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Explicit relations on the degenerate type 2-unified Apostol–Bernoulli, Euler and Genocchi polynomials and numbers

Burak Kurt

Department of Mathematics and Science Education, Faculty of Education,
Akdeniz University, Antalya TR-07058, Turkey
e-mail: burakkurt@akdeniz.edu.tr

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Abstract: The main aim of this paper is to introduce and investigate the degenerate type 2-unified Apostol–Bernoulli, Euler and Genocchi polynomials by using monomiality principle and operational methods. We give explicit relations and some identities for the degenerate type 2-unified Apostol–Bernoulli, Euler and Genocchi polynomials.
Keywords: Type 2-Bernoulli, Euler and Genocchi polynomials, Degenerate Bernoulli, Euler and Genocchi polynomials, Unified degenerate Apostol–Bernoulli, Euler and Genocchi polynomials, Monomiality principle, Multiplicative operators.
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1 Introduction

The degenerate versions of special polynomials and numbers initiated by Carlitz [2, 3] have regained the attention of some mathematics by replacing the usual exponential function in the generating function of special polynomials with the degenerate exponential functions. The study of special polynomials provide many useful identities, their relations and representations associated with special numbers and polynomials. Belingeri et al. [1] and Dattoli et al. [4, 5] considered monomiality principle for the Appell polynomials. Using monomiality principle and operational methods. Khan et al. [11–13] introduced and investigated the 2-variable Apostol-type polynomials and degenerate Apostol-type Bernoulli, Euler and Genocchi polynomials.

Luo [5], Srivastava [20] and Luo and Srivastava [16] introduced and investigated Apostol–Bernoulli, Euler and Genocchi polynomials and gave some explicit relations and identities for these polynomials. The type 2-degenerate Apostol–Bernoulli, Euler and Genocchi polynomials have been introduced in [6–10, 12] and the coauthors of these articles gave recurrence relations and identities and symmetry properties for these polynomials. Motivated by Khan et al. [11], we define and investigate the type 2-unified degenerate Apostol–Bernoulli, Euler and Genocchi polynomials.

We use the usual notations; $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ for the set of natural integers, integers, real numbers and complex numbers, respectively. Also, we let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^- := \{-1, -2, -3, \cdots\}$.

We begin by introducing the following definition and notation (see also [1–22]). The classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$ are defined by the following generating functions, respectively,

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t - 1}e^{xt}, |t| < 2\pi;$$
$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t + 1}e^{xt},\ |t| < \pi;$$
$$\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} = \frac{2t}{e^t + 1}e^{xt}, |t| < \pi.$$

When $x = 0$, we get the Bernoulli numbers $B_n(0) := B_n$, the Euler numbers $E_n(0) := E_n$ and the Genocchi numbers $G_n(0) := G_n$, respectively.

Kim and Lee [8], Kim D. S. and three coauthors [7] and Kim and two coauthors [6] defined the type 2-Bernoulli polynomials $B_n^*(x)$ and the type 2-Euler polynomials $E_n^*(x)$ as

$$\sum_{n=0}^{\infty} B_n^*(x)\frac{t^n}{n!} = \frac{t}{e^t - e^{-t}}e^{xt},\ |t| < \pi \tag{1}$$

and

$$\sum_{n=0}^{\infty} E_n^*(x)\frac{t^n}{n!} = \frac{2}{e^t + e^{-t}}e^{xt},\ |t| < \frac{\pi}{2}, \tag{2}$$

respectively.

