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On certain inequalities for the prime counting function – Part III

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Abstract: As a continuation of [10] and [11], we offer some new inequalities for the prime counting function $\pi(x)$. Particularly, a multiplicative analogue of the Hardy–Littlewood conjecture is provided. Improvements of the converse of Landau's inequality are given. Some results on $\pi(p_n^2)$ are offered, p_n denoting the n-th prime number. Results on $\pi(\pi(x))$ are also considered.
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1 Introduction

Let $\pi(x)$ denote the number of primes $\leq x$, where $x \geq 1$ is a positive integer. In Parts I and II [10, 11] we have proved some inequalities of a new type for $\pi(x)$.

For example, in [10] we established the following counterpart of the Hardy–Littlewood conjecture:

$$\pi(x+y) \geq \frac{2}{3} \cdot [\pi(x) + \pi(y)] \quad (x, y \geq 2) \tag{1}$$

Another inequality from [10] is the following (see relation (15))

$$(x+y)\pi(x+y) \le 2[x\pi(x) + y\pi(y)]. \tag{2}$$

In [11] we proved that

$$\sqrt{\pi(x+y)} < \sqrt{\pi(x)} + \sqrt{\pi(y)} \tag{3}$$

and

$$\sqrt{3\pi(x+y)} \ge \sqrt{\pi(x)} + \sqrt{\pi(y)} \tag{4}$$

where $x, y \ge 2$, and that

$$\sqrt{2\pi(x+y)} \ge \sqrt{\pi(x)} + \sqrt{\pi(y)} \tag{5}$$

for infinitely many (x, y), and

$$\sqrt{2\pi(x+y)} \le \sqrt{\pi(x)} + \sqrt{\pi(y)} \tag{6}$$

for infinitely many (x, y).

Among the inequalities from [11] we mention also:

$$(x+y)\sqrt{\pi(x+y)} \le x\sqrt{2\pi(x)} + y\sqrt{2\pi(y)} \tag{7}$$

for all $x, y \ge 2$; and

$$\sqrt{x+y} \cdot \pi(x+y) \le \sqrt{2x} \cdot \pi(x) + \sqrt{2y} \cdot \pi(y) \tag{8}$$

for any $2 \le y \le x$ with the exception of $(x, y) = (4, 3); (10, 9)$.

In this paper, will improve relation (1). This will give also improvements of Landau's converse inequality, considered in [10] and [2]. We will consider also the iteration function $\pi(\pi(x))$, as well as the sequence $\pi(p_n^2)$, where p_n denotes the n^{th} prime number.

2 Main results

The following auxiliary results will be used:

Lemma 1. *For $x, y \ge 67$ one has*

$$\frac{x}{\log x - \frac{1}{2}} < \pi(x) < \frac{x}{\log x - 1.12}. \tag{9}$$

For references to this results, see [10, 11].

Lemma 2. *For $x \ge 5393$ one has*

$$\pi(x) \ge \frac{x}{\log x - 1}. \tag{10}$$

The author [9] proved this inequality in 2006, based on earlier results by P. Dusart [3] and L. Panaitopol [5]. The first result contains the following improvement of (1):

Theorem 1. *One has, for any* $2 \le y \le x$ *the inequality*

$$\pi(x+y) \ge \frac{3}{4} \cdot [\pi(x) + \pi(y)], \tag{11}$$

with the exceptions of $(x,y) = (7,3); (5,5); (7,5); (23,13); (19,17)$.

There is equality in (11) for $(x,y) = (3,3); (13,3); (11,5); (23,5); (7,7); (8,7); (9,7); (19,7); (20,7); (21,7); (8,8); (19,8); (20,8); (19,9); (17,11); (13,13); (14,13); (15,13); (14,14); (23,17); (19,19); (20,19); (21,19); (20,20)$.

Corollary 1. *For any* $x \ne 5$, $x \ge 2$ *one has*

$$\pi(2x) \ge \frac{3}{2}\pi(x), \tag{12}$$

with equality only for $x = 3, 7, 8, 13, 14, 19, 20$.

Proof. Let

$$f(x) = \frac{x}{\log x - 1.12}.$$

A simple computation (which we omit here) gives the second derivative of this function:

$$x \cdot (\log x - 1.12)^2 \cdot f''(x) = -\log x + 3.12 < 0$$

if $\log x > 3.12$, i.e., $x > e^{3.12} = 22.64\ldots$. Thus the function f is concave, which gives

$$f(x) + f(y) \le 2f\left(\frac{x+y}{2}\right) \text{ for any } x, y \ge 23. \tag{13}$$

Now, using the right side of (9), and by (13) we get

$$\pi(x) + \pi(y) < f(x) + f(y) \le \frac{x+y}{\log\left(\frac{x+y}{2}\right) - 1.12}. \tag{14}$$

On the other hand, by the left side of (9) we get

$$\frac{4}{3}\pi(x+y) > \frac{4}{3} \cdot \frac{(x+y)}{\log(x+y) - \frac{1}{2}}.$$

Now we have to considered the inequality

$$\frac{(x+y)}{\log\frac{(x+y)}{2} - 1.12} < \frac{4}{3} \cdot \frac{(x+y)}{\log(x+y) - \frac{1}{2}} \tag{15}$$

which can be written , after elementary computations as

$$\log(x+y) > 5.75\ldots; \; i.e., \;\; x+y > e^{5.75\ldots} \approx 317.34\ldots$$

Therefore, inequality (11) is true for $x, y \ge 67$ and $x + y \ge 318$. A computer verification shows that (11) is true also (with strict inequality) for $67 \le y \le x$ and $x + y \le 317$. Therefore (11) is valid with strict inequality for any $x, y \ge 67$.

