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Solution to a pair of linear, two-variable, Diophantine equations with coprime coefficients from balancing and Lucas-balancing numbers

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Abstract: Let B_n and C_n be the n-th balancing and Lucas-balancing numbers, respectively. We consider the Diophantine equations $ax+by = \frac{1}{2}(a-1)(b-1)$ and $1+ax+by = \frac{1}{2}(a-1)(b-1)$ for $(a, b) \in \{(B_n, B_{n+1}), (B_{2n-1}, B_{2n+1}), (B_n, C_n), (C_n, C_{n+1})\}$ and present the non-negative integer solutions of the Diophantine equations in each case.
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1 Introduction

As defined by Behera and Panda [1], a natural number B is a balancing number if

$$1 + 2 + \cdots + (B - 1) = (B + 1) + (B + 2) + \cdots + (B + R)$$

for some natural number R, which is the balancer corresponding to B. The n-th balancing number is denoted by B_n and $C_n = \sqrt{8B_n^2 + 1}$ is called the n-th Lucas-balancing number [11, p. 25]. Customarily, 1 is accepted as the first balancing number, i.e., $B_1 = 1$. The balancing and Lucas-balancing numbers satisfy the recurrence relations $B_1 = 1, B_2 = 6, B_{n+1} = 6B_n - B_{n-1}$

and $C_1=3, C_2=17, C_{n+1}=6C_n-C_{n-1}$ for $n\geq 2$. On other hand, b is called a cobalancing number with cobalancer r [11] if

$$1+2+\cdots+b=(b+1)+(b+2)+\cdots+(b+r).$$

The n-th cobalancing number is denoted by b_n and cobalancing numbers satisfy the nonhomogeneous recurrence $b_1=0, b_2=2, b_{n+1}=6b_n-b_{n-1}+2$ for $n\geq 2$. The Binet forms are

$$B_n=\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}, \qquad C_n=\frac{\alpha^{2n}+\beta^{2n}}{2}, \qquad b_n=\frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}}-\frac{1}{2}.$$

where $\alpha=1+\sqrt{2}$ and $\beta=1-\sqrt{2}$.

Cyclotomy is the process of dividing a circle into equal parts, which is precisely the effect obtained by plotting the n-th roots of the unity in the complex plane. For $n\geq 1$, the n-th cyclotomic polynomial is defined as $\Phi_n(X)=\Pi_{m=1,\ (m,n)=1}^{n}\big(X-e^{\frac{2m\pi i}{n}}\big)$, where $e^{\frac{2m\pi i}{n}}$ is the primitive n-th roots of the unity. When $n=pq$ for some distict primes p and q, while computing the middle term of $\Phi_n(X)$, Beiter [2] sketched a proof that $\frac{1}{2}(p-1)(q-1)$ can be uniquely written as $\alpha q+\beta p+\delta$, where $0\leq\alpha\leq p-1$, $\beta\geq 0$, and $\delta\in\{0,1\}$.

Generalizing the result of Beiter [2], in a recent study by Chu [3] proved that, for any positive and relatively prime integers a and b, exactly one of the two equations $ax+by=\frac{1}{2}(a-1)(b-1)$ and $1+ax+by=\frac{1}{2}(a-1)(b-1)$ has a unique non-negative integer solution. In the same paper, he considered the above Diophantine equations for a and b chosen from the Fibonacci sequence.

The main results of this paper gives the unique non-negative integer solutions of the Diophantine equations $ax+by=\frac{1}{2}(a-1)(b-1)$ and $1+ax+by=\frac{1}{2}(a-1)(b-1)$ for each $(a,b)\in\{(B_n,B_{n+1}),(B_{2n-1},B_{2n+1}),(\frac{B_{2n}}{6},\frac{B_{2n+2}}{6}),(B_n,C_n),(C_n,C_{n+1})\}$.

The sums of balancing and Lucas-balancing numbers has been extensively studied by many authors (e.g., see [4–9, 12, 13]). For any non-negative integers m and n, the following known identities will be helpful and used in the main results without further reference.

1. $B_{m\pm 1}=3B_m\pm C_m$ [10, Theorem 2.5]
2. $C_{m\pm 1}=3C_m\pm 8B_m$ [10, Theorem 2.5]
3. $B_{m+n}B_{m-n}=B_m^2-B_n^2$ [10, Theorem 2.1]
4. $\sum_{i=0}^{n}C_{2i}=C_nB_{n+1}$ [8, Theorem 2.1]
5. $\sum_{i=1}^{n}C_{4i}=\frac{1}{12}(B_{4n+2}-6)$ [12, Theorem 4.1]
6. $\sum_{i=0}^{2n}(-1)^iC_{2i}=\frac{1}{6}(C_{4n+1}+3)$ [8, Theorem 2.1]
7. $(B_m,B_n)=B_{(m,n)}$ [10, Theorem 2.13].