Ryoo [19] defined the type 2-Genocchi polynomials as

$$\sum_{n=0}^{\infty} G_n^*(x)\frac{t^n}{n!} = \frac{2t}{e^t + e^{-t}}e^{xt},\ |t| < \frac{\pi}{2}. \tag{3}$$

We define the type 2-Apostol–Bernoulli polynomials $B_{n,\gamma}^{*(\alpha)}(x)$ of order $\alpha \in \mathbb{N}$, the type 2-Apostol-Euler polynomials $E_{n,\gamma}^{*(\alpha)}(x)$ of order $\alpha \in \mathbb{N}$ and the type 2-Apostol–Genocchi polynomials $G_{n,\gamma}^{*(\alpha)}(x)$ of order $\alpha \in \mathbb{N}$ by the following generating functions as follows

$$\sum_{n=0}^{\infty} B_{n,\gamma}^{*(\alpha)}(x)\frac{t^n}{n!}=\left(\frac{t}{\gamma e^t-e^{-t}}\right)^{\alpha}e^{xt}, \tag{4}$$

$$\left(|t|<\pi \text{ when } \gamma=1;\ |t|<|\log\gamma| \text{ when } \gamma\neq 1\right),$$

$$\sum_{n=0}^{\infty} E_{n,\gamma}^{*(\alpha)}(x)\frac{t^n}{n!}=\left(\frac{2}{\gamma e^t+e^{-t}}\right)^{\alpha}e^{xt}, \tag{5}$$

$$\left(|t|<\frac{\pi}{2} \text{ when } \gamma=1;\ |t|<|\log(-\gamma)| \text{ when } \gamma\neq 1\right)$$

and

$$\sum_{n=0}^{\infty} G_{n,\gamma}^{*(\alpha)}(x)\frac{t^n}{n!}=\left(\frac{2t}{\gamma e^t+e^{-t}}\right)^{\alpha}e^{xt}, \tag{6}$$

$$\left(|t|<\frac{\pi}{2} \text{ when } \gamma=1;\ |t|<|\log(-\gamma)| \text{ when } \gamma\neq 1\right).$$

Carlitz [2,3] defined the degenerate Bernoulli polynomials $\mathcal{B}_n(x\mid\lambda)$ and the degenerate Euler polynomials $\mathcal{E}_n(x\mid\lambda)$ by means of the following generating functions, respectively

$$\sum_{n=0}^{\infty}\mathcal{B}_n(x\mid\lambda)\frac{t^n}{n!}=\frac{t}{(1+\lambda t)^{1/\lambda}-1}(1+\lambda t)^{x/\lambda} \tag{7}$$

and

$$\sum_{n=0}^{\infty}\mathcal{E}_n(x\mid\lambda)\frac{t^n}{n!}=\frac{2}{(1+\lambda t)^{1/\lambda}+1}(1+\lambda t)^{x/\lambda}. \tag{8}$$

The degenerate Genocchi polynomials $\mathcal{G}_n(x\mid\lambda)$ in [14] are defined as follows

$$\sum_{n=0}^{\infty}\mathcal{G}_n(x\mid\lambda)\frac{t^n}{n!}=\frac{2t}{(1+\lambda t)^{1/\lambda}+1}(1+\lambda t)^{x/\lambda}, \tag{9}$$

which in the special case when $x=0$ in (7), (8) and (9) reduce to the generating function of the degenerate Bernoulli numbers $\mathcal{B}_n(\lambda)$, the degenerate Euler numbers $\mathcal{E}_n(\lambda)$ and the degenerate Genocchi numbers $\mathcal{G}_n(\lambda)$, respectively.

We define the degenerate type 2-Apostol Bernoulli polynomials $B_{n,\gamma}^{*(\alpha)}(x\mid\lambda)$ of order $\alpha\in\mathbb{N}$, the degenerate type 2-Apostol Euler polynomials $E_{n,\gamma}^{*(\alpha)}(x\mid\lambda)$ of order $\alpha\in\mathbb{N}$ and the degenerate type 2-Apostol Genocchi polynomials $G_{n,\gamma}^{*(\alpha)}(x\mid\lambda)$ of order $\alpha\in\mathbb{N}$ by means of the following generating functions as follows

$$\sum_{n=0}^{\infty} B_{n,\gamma}^{*(\alpha)}(x\mid\lambda)\frac{t^n}{n!}=\left(\frac{t}{\gamma(1+\lambda t)^{1/\lambda}-(1+\lambda t)^{-1/\lambda}}\right)^{\alpha}(1+\lambda t)^{x/\lambda}, \tag{10}$$

