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The Dirichlet divisor problem over square-free integers and unitary convolutions

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Abstract: We obtain an asymptotic formula for the sum $\tilde{D}_2$ of the divisors of all square-free integers less than or equal to x, with error term $O(x^{1/2+\epsilon})$. This improves the error term $O(x^{3/4+\epsilon})$ presented in [7] obtained via analytical methods. Our approach is elementary and it is based on the connections between the function $\tilde{D}_2$ and unitary convolutions.
Keywords: Dirichlet divisor problem, Square-free integers, Unitary convolutions.
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1 Introduction

One of the oldest unsolved problems in Analytic Number Theory (the classical Dirichlet divisor problem) is determining the smallest positive number η such that the error term $\Delta(x)$ in

$$D(x) \;:=\; \sum_{n \,\leq\, x} \sum_{d|n} 1 \;=\; x\log(x) + (2\gamma - 1)x + \Delta(x) \tag{1}$$

satisfies $\Delta(x) \;=\; O(x^{\eta+\epsilon})$ for every $\epsilon > 0$ (γ is the Euler–Mascheroni constant). In 1849, Dirichlet showed that

$$\Delta(x) = O(\sqrt{x}) \tag{2}$$

and many mathematicians have worked on improving Dirichlet's estimate since. Hardy proved that η can not be smaller than $1/4$ and it is widely conjectured that $\Delta(x) \;=\; O(x^{1/4+\epsilon})\ \forall\epsilon > 0$.

The sharpest known bound $\Delta(x) = O\left(x^{131/416+\epsilon}\right)\ \forall\, \epsilon > 0$ is due to Huxley (see [2] for a recent survey of the subject).

Variants of the Dirichlet divisor problem can be obtained by imposing some conditions over the summation index n or/and considering only the divisors d of n that fulfill some requirements. For instance, in 1874, Mertens considered the problem of estimating the sum

$$D_2(x) := \sum_{n \leq x} \sum_{d|n} \mid \mu(d) \mid$$

in the left-hand side of (1) only for square-free divisors d of n [9]. In 1932, Hölder [6] considered the Dirichlet divisor problem for k–free divisors, an extension of the square-free ($k = 2$) case (a positive integer n is k-free if n is not divisible by the k-th power of any prime number). Let us also mention some problems concerning the estimation of sums like

$$\sum_{\substack{n \leq x \\ n \in \mathcal{A}}} \sum_{d|n} 1,$$

when $\mathcal{A}$ is a residue class [10] (or some union of residue classes [8]), or, more generally, when $\mathcal{A}$ is the image of some polynomial with positive integer coefficients (see [12], pp. 84–85, or [3] and the references therein).

Recently, Jakimczuk and Lalín [7] estimated the number $\tilde{D}_2(x)$ of the divisors of all square-free integers that do not exceed x:

$$\tilde{D}_2(x) = \sum_{n \leq x} \mid \mu(n) \mid \sum_{d|n} 1 = \sum_{n \leq x} \mid \mu(n) \mid \sum_{d|n} \mid \mu(d) \mid = \sum_{ij \leq x} \mid \mu(ij) \mid . \tag{3}$$

Combining Perron's formula with an Euler-type-product formula for the Dirichlet series with coefficients $a_n = \mid \mu(n) \mid \sum_{d|n} 1$, they proved the following result.

Theorem 1.1 ([7]). *There is $\beta \in \mathbb{R}$ such that, for every $\epsilon > 0$,*

$$\tilde{D}_2(x) = \prod_{p \text{ prime}} \left[1 - \frac{3}{p^2} + \frac{2}{p^3}\right] x \log(x) + \beta x + O_\epsilon\left(x^{3/4+\epsilon}\right). \tag{4}$$

In this note we present an elementary approach for estimating $\tilde{D}_2$ based on its connections with unitary convolutions [5]. We express the summatory functions of unitary convolutions in terms of the summatory functions of ordinary Dirichlet convolutions. Using this result, we write $\tilde{D}_2$ in terms of the Dirichlet function (1) and obtain the following improvement over (4).