Let now consider $x \geq y$ and $y \leq 66$. Then $\pi(y) \leq 18$ and $\frac{3}{4} \cdot [\pi(x) + \pi(y)] \leq \frac{3}{4} \cdot [\pi(x) + 18]$. Since $\pi(x) \leq \pi(x+y)$, it will be sufficient to consider the inequality $\frac{3}{4} \cdot [\pi(x) + 18] \leq \pi(x)$, or $\pi(x) \geq 54$. This is true if $x \geq 257$.

Now, for

$$2 \leq y \leq x \leq 256, \ \ y \leq 66 \tag{16}$$

a computer verification shows the exceptions listed in Theorem 1, as well the equality cases. □

Remark 1. *By letting $x = p_n$, the n-th prime number, we get that for $n \neq 3$ one has*

$$\pi(2p_n) \geq \frac{3}{2} \cdot n. \tag{17}$$

Particularly, as $\frac{3}{2} > \sqrt{2}$, we get that for $n \neq 3$

$$\pi(2p_n) > \sqrt{2} \cdot n, \tag{18}$$

which was an open problem stated in [4].

The following result gives multiplicative analogues of the Hardy–Littlewood conjecture.

Theorem 2. *For any $x, y \geq 3$ one has*

$$\pi(x+y) \leq \pi(x) \cdot \pi(y), \tag{19}$$

with equality only for $(x, y) = (4, 3)$ for $y \leq x$.

One has

$$\pi(x+y) \leq \frac{2}{3}\pi(x) \cdot \pi(y) \tag{20}$$

with the exceptions of $(x, y) = (3, 3); (4, 3); (4, 4)$; when $y \leq x$. There is equality in (20) *only for* $(x, y) = (5, 3); (6, 3)$ $(y \leq x)$.

Proof. In [2] the following inequality is proved (see Theorem 6, left side):

$$\frac{1}{2} \leq \frac{\pi(x)^{x/(x+y)} \cdot \pi(y)^{y/(x+y)}}{\pi(x+y)}. \tag{21}$$

Now, (21) can be written as

$$\pi(x+y) \leq 2 \cdot \pi(x)^{x/(x+y)} \cdot \pi(y)^{y/(x+y)}. \tag{22}$$

In order to prove (19), it is sufficient to show that

$$\pi(x)^y \cdot \pi(y)^x \geq 2^{x+y}. \tag{23}$$

Clearly, (23) is true, if $\pi(x) \geq 2$, $\pi(y) \geq 2$; i.e., when $x, y \geq 3$. As $\pi(x) = 2$ only for $x \in \{3, 4\}$, simple considerations show the cases of equality in (19).

Now, inequality (20) is true, if we can show that

$$\pi(x)^y \cdot \pi(y)^x \geq 3^{x+y}. \tag{24}$$

This is valid, if $\pi(x) \geq 3$ and $\pi(y) \geq 3$; i.e., when $x, y \geq 7$. As we supposed $x, y \geq 3$; the cases of exceptions can be verified, and also the cases of equality can be verified, and also the cases of equality can be easily shown. □

Remark 2. *As $x+y \leq xy$, or equivalently $(x-1)(y-1) \geq 1$, valid for $x \geq 2, y \geq 2$; we can write $\pi(x+y) \leq \pi(xy)$. In* [6] *L. Panaitopol proved that*

$$\pi(x) \cdot \pi(y) \leq \pi(xy) \tag{25}$$

with the exceptions of $(x, y) = (7, 5)$ and $(7, 7)$ for $2 \leq y \leq x$. Thus, by (25) *and* (19) *we have:*

$$\pi(x+y) \leq \pi(x) \cdot \pi(y) \leq \pi(xy) \tag{26}$$

with the above mentioned exceptions.

Theorem 3. *If $2 \leq y \leq x$, then*

$$\pi(x+y) \leq \frac{x}{y} \cdot \pi(x) + \pi(y); \tag{27}$$

$$\pi(x+y) \leq 2\sqrt{\pi(y) \cdot \pi(x)^{x/y}} \leq \pi(y) + \pi(x)^{x/y}. \tag{27'}$$

Proof. By relation (2) we get

$$\pi(x+y) \leq \frac{2x}{x+y} \cdot \pi(x) + \frac{2y}{x+y} \cdot \pi(y) \leq \frac{x}{y}\pi(x) + \pi(y),$$

as for $y \leq x$ one has $\frac{2y}{x+y} \leq 1$ and $\frac{2x}{x+y} \leq \frac{x}{y}$. Inequality (27) follows.