Note: Throughout the paper, consider the numerical value of $\sum_{i=1}^{0}t_i$ as zero and greatest common divisor of a and b is denoted by (a,b).

2 Non-negative integer solutions of a few Diophantine equations

For a given pair of consecutive balancing numbers, we have $(B_n, B_{n+1}) = 1$. So we investigate the non-negative integer solutions to the following Diophantine equations:

$$B_n x + B_{n+1} y = \frac{(B_n - 1)(B_{n+1} - 1)}{2} \tag{1}$$

$$1 + B_n x + B_{n+1} y = \frac{(B_n - 1)(B_{n+1} - 1)}{2}. \tag{2}$$

The following table provides two cases:

n	B_n	B_{n+1}	in which equation	x	y
1	1	6	(1)	0	0
2	6	35	(2)	14	0
3	35	204	(1)	17	14
4	204	1189	(2)	492	17
5	1189	6930	(1)	594	492
6	6930	40391	(2)	16730	594

Firstly, we observe Equation (1) and Equation (2) are used alternatively, and secondly there is a pattern in the values of x and y. This pattern in the table is summarized in the following theorem.

Theorem 2.1. *For $n \geq 1$, the following equalities are correct*

$$B_{2n-1}\left(\frac{B_{2n-1} - 1}{2}\right) + B_{2n} b_{2n-1} = \frac{(B_{2n-1} - 1)(B_{2n} - 1)}{2} \tag{3}$$

$$1 + B_{2n} b_{2n+1} + B_{2n+1}\left(\frac{B_{2n-1} - 1}{2}\right) = \frac{(B_{2n} - 1)(B_{2n+1} - 1)}{2}. \tag{4}$$

Proof. Firstly, we prove the equality $2b_{2n+1} = B_{2n+1} - B_{2n} - 1$ using the Corollary 3.4.2 by Ray [11], which states $b_{n+1} - b_n = 2B_n$.

Consider

$$\begin{aligned} 2B_{2n-1} - 2B_{2n-2} - 2 &= (b_{2n} - b_{2n-1}) - (b_{2n-1} - b_{2n-2}) - 2 \\ &= b_{2n} - 2b_{2n-1} + b_{2n-2} - 2 \\ &= (b_{2n} + b_{2n-2} - 2) - 2b_{2n-1} \\ &= 6b_{2n-1} - 2b_{2n-1} \\ &= 4b_{2n-1}. \end{aligned}$$

The proof of (3) follows by considering

$$\begin{aligned} B_{2n-1}\big(B_{2n-1} - 1\big) + 2B_{2n} b_{2n-1} &= B_{2n-1}^2 - B_{2n-1} + B_{2n}[B_{2n-1} - B_{2n-2} - 1] \\ &= [B_{2n-1}^2 - B_{2n} B_{2n-2}] + B_{2n-1}(B_{2n} - 1) - B_{2n} \\ &= 1 + B_{2n-1} B_{2n} - B_{2n-1} - B_{2n} \\ &= (B_{2n-1} - 1)(B_{2n} - 1) \end{aligned}$$

and the proof of (4) follows by considering

$$\begin{aligned}2B_{2n}b_{2n+1} + B_{2n+1}\big(B_{2n-1} - 1\big) &= 2B_{2n}b_{2n+1} + B_{2n+1}B_{2n-1} - B_{2n+1}\\ &= 2B_{2n}b_{2n+1} + B_{2n}^2 - B_1^2 - B_{2n+1}\\ &= B_{2n}(2b_{2n+1} + B_{2n}) - 1 - B_{2n+1}\\ &= B_{2n}(B_{2n+1} - 1) - 1 - B_{2n+1}\\ &= (B_{2n} - 1)(B_{2n+1} - 1) - 2. \qquad \square\end{aligned}$$

Given a pair of consecutive odd indexed balancing numbers, we have $(B_{2n-1}, B_{2n+1}) = 1$. So we investigate the non-negative integer solutions to the following Diophantine equations:

$$B_{2n-1}x + B_{2n+1}y = \frac{(B_{2n-1} - 1)(B_{2n+1} - 1)}{2} \tag{5}$$

$$1 + B_{2n-1}x + B_{2n+1}y = \frac{(B_{2n-1} - 1)(B_{2n+1} - 1)}{2}. \tag{6}$$

The following table provides two cases:

n	B_{2n-1}	B_{2n+1}	**in which equation**	x	y
1	1	35	(5)	0	0
2	35	1189	(6)	577	0
3	1189	40391	(5)	577	577
4	40391	1372105	(6)	666434	577
5	1372105	46611179	(5)	666434	666434
6	46611179	1583407981	(6)	769064835	666434

The patterns in the table are summarized by the following theorem.