Theorem 1.2. *There is $\beta \in \mathbb{R}$ such that, for every $\epsilon > 0$,*

$$\tilde{D}_2(x) = \frac{1}{\zeta^2(2)} \left[\prod_{p \text{ prime}} \left(1 - \frac{1}{(p+1)^2}\right)\right] x \log(x) + \beta x + O_\epsilon\left(x^{1/2+\epsilon}\right) \tag{5}$$

(ζ is the Riemann zeta function).

Using the Euler product for ζ, one can easily check that the leading coefficients of $\tilde{D}_2$ in (4) and (5) are identical. However, the representation of the coefficient c of the leading term $x\log(x)$ of $\tilde{D}_2$ in (5) looks more informative because it immediately tells us that $c < \frac{1}{\zeta(2)^2}$. This is already expected because

$$\tilde{D}_2(x) \;\le\; \sum_{ij\,\le\,x} \mid \mu(i) \mid\mid \mu(j) \mid \quad \forall j \ge 1 \tag{6}$$

and the right-hand side of (6) is easily seen to be asymptotic to $\frac{1}{\zeta(2)^2}\, x\log(x)$.

2 Summatory functions of unitary convolutions

Let $\chi_{i,.} : j \longmapsto \chi_{i,j}$ denote the Dirichlet principal character modulus i

$$\chi_{i,j} \;=\; \begin{cases} 1, & (i,j) \;=\; 1, \\ 0, & (i,j) \;>\; 1. \end{cases}$$

In the beginning of the sixties, Cohen [5] studied the properties of unitary convolutions. The unitary convolution of the arithmetic functions g and h is defined by

$$f(n) \;=\; \sum_{ij\,=\,n} g(i)h(j)\chi_{i,j}, \;\; n \ge 1. \tag{7}$$

This subject is very close to the divisor problem we are concerned with. In fact,

$$\tilde{D}_2(x) \;=\; \sum_{i\,\le\,x}\;\sum_{j\,\le\,x/i} \mid \mu(i) \mid \mid \mu(j) \mid \chi_{i,j} \tag{8}$$

is the summatory function of the unitary convolution of the function $\mid \mu \mid$ with itself. Cohen presented asymptotic formulae for the sums

$$\sum_{j\,\le\,x} \mid \mu(j) \mid \chi_{i,j} \quad \text{and} \quad \sum_{j\,\le\,x} j \mid \mu(j) \mid \chi_{i,j}$$

([5], Lemmas 5.2 and 5.3). For instance, we have

$$\sum_{j\,\le\,x} \mid \mu(j) \mid \chi_{i,j} \;=\; x\frac{1}{\zeta_i(2)}\Bigg(\sum_{d|i}\frac{\mu(d)}{d}\Bigg) \;+\; \Bigg(\sum_{d|i}1\Bigg)O\left(\sqrt{x}\right),$$

$\zeta_i(z) \;:=\; \sum\limits_{j\,=\,1}^{\infty} \frac{\chi_{i,j}}{j^z}$, $\mathfrak{Re}(z) > 1, i \ge 1$.

Using this information in (8), we obtain

$$\tilde{D}_2(x) \;\sim\; x\sum_{i\,\le\,x}\frac{\mid \mu(i) \mid}{i\zeta_i(2)}\Bigg(\sum_{d|i}\frac{\mu(d)}{d}\Bigg). \tag{9}$$

The main problem with this approach is that it is not much clear how to interpret the sum in the right-hand side of (9). In order to avoid this difficulty, we express the summatory functions of unitary convolutions in a more convenient way. Given two arithmetic functions g, h and $r \geq 1$, let

$$V_r[g,h](x) \;=\; \sum_{ij \,\leq\, x/r^2} g(ri)h(rj), \;\; x \geq 1.$$

Lemma 2.1. *Let $g, h : \mathbb{N} \to \mathbb{C}$ be two arithmetic functions and let f be the unitary convolution of g and h defined by* (7)*. For $x \geq 1$,*

$$\sum_{n \leq x} f(n) \;=\; \sum_{r \,\leq\, \sqrt{x}} \mu(r) V_r[g,h](x). \tag{10}$$