$$\sum_{n=0}^{\infty} E_{n,\gamma}^{*(\alpha)}(x\mid\lambda)\frac{t^n}{n!}=\left(\frac{2}{\gamma(1+\lambda t)^{1/\lambda}+(1+\lambda t)^{-1/\lambda}}\right)^{\alpha}(1+\lambda t)^{x/\lambda} \tag{11}$$

and

$$\sum_{n=0}^{\infty} G_{n,\gamma}^{*(\alpha)}(x\mid\lambda)\frac{t^n}{n!}=\left(\frac{2t}{\gamma(1+\lambda t)^{1/\lambda}+(1+\lambda t)^{-1/\lambda}}\right)^{\alpha}(1+\lambda t)^{x/\lambda}, \tag{12}$$

respectively.

We define the generalized degenerate type 2-unified Apostol–Bernoulli, Euler and Genocchi polynomials $\mathcal{R}_{n,\beta}^{*(\alpha)}(x;\lambda,k,a,b)$ of order $\alpha\in\mathbb{N}$ by means of the following generating functions

$$\sum_{n=0}^{\infty}\mathcal{R}_{n,\beta}^{*(\alpha)}(x\mid\lambda;k,a,b)\frac{t^n}{n!} = \left(\frac{2^{1-k}t^k}{\beta^b(1+\lambda t)^{1/\lambda}-a^b(1+\lambda t)^{-1/\lambda}}\right)^{\alpha}(1+\lambda t)^{x/\lambda}$$
$$(\beta\in\mathbb{C},\ \alpha,k\in\mathbb{N},\ a,b\in\mathbb{R}\text{ and }0\neq\lambda\in\mathbb{R}). \tag{13}$$

For $x=0$, we have the degenerate type 2-unified Apostol–Bernoulli, Euler and Genocchi numbers $\mathcal{R}_{n,\beta}^{*(\alpha)}(0\mid\lambda;k,a,b)=\mathcal{R}_{n,\beta}^{*(\alpha)}(\lambda;k,a,b)$.

Corollary 1.1. *Setting $k=a=b=1$ and $\beta=\gamma$ in (13), we get*

$$\mathcal{R}_{n,\gamma}^{*(\alpha)}(x\mid\lambda;1,1,1)=\mathrm{B}_{n,\gamma}^{*(\alpha)}(x\mid\lambda).$$

Corollary 1.2. *Choosing $k=0$, $b=-a=1$ and $\beta=\gamma$ in (13), we get*

$$\mathcal{R}_{n,\gamma}^{*(\alpha)}(x\mid\lambda;0,-1,1)=\mathrm{E}_{n,\gamma}^{*(\alpha)}(x\mid\lambda).$$

Corollary 1.3. *Letting $k=-2a=b=1$ and $2\beta=\gamma$ in (13), we get*

$$\mathcal{R}_{n,\frac{\gamma}{2}}^{*(\alpha)}\left(x\mid\lambda;1,-\frac{1}{2},1\right)=\mathrm{G}_{n,\gamma}^{*(\alpha)}(x\mid\lambda).$$

From the Binomials theorems, we have

$$(1+\lambda t)^{x/\lambda}=\sum_{n=0}^{\infty}(x\mid\lambda)_n\frac{t^n}{n!}, \tag{14}$$

where $(x\mid\lambda)_n=x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$, $(x\mid\lambda)_0=1$.

The notation of quasi-monomiality was introduced and studied Dattoli [4], Dattoli et al. [5] and Belingeri et al. [1], in details S. Khan et al. in [11–13].