Theorem 2.2. *For $n \geq 1$, the following equalities hold*

1. $B_{4n-3}\left(\sum_{i=1}^{n-1} C_{4i}\right) + B_{4n-1}\left(\sum_{i=1}^{n-1} C_{4i}\right) = \dfrac{(B_{4n-3} - 1)(B_{4n-1} - 1)}{2}$
2. $1 + B_{4n-1}\left(\sum_{i=1}^{n} C_{4i}\right) + B_{4n+1}\left(\sum_{i=1}^{n-1} C_{4i}\right) = \dfrac{(B_{4n-1} - 1)(B_{4n+1} - 1)}{2}.$

Proof. Noting that $\sum_{i=1}^{n} C_{4i} = \frac{1}{12}\left(B_{4n+2} - 6\right)$, we prove the first identity

$$\begin{aligned}B_{4n-3}\left(\frac{B_{4n-2} - 6}{12}\right) + B_{4n-1}\left(\frac{B_{4n-2} - 6}{12}\right) &= \frac{1}{12}(B_{4n-3} + B_{4n-1})B_{4n-2} - \frac{1}{2}(B_{4n-3} + B_{4n-1})\\ &= \frac{1}{12}(6B_{4n-2}^2) - \frac{1}{2}(B_{4n-3} + B_{4n-1})\\ &= \frac{1}{2}(B_{4n-2}^2 - B_{4n-3} - B_{4n-1})\\ &= \frac{1}{2}(1 + B_{4n-1}B_{4n-3} - B_{4n-3} - B_{4n-1})\\ &= \frac{1}{2}(B_{4n-1} - 1)(B_{4n-3} - 1).\end{aligned}$$

The proof of second identity follows by considering

$$
\begin{aligned}
&B_{4n-1}\left(\frac{B_{4n+2}-6}{12}\right)+B_{4n+1}\left(\frac{B_{4n-2}-6}{12}\right)\\
&=\frac{1}{12}[B_{4n-1}B_{4n+2}+B_{4n+1}B_{4n-2}]-\frac{1}{2}B_{4n-1}-\frac{1}{2}B_{4n+1}\\
&=\frac{1}{12}[B_{4n-1}(3B_{4n+1}+C_{4n+1})+B_{4n+1}(3B_{4n-1}-C_{4n-1})]-\frac{1}{2}B_{4n-1}-\frac{1}{2}B_{4n+1}\\
&=\frac{1}{12}[6B_{4n-1}B_{4n+1}+(B_{4n-1}C_{4n+1}-B_{4n+1}C_{4n-1})]-\frac{1}{2}B_{4n-1}-\frac{1}{2}B_{4n+1}\\
&=\frac{1}{12}[6B_{4n-1}B_{4n+1}-6]-\frac{1}{2}B_{4n-1}-\frac{1}{2}B_{4n+1}\\
&=-\frac{1}{2}+\frac{1}{2}B_{4n-1}B_{4n+1}-\frac{1}{2}B_{4n-1}-\frac{1}{2}B_{4n+1}\\
&=\frac{(B_{4n-1}-1)(B_{4n+1}-1)}{2}-1. \qquad \square
\end{aligned}
$$

Given a pair of consecutive even indexed balancing numbers, we have $(B_{2n}, B_{2n+2}) = 6$. So we investigate the non-negative integer solutions to the following Diophantine equations:

$$\frac{B_{2n}}{6}x+\frac{B_{2n+2}}{6}y=\frac{\left(\frac{B_{2n}}{6}-1\right)\left(\frac{B_{2n+2}}{6}-1\right)}{2} \tag{7}$$

$$1+\frac{B_{2n}}{6}x+\frac{B_{2n+2}}{6}y=\frac{\left(\frac{B_{2n}}{6}-1\right)\left(\frac{B_{2n+2}}{6}-1\right)}{2}. \tag{8}$$

The following table provides two cases:

n	$\frac{1}{6}B_{2n}$	$\frac{1}{6}B_{2n+2}$	**in which equation**	x	y
1	1	34	(7)	0	0
2	34	1155	(8)	560	0
3	1155	39236	(7)	577	560
4	39236	1332869	(8)	646816	577

The patterns in the table are summarized by the following theorem. The proof follows in a similar manner, so we omit the proof.