Proof. For $x \geq 1$, $r \leq \sqrt{x}$ and $r' \leq \sqrt{x}/r$, we group all i, j with $ij \leq x/r^2$ and $\gcd(i,j) = r'$:

$$\sum_{ij \,\leq\, x/r^2} g(ri)h(rj) \;=\; \sum_{rr' \,\leq\, \sqrt{x}} \;\; \sum_{i'j' \,\leq\, x/\,(rr')^2} g\left(rr'i'\right) h\left(rr'j'\right) \chi_{i',j'}. \tag{11}$$

In order to simplify the notation, for $\ell = 1, 2, \ldots, \tau := \lfloor \sqrt{x} \rfloor$, let

$$z_\ell \;=\; \sum_{ij \,\leq\, x/\ell^2} g(\ell i)h(\ell j)\chi_{i,j}, \qquad w_\ell \;=\; \sum_{ij \,\leq\, x/\ell^2} g(\ell i)h(\ell j).$$

The relations (11) for $r = 1, 2, \ldots, \tau$ can be expressed as the system of linear equations

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\ 0 & 1 & 0 & 1 & 0 & 1 & \ldots \\ 0 & 0 & 1 & 0 & 0 & 1 & \ldots \\ 0 & 0 & 0 & 1 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \end{bmatrix} \times \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ \vdots \\ z_\tau \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ \vdots \\ w_\tau \end{bmatrix}.$$

By Cramer's rule,

$$z_1 \;=\; \begin{vmatrix} w_1 & w_2 & w_3 & w_4 & \ldots & w_\tau \\ 1 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 1 & 0 & \ldots & 0 \\ 1 & 1 & 0 & 1 & \ldots & 0 \\ 1 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{vmatrix}. \tag{12}$$

The right-hand side of (12) is a Redheffer determinant [1, 4, 11]. Hence,

$$z_1 \;=\; \sum_{r\,=\,1}^{\tau} \mu(r) w_r. \qquad \square$$

Applying Lemma 2.1 to the functions g and h defined by

$$g(i) \;=\; h(i) \;= \mid \mu(i) \mid, \tag{13}$$

we obtain the following result.

Corollary 1.

$$\tilde{D}_2(x) \;=\; \sum_{r\,\le\,\sqrt{x}} \mu(r)V_r(x), \tag{14}$$

with

$$V_r(x) \;=\; \sum_{ij\,\le\,x/r^2} |\,\mu(ri)\,|\,|\,\mu(rj)\,|\,. \tag{15}$$

Remark 1. *Note that the indexes i and j do not appear simultaneously as arguments of χ in* (15) *(as they do in* (8)*) and this avoids dealing with expressions like the one in the right-hand side of* (9).

3 Proof of Theorem 1.2

In some previous investigations, we combined (15) and some asymptotic formulae for

$$\sum_{i\,\le\,x} |\,\mu(ri)\,|, \quad \sum_{i\,\le\,x} i\,|\,\mu(ri)\,|, \quad \sum_{i\,\le\,x} |\,\frac{\mu(ri)}{i}\,|$$

to estimate $\tilde{D}_2$. Curiously, that attempt led to same estimate [7]

$$O(x^{3/4}+\epsilon)$$

obtained by Jakimczuk and Lalín for the error term. In order to obtain sharper results, we express $V_r(x)$ directly (see the proof at the end of this section) in terms of the Dirichlet function (1).