According to monomiality principle, a polynomial set $\{P_n(x)\}_{n\in\mathbb{N}}$ is “quasi-monomial” provided there exist two operators $\widehat{M}$ and $\widehat{P}$ playing, respectively the role of multiplicative and derivative operators for the given set of polynomials. These operator satisfy the following identities for all $n\in\mathbb{N}$.

$$\widehat{M}\{P_n(x)\}=P_{n+1}(x), \tag{15}$$

$$\widehat{P}\{P_n(x)\}=nP_{n-1}(x), \tag{16}$$

$$\widehat{M}\,\widehat{P}\{P_n(x)\}=n\,P_n(x) \tag{17}$$

and

$$P_n(x)=\widehat{M}^{(n)}\{P_0(x)\}=\widehat{M}\{1\}. \tag{18}$$

2 Main theorems and related relations

In this section, we consider the generalized type 2-unified degenerate Apostol–Bernoulli, Euler and Genocchi polynomials. By applying the monomiality principle and operator methods, we give an explicit relations and identities for these polynomials.

Theorem 2.1. *The following relation holds true:*

$$\mathcal{R}_{n,\beta}^{*(\alpha)}(x \mid \lambda; k, a, b) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{R}_{n-k,\beta}^{*(\alpha)}(\lambda; k, a, b) \ (x \mid \lambda)_k . \tag{19}$$

Proof. From (12) and (13), we find

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{R}_{n,\beta}^{*(\alpha)}(x \mid \lambda; k, a, b) \frac{t^n}{n!} &= \left(\frac{2^{1-k} t^k}{\beta^b (1+\lambda t)^{1/\lambda} - a^b (1+\lambda t)^{-1/\lambda}} \right)^{\alpha} (1+\lambda t)^{x/\lambda} \\ &= \sum_{m=0}^{\infty} \mathcal{R}_{m,\beta}^{*(\alpha)}(\lambda; k, a, b) \frac{t^m}{m!} \sum_{p=0}^{\infty} (x \mid \lambda)_p \frac{t^p}{p!}.\end{aligned}$$

By using Cauchy product and comparing the coefficient of the both sides, we get (19). □

Theorem 2.2. *The following relation holds true:*

$$\mathcal{R}_{n,\beta}^{*(\alpha)}(x+y \mid \lambda; k, a, b) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{R}_{n-k,\beta}^{*(\alpha)}(x \mid \lambda; k, a, b) \ \mathcal{R}_{k,\beta}^{*(\alpha)}(y \mid \lambda; k, a, b) . \tag{20}$$

The proof of this theorem is similar to (19). We omit it.

Theorem 2.3. *The generalized degenerate type* 2*-unified Apostol–Bernoulli, Euler and Genocchi polynomials* $\mathcal{R}_{n,\beta}^{*(\alpha)}(x \mid \lambda; k, a, b)$ *are quasi-monimial with respect to the following multiplicative and derivative operators*

$$\widehat{M}\, \mathcal{R}_{n,\beta}^{*(\alpha)} = \frac{x}{e^{\lambda D_x}} + \frac{\alpha \lambda k}{e^{\lambda D_x} - 1} - \frac{\alpha e^{-\lambda D_x} \left(\beta^b e^{D_x} + a^b e^{-D_x} \right)}{\beta^b e^{D_x} - a^b e^{-D_x}} \tag{21}$$

and

$$\widehat{P}\, \mathcal{R}_{n,\beta}^{*(\alpha)} = \frac{e^{\lambda D_x} - 1}{\lambda}. \tag{22}$$

Proof. Differentiating the generating function (13) partially with respect to t

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{R}_{n+1,\beta}^{*(\alpha)}(x \mid \lambda; k, a, b) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \left\{ \frac{x}{1+\lambda t} + \frac{\alpha k}{t} - \alpha \frac{\beta^b (1+\lambda t)^{\frac{1-\lambda}{\lambda}} + a^b (1+\lambda t)^{-\frac{1-\lambda}{\lambda}}}{\beta^b (1+\lambda t)^{\frac{1}{\lambda}} - a^b (1+\lambda t)^{-\frac{1}{\lambda}}} \right\} \\ &\quad \times \mathcal{R}_{n,\beta}^{*(\alpha)}(x \mid \lambda; k, a, b) \frac{t^n}{n!}.\end{aligned} \tag{23}$$