Theorem 2.3. *For $n \geq 1$, the following equalities are true*

1. $1+\frac{B_{4n}}{6}\left(\sum_{i=1}^{2n}(-1)^iC_{2i}\right)+\frac{B_{4n+2}}{6}\left(\sum_{i=1}^{n-1}C_{4i}\right)=\frac{1}{2}\left(\frac{B_{4n}}{6}-1\right)\left(\frac{B_{4n+2}}{6}-1\right)$

2. $\frac{B_{4n-2}}{6}\left(\sum_{i=1}^{n-1}C_{4i}\right)+\frac{B_{4n}}{6}\left(\sum_{i=1}^{2n-2}(-1)^iC_{2i}\right)=\frac{1}{2}\left(\frac{B_{4n-2}}{6}-1\right)\left(\frac{B_{4n}}{6}-1\right).$

For a pair of balancing and Lucas-balancing numbers of the same index, we know that $(B_n, C_n) = 1$ [10, Lemma 2.9]. So we investigate the non-negative integer solutions to the following Diophantine equations:

$$B_nx+C_ny=\frac{(B_n-1)(C_n-1)}{2} \tag{9}$$

$$1 + B_n x + C_n y = \frac{(B_n - 1)(C_n - 1)}{2}. \tag{10}$$

The following table provides a single case only.

n	B_n	C_n	**in which equation**	x	y
1	1	3	(9)	0	0
2	6	17	(9)	1	2
3	35	99	(9)	8	14
4	204	577	(9)	49	84

The table results in the following two theorems.

Theorem 2.4. *For* $n \geq 1$, $B_n\big(B_{n-1} + b_{n-1}\big) + C_n b_n = \dfrac{(B_n - 1)(C_n - 1)}{2}$.

Proof. Since

$$\begin{aligned} 2B_n\big(B_{n-1} + b_{n-1}\big) + 2C_n b_n &= 2B_n B_{n-1} + B_n(B_{n-1} - B_{n-2} - 1) + C_n(B_n - B_{n-1} - 1) \\ &= B_{n-1}(3B_n - C_n) - B_n B_{n-2} + B_n C_n - B_n - C_n \\ &= B_{n-1}^2 - B_n B_{n-2} + B_n C_n - B_n - C_n \\ &= 1 + B_n C_n - B_n - C_n \\ &= (B_n - 1)(C_n - 1), \end{aligned}$$

the theorem follows. □

Since the Diophantine equation $B_n x + C_n y = \frac{1}{2}(B_n - 1)(C_n - 1)$ has non-negative integer solution, we have the following theorem due to Chu's theorem [3, Theorem 1.1].

Theorem 2.5. *For* $n \geq 1$, *the Diophantine equation* $1 + B_n x + C_n y = \frac{(B_n-1)(C_n-1)}{2}$ *has no solution in non-negative integers.*

Since $(C_n, C_{n+1}) = 1$, we can also investigate the non-negative integer solutions of the following Diophantine equations:

$$C_n x + C_{n+1} y = \frac{(C_n - 1)(C_{n+1} - 1)}{2} \tag{11}$$

$$1 + C_n x + C_{n+1} y = \frac{(C_n - 1)(C_{n+1} - 1)}{2}. \tag{12}$$

The following table provides two cases:

n	C_n	C_{n+1}	**in which equation**	x	y
1	3	17	(12)	5	0
2	17	99	(11)	17	5
3	99	577	(12)	186	17
4	577	3363	(11)	594	186

The pattern in the table is summarized in the following theorem. The proof follows in a similar manner, so we omit the proof.

Theorem 2.6. *For $n \geq 1$, the following equalities are true*

1. $1 + C_{2n-1}\Bigg(B_{2n} - \sum_{i=0}^{n-1} C_{2i}\Bigg) + C_{2n}\Bigg(\sum_{i=1}^{n-1} C_{2i}\Bigg) = \frac{(C_{2n-1}-1)(C_{2n}-1)}{2}$

2. $C_{2n}\Bigg(\sum_{i=1}^{n} C_{2i}\Bigg) + C_{2n+1}\Bigg(B_{2n} - \sum_{i=0}^{n-1} C_{2i}\Bigg) = \frac{(C_{2n}-1)(C_{2n+1}-1)}{2}$.

3 Future work

For any non-negative integer n and k, one can easily verify the following:

- $(B_n, C_{nk}) = 1$
- $(B_{4n-1} + 2, B_{4n} + 12) = 1$
- $(B_{4n} + 2, B_{4n+1} + 12) = 1$
- $(B_{4n} - 2, B_{4n+1} - 12) = 1$
- $(B_{4n+1} - 2, B_{4n+2} - 12) = 1$
- $(B_{4n+2} - 2, B_{4n+3} - 12) = 1$
- $(B_{4n+2} + 2, B_{4n+3} + 12) = 1$.

One can use the above results or can generate similar results by considering suitable integers a and b such that $(B_n - a, B_{n+k} - b) = 1$ and solve the Diophantine equations:

$$(B_n - a)x + (B_{n+k} - b)y = \frac{1}{2}(B_n - a - 1)(B_{n+k} - b - 1)$$

and

$$1 + (B_n - a)x + (B_{n+k} - b)y = \frac{1}{2}(B_n - a - 1)(B_{n+k} - b - 1).$$

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