Lemma 3.1. *If $\mu(r)\neq 0$, the function V_r defined in* (15) *satisfies*

$$\begin{aligned} V_r(x) \;&=\; \sum_{(d,d',n,n')\,\in\,\mathcal{A}} \mu(d)\mu(d')\mu(n)\mu(n')\,\chi_{r,n}\,\chi_{r,n'} D\left(\frac{x/r^2}{d'dn^2(n')^2}\right), \\ \mathcal{A} \;&=\; \left\{(d,d',n,n') : d,d' \mid r,\; n \le \sqrt{\tfrac{x}{r^2}},\; n' \le \sqrt{\frac{\frac{x}{r^2}}{dd'n^2}}\right\}. \end{aligned}$$

The proof of Theorem 1.2 follows directly by Corollary 1 and Lemma 3.1, combined with Dirichlet estimates (2), after elementary, but somewhat tedious, handwork. For instance, the coefficient c of the leading term $x\log(x)$ in $\tilde{D}_2$ is

$$\begin{aligned} c \;&=\; \sum_{r=1}^{\infty} \frac{\mu(r)}{r^2} \sum_{d,\,d'\,\mid\,r} \frac{\mu(d)}{d}\frac{\mu(d')}{d'} \sum_{n'=1}^{\infty} \frac{\mu(n)\,\chi_{r,n}}{n^2} \sum_{n'=1}^{\infty} \frac{\mu(n')\chi_{r,n'}}{(n')^2} \\ &=\; \sum_{r=1}^{\infty} \frac{\mu(r)}{r^2} \prod_{\substack{p\,\mid\,r \\ p \text{ prime}}} \left(1-\frac{1}{p}\right)^2 \prod_{\substack{p\,\nmid\,r \\ p \text{ prime}}} \left(1-\frac{1}{p^2}\right)^2 \\ &=\; \frac{1}{\zeta(2)^2} \sum_{r=1}^{\infty} \mu(r) \prod_{\substack{p\,\mid\,r \\ p \text{ prime}}} \frac{1}{p^2}\left(\frac{1-\frac{1}{p}}{1-\frac{1}{p^2}}\right)^2 \\ &=\; \frac{1}{\zeta(2)^2} \prod_{\substack{p\,\mid\,r \\ p \text{ prime}}} \left(1-\frac{1}{(p+1)^2}\right). \end{aligned}$$

In the same vein, the coefficient β of x in $\tilde{D}_2$ is $\beta = c - c'$, with

$$c' \;=\; \sum_{r=1}^{\infty} \frac{\mu(r)}{r^2} \sum_{d,\, d' \mid r} \frac{\mu(d)}{d}\frac{\mu(d')}{d'} \sum_{n'=1}^{\infty} \frac{\mu(n)\,\chi_{r,n}}{n^2} \sum_{n'=1}^{\infty} \frac{\mu(n')\chi_{r,n'}}{(n')^2} \log\left(r^2 dd' n^2 n'^2\right). \tag{16}$$

Using that $\sum_{d \mid r} 1 = O_\epsilon(r^\epsilon)\ \forall \epsilon > 0$, one can readily see that the series in the right-hand side of (16) is absolutely convergent. In addition, the overall error term $E(x)$ for $\tilde{D}_2$ associated to the error term $O(\sqrt{x})$ in Dirichlet formula satisfies

$$\begin{aligned} E(x) &\ll \sum_{r \le \sqrt{x}} \mid \mu(r) \mid \sum_{\substack{d,\, d' \mid r, \\ n \le \sqrt{(x/r^2)} \\ n' \le \sqrt{\frac{x/r^2}{dd'n^2}}}} \left(\frac{x/r^2}{(n')^2 dd' n^2}\right)^{1/2} \\ &\ll_\epsilon \sum_{r \le \sqrt{x}} \mid \mu(r) \mid \sum_{\substack{d,\, d' \mid r, \\ n \le \sqrt{(x/r^2)}}} \left(\frac{x/r^2}{dd'n^2}\right)^{1/2+\epsilon} \\ &\ll_\epsilon \sum_{r \le \sqrt{x}} \mid \mu(r) \mid \sum_{d,\, d' \mid r} \left(\frac{x/r^2}{dd'}\right)^{1/2+\epsilon} \\ &\ll_\epsilon x^{1/2+\epsilon} \sum_{r \le \sqrt{x}} \mid \mu(r) \mid \prod_{\substack{p \mid r \\ p \text{ prime}}} \frac{1}{p^{1+2\epsilon}} \left(1 + \frac{1}{p^{1/2+\epsilon}}\right)^2 \\ &\ll_\epsilon x^{1/2+\epsilon} \prod_{p \text{ prime}} \left[1 + \frac{1}{p^{1+2\epsilon}}\left(1 + \frac{1}{p^{1/2+\epsilon}}\right)^2\right] \\ &\ll_\epsilon x^{1/2+\epsilon}. \end{aligned} \tag{17}$$