By using the following identities in (23)

$$t \left\{ (1+\lambda t)^{\frac{x}{\lambda}} \right\} = \frac{e^{\lambda D_x} - 1}{\lambda} \left\{ (1+\lambda t)^{\frac{x}{\lambda}} \right\}, \tag{24}$$

we get

$$\sum_{n=0}^{\infty}\mathcal{R}_{n+1,\beta}^{*(\alpha)}(x\mid\lambda;k,a,b)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left\{\frac{x}{e^{\lambda D_x}}+\frac{\alpha\lambda k}{e^{\lambda D_x}-1}-\frac{\alpha e^{-\lambda D_x}\left(\beta^b e^{D_x}+a^b e^{-D_x}\right)}{\beta^b e^{D_x}-a^b e^{-D_x}}\right\} \times\mathcal{R}_{n,\beta}^{*(\alpha)}(x\mid\lambda;k,a,b)\frac{t^n}{n!}. \tag{25}$$

Comparing the coefficients of both sides of equation (25), we have (21).

Using generating function (13) after some simplification, we have (22). □

Theorem 2.4. *The generalized degenerate type* 2*-unified Apostol–Bernoulli, Euler and Genocchi polynomials* $\mathcal{R}_{n,\beta}^{*(\alpha)}(x\mid\lambda;k,a,b)$ *satisfy the following differential equation*

$$\left\{\alpha k+\frac{x\left(e^{\lambda D_x}-1\right)}{\lambda e^{\lambda D_x}}-\frac{\alpha e^{-\lambda D_x}\left(\beta^b e^{D_x}+a^b e^{-D_x}\right)}{\beta^b e^{D_x}-a^b e^{-D_x}}\left(\frac{e^{\lambda D_x}-1}{\lambda}\right)-n\right\}\mathcal{R}_{n,\beta}^{*(\alpha)}(x\mid\lambda;k,a,b)=0. \tag{26}$$

Proof. Using operators (21) and (22) and in view of the monomiality principle $\widehat{M}\ \widehat{P}\left\{P_n(x)\right\}=nP_n(x)$, we get (26). □

Theorem 2.5. *The following relation holds true*

$$\begin{aligned}&\mathcal{R}_{n,\beta}^{*(\alpha)}(x\mid\lambda;k,a,b)\\&=\sum_{l=0}^{n+1}\binom{n+1}{l}\mathcal{R}_{n+1-l,\beta}^{*(\alpha)}(x\mid\lambda;k,a,b)\left\{\frac{\beta^b}{a^b}\mathcal{B}_{l,\frac{\beta^b}{a^b}}^{*}(x+1\mid\lambda)-\mathcal{B}_{l,\frac{\beta^b}{a^b}}^{*}(x-1\mid\lambda)\right\}.\end{aligned} \tag{27}$$

Proof. From (13) and (10), we can write

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{R}_{n,\beta}^{*(\alpha)}(x\mid\lambda;k,a,b)\frac{t^n}{n!}&=\frac{1}{t}\left(\frac{2^{(1-k)}t^k}{\beta^b(1+\lambda t)^{\frac{1}{\lambda}}-a^b(1+\lambda t)^{-\frac{1}{\lambda}}}\right)^{\alpha}\\&\quad\times\left\{\frac{\beta^b}{a^b}\frac{t\ (1+\lambda t)^{\frac{x+1}{\lambda}}}{\frac{\beta^b}{a^b}(1+\lambda t)^{\frac{1}{\lambda}}-(1+\lambda t)^{-\frac{1}{\lambda}}}-\frac{t\ (1+\lambda t)^{\frac{x-1}{\lambda}}}{\frac{\beta^b}{a^b}(1+\lambda t)^{\frac{1}{\lambda}}-(1+\lambda t)^{-\frac{1}{\lambda}}}\right\}\\&=\frac{1}{t}\sum_{n=0}^{\infty}\mathcal{R}_{n,\beta}^{*(\alpha)}(\lambda;k,a,b)\frac{t^n}{n!}\left\{\frac{\beta^b}{a^b}\sum_{n=0}^{\infty}\mathcal{B}_{n,\frac{\beta^b}{a^b}}^{*}(x+1\mid\lambda)\frac{t^n}{n!}\right.\\&\quad\left.-\sum_{n=0}^{\infty}\mathcal{B}_{n,\frac{\beta^b}{a^b}}^{*}(x-1\mid\lambda)\frac{t^n}{n!}\right\}.\end{aligned}$$

