odd,

$$4(y_n + y_{n+1}) - 2 = y_{n+2} + y_{n+3}\,.$$

However, from (2.5), through elementary operations on rows and columns, we can state

$$y_{n+1} = 4y_{n-1} - 1\,, \tag{2.6}$$

for any n.

In [8], as one of the motivations to introduce this sequence, we find that the symmetric of the subsequence of the odd terms of (y_n) satisfies the recurrence relation

$$k_n = 5k_{n-1} - 4_{n-2}\,, \quad \text{for } n > 2,$$

with initial conditions $k_1 = 1$ and $k_2 = 5$. This sequence satisfies the closed formula

$$k_n = \frac{1}{3}\left(2^{2n} - 1\right)$$

and emerges in a wide variety of interesting problems, from interpolation [23, p.35] to the central factorial numbers [10, Table 1].

The purpose of this brief note is to draw the reader's attention to the even-term subsequence of (y_n), even though it is part of the much-studied Jacobsthal sequence.

2.1 The even terms of (y_n)

The first terms of the subsequence of the even terms of (y_n), say (ℓ_n), are

$$1,\ 3,\ 11,\ 43,\ 171,\ 683,\ 2731,\ 10923,\ \ldots$$

From (2.6) we can deduce

$$\ell_n = 4\ell_{n-1} - 1\,.$$

Of course, if we consider only the even terms of y_n, an explicit formula in terms of Chebyshev polynomials of the second kind is

$$\ell_n = (-2)^{n-1} U_{2n-2}\left(\frac{\mathrm{i}\sqrt{2}}{4}\right).$$

From (2.2), we can also obtain

$$\begin{aligned}
\ell_n &= (-2)^{n-1}\sum_{k=0}^{n-1}(-1)^k\binom{2n-2-k}{k}\left(-\frac{1}{2}\right)^{n-1-k} \\
&= \sum_{k=0}^{n-1}\binom{2n-2-k}{k}2^k \\
&= \sum_{k=0}^{n-1}\binom{n-1+k}{2k}2^{n-1-k} \\
&= \frac{1}{3}\left(2^{2n-1}+1\right)
\end{aligned}$$

according to [15, (1.77)].

3 Concluding comments

It is once more surprising the extent to which these particular sequences have connections in different mathematical contexts. Because there is so much overlap among the topics, Figure 1 illustrates some of the scope of the mathematical territory covered by these topics in the papers to which reference has been made in this note. 'Roadmaps' such as this can be important teaching and learning devices in achieving comprehensive literature reviews in postgraduate learning, and integrated understanding, as distinct from isolated silos, in undergraduate teaching. They are particularly useful in 'Capstone' subjects towards the end of a course.

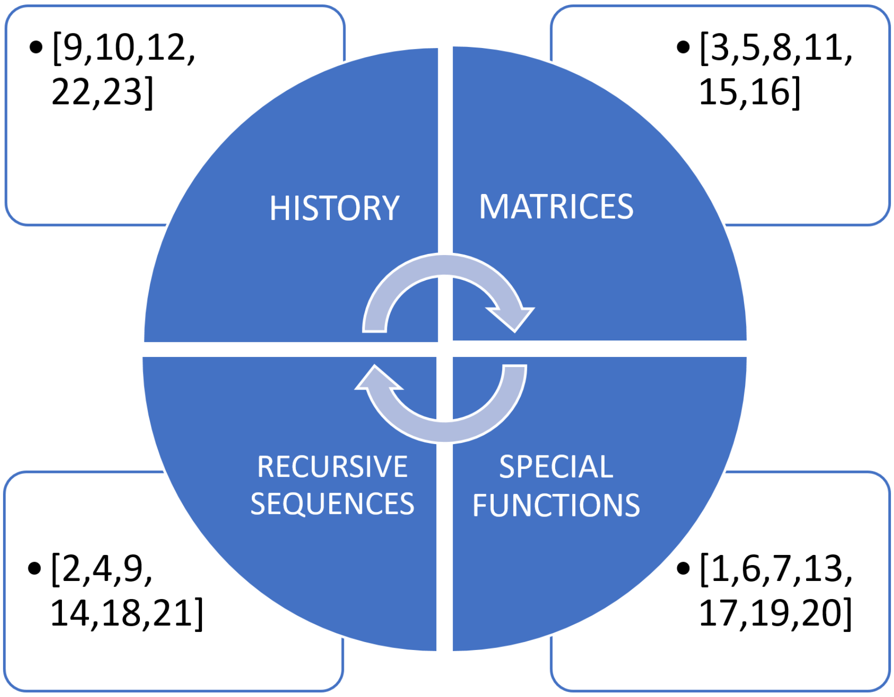

Figure 1. Roadmap of connections among topics

Cvetković et al. [7] have also utilized matricial transforms in relation to special functions and Fibonacci numbers, and Shannon [21] has suggested some Fibonacci analogs of special functions. Shannon and Horadam [19] and Melham [20] have further extended some connections among special functions and Fibonacci numbers.

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