We leave the rest of the details to the interested reader. □

3.1 Proof of Lemma 3.1

Lemma 3.2. *Let $g : \mathbb{N} \to \mathbb{C}$ be an arithmetic function. For $x \ge 1$,*

$$\sum_{j \le x} g(j)\,|\mu(j)| \;=\; \sum_{n \le \sqrt{x}} \mu(n) \sum_{i \le x/n^2} g\left(in^2\right). \tag{18}$$

Proof. Using

$$\sum_{n^2 | j} \mu(n) \;=\; |\mu(j)|,$$

we obtain

$$\sum_{n \le \sqrt[2]{x}} \mu(n) \sum_{i \le x/n^2} g\left(in^2\right) \;=\; \sum_{j \le x} g(j) \sum_{n^2 | j} \mu(n) \;=\; \sum_{j \le x} g(j)|\mu(j)|.$$ □

Let $r \geq 1$ with $\mu(r) \neq 0$. For $x \geq 1$, let

$$f(x) \;=\; \sum_{j \,\leq\, x} \mid \mu(rj) \mid .$$

We have

$$\begin{aligned} V_r(x) \;&=\; \sum_{i \,\leq\, (x/r^2)} \mid \mu(i) \mid f\left(\frac{x/r^2}{i}\right) \chi_{r,i} \\ &\overset{(18)}{=}\; \sum_{n \,\leq\, \sqrt{x/r^2}} \mu(n)\chi_{r,n} \sum_{i \,\leq\, (x/r^2)/n^2} f\left(\frac{x/r^2}{i\,n^2}\right) \chi_{r,i} \\ &=\; \sum_{n \,\leq\, \sqrt{x/r^2}} \mu(n)\chi_{r,n} \sum_{d|r} \mu(d) \sum_{i \,\leq\, \frac{x/r^2}{dn^2}} f\left(\frac{x/r^2}{din^2}\right). \end{aligned}$$

In addition,

$$\begin{aligned} f(x) \;&=\; \sum_{j \,\leq\, x} \mid \mu(j) \mid \chi_{r,j} \overset{(18)}{=} \sum_{n' \leq \sqrt{x}} \mu(n')\chi_{r,n'} \sum_{i \,\leq\, x/(n')^2} \chi_{r,i} \\ &=\; \sum_{n' \,\leq\, \sqrt{x}} \mu(n')\chi_{r,n'} \left(\sum_{d'|r} \mu(d') \left\lfloor \frac{x}{(n')^2 d'} \right\rfloor \right). \end{aligned}$$

Therefore,

$$\begin{aligned} V_r(x) \;&=\; \sum_{\substack{d,\, d' \,\mid\, r, \\ n \leq \sqrt{x/r^2} \\ i \leq \frac{x/r^2}{dn^2} \\ n' \leq \sqrt{\frac{x/r^2}{dd'in^2}}}} \mu(d)\mu(d')\mu(n)\mu(n')\; \chi_{r,n}\; \chi_{r,n'} \left\lfloor \frac{x/r^2}{(n')^2 d' din^2} \right\rfloor \\ &=\; \sum_{\substack{d,\, d' \,\mid\, r, \\ n \leq \sqrt{x/r^2} \\ n' \leq \sqrt{\frac{x/r^2}{dd'n^2}}}} \mu(d)\mu(d')\mu(n)\mu(n')\; \chi_{r,n}\; \chi_{r,n'} D\left(\frac{x/r^2}{(n')^2 dd'n^2}\right). \end{aligned}$$

□

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