By using Cauchy product and comparing the coefficients of both sides, we have (27). □

Theorem 2.6. *The following relation holds true:*

$$\mathcal{R}_{n,\beta}^{*(\alpha)}(x\mid\lambda;k,a,b) = \frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}\mathcal{R}_{n-l,\beta}^{*(\alpha)}(\lambda;k,a,b) \times\left\{\frac{\beta^b}{a^b}\,\mathcal{E}_{l,\frac{\beta^b}{a^b}}^{*}(x+1\mid\lambda)+\mathcal{E}_{l,\frac{\beta^b}{a^b}}^{*}(x-1\mid\lambda)\right\}. \tag{28}$$

Proof. From (13) and (11), we find

$$\sum_{n=0}^{\infty} \mathcal{R}_{n,\beta}^{*(\alpha)}(x \mid \lambda; k, a, b)\frac{t^n}{n!} = \frac{1}{2}\left(\frac{2^{1-k}\, t^k}{\beta^b (1+\lambda t)^{\frac{1}{\lambda}} - a^b (1+\lambda t)^{-\frac{1}{\lambda}}}\right)^{\alpha}$$

$$\left(\frac{\beta^b}{a^b}\frac{2\,(1+\lambda t)^{\frac{x+1}{\lambda}}}{\frac{\beta^b}{a^b}(1+\lambda t)^{\frac{1}{\lambda}} + (1+\lambda t)^{-\frac{1}{\lambda}}} + \frac{2\,(1+\lambda t)^{\frac{x-1}{\lambda}}}{\frac{\beta^b}{a^b}(1+\lambda t)^{\frac{1}{\lambda}} + (1+\lambda t)^{-\frac{1}{\lambda}}}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\mathcal{R}_{n,\beta}^{*(\alpha)}(\lambda; k, a, b)\frac{t^n}{n!}\left\{\frac{\beta^b}{a^b}\sum_{n=0}^{\infty}\mathcal{E}^{*}_{n,\frac{\beta^b}{a^b}}(x+1 \mid \lambda)\frac{t^n}{n!}\right.$$

$$\left. + \sum_{n=0}^{\infty}\mathcal{E}^{*}_{n,\frac{\beta^b}{a^b}}(x-1 \mid \lambda)\frac{t^n}{n!}\right\}.$$

Using Cauchy product and comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we get (28). □

Theorem 2.7. *The following relation holds true*

$$\mathcal{R}_{n,\beta}^{*(\alpha)}(x \mid \lambda; k, a, b) = \frac{1}{2}\sum_{l=0}^{n+1}\binom{n+1}{l}\mathcal{R}_{n+1-l,\beta}^{*(\alpha)}(\lambda; k, a, b) \times\left\{\frac{\beta^b}{a^b}\mathcal{G}^{*}_{l,\frac{\beta^b}{a^b}}(x+1 \mid \lambda) + \mathcal{G}^{*}_{l,\frac{\beta^b}{a^b}}(x-1 \mid \lambda)\right\}. \quad (29)$$

3 Conclusion

We introduce and investigate the generalized degenerate type 2-unified Apostol–Bernoulli, Euler and Genocchi polynomials and give explicit relations by using the monomiality principle. Also, by using to monomiality principle and operators methods to the generalized the degenerate type 2-unified Apostol–Bernoulli, Euler and Genocchi polynomials, we have shown give some relations and identities for these polynomials.

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