By inequality (21) one has, for $2 \leq y \leq x$, by $\frac{x}{x+y} \leq \frac{1}{2} \cdot \frac{x}{y}$ and $\frac{y}{x+y} \leq \frac{1}{2}$ that $\pi(x+y) \leq 2\pi(x)^{x/2y} \cdot \pi(y)^{1/2}$, so the first inequlity of (27') follows.

The second one is the consequence of $2\sqrt{ab} \leq a+b$ for $a = \pi(y)$, $b = \pi(x)^{x/y}$. □

Remark 3. (27) *and* (27') *are extensions of Landau's inequality*

$$\pi(2x) \leq 2\pi(x), \tag{28}$$

as for $y = x$ from (27) *and* (27') *we get* (28).

In [1] *is proved a refinement of* (28)*:*

$$2\pi(x) - \pi(2x) \geq 2\omega(x), \tag{29}$$

for $x \geq 71$, where $\omega(x)$ denotes the number of distinct prime factors of x. As $\omega(x) \geq 1$, clearly (29) *is an improvement of* (28)*. Now, as inequality* (12) *of Corollary 1 can be rewritten as $2\pi(x) - \pi(2x) \leq \pi(2x) - \pi(x)$, by* (29) *we get*

$$\pi(2x) - \pi(x) \geq 2\pi(x) - \pi(2x) \geq 2\omega(x),\ x \geq 71. \tag{30}$$

Particularly, (30) *shows the following nice improvement of Bertrand's postulate (which states that between x and $2x$ there exists at least a prime (see* [7]*):*

Proposition 1. *For $x \geq 71$, between x and $2x$ there are at least $2\omega(x)$ primes.*

It is known that (see [12]*) for any $k, x \geq 3$,*

$$\pi(kx) < k\pi(x). \tag{31}$$

This easily implies that

$$\pi(3x) \le 3\pi(x), \quad x \ge 2 \tag{32}$$

with equality only for $x = 2$.

Now, concerning the iteration of $\pi(x)$, *from* (31) *we get* $\pi(\pi(kx)) \le \pi(k\pi(x)) \le k\pi(x)$. *For the particular cases of* $k = 2$ *and* $k = 3$ *one has a more precise result:*

Theorem 4. *For any* $x \ge 3$ *one has*

$$\frac{5}{4} \le \frac{\pi(\pi(2x))}{\pi(\pi(x))} \le 2, \tag{33}$$

in the left side with the exception of $x = 5$. *There is equality in the right side of* (33) *for* $x \in \{3, 4, 9, 10\}$; *while in the left side for* $x \in \{17, 18, 19, 20\}$

$$\frac{3}{2} \le \frac{\pi(\pi(3x))}{\pi(\pi(x))} \le 3, \tag{34}$$

with equalities in the right side of (34) *for* $x = 4$, *while in the left side for* $x \in \{17, 18, 19\}$.

Proof. The right sides of (33) and (34) are consequences of (28) and (32), by remarking, that in (28) there is strict inequality for $x > 10$. Thus the equality in the right side of (33) should be considered only for $\pi(x) \le 10$, and an easy verification gives the cases of equalities. A similar argument shows that in the right side of (34) there is equality only for $\pi(x) = 2$, and the result follows.

Now, for the left side of (33) we first prove that

$$\frac{\pi(2x)}{\pi(x)} > \frac{9}{5} \text{ for } x \ge 4628. \tag{35}$$

Indeed, using Lemma 2 for $2x \ge 5393$ (i.e., $x \ge 2697$) and the right side of Lemma 1, we can write

$$\frac{\pi(2x)}{\pi(x)} > \frac{2x}{\log 2x - 1} \cdot \frac{\log x - 1.12}{x} \ge \frac{9}{5}$$

iff $10(\ln x - 1.12) > 9(\log 2x - 1)$, i.e., $\log x > 8.438\ldots$, which is true for $x \ge e^{8.44} = 4628.\ldots$

Now, we will show that

$$\frac{9}{5} > \frac{5}{4} \cdot \left(\frac{\log \pi(2x) - 1}{\log \pi(x) - 1.12}\right), \tag{36}$$

or equivalently $36 \log \pi(x) - 40.32 > 25 \log \pi(2x) - 25$, or $36 \log \pi(x) - 25 \log \pi(2x) > 15.32$. Now, by (28) one has $\log \pi(2x) < \log 2 + \log \pi(x)$, so $25 \log \pi(2x) < 25 \log 2 + 25 \log \pi(x)$, and therefore $36 \log \pi(x) - 25 \log \pi(2x) > 11 \log \pi(x) - 25 \log 2$ and $11 \log \pi(x) - 25 \log 2 > 15.32$ for $11 \log \pi(x) > 15.32 + 25 \log 2 \approx 15.32 + 17.25 = 32.57$; i.e., $\log \pi(x) > 2.96\ldots$. This is valid for $x \ge 73$.

Now, having in mind the validity of (35) for $x \ge 4628$, a computer verification for $3 \le x \le 4627$ shows that the left side is true excepting $x = 5$, and with equalitiess only for $x = 3, 4, 9, 10$.

The proof of left side of (34) could be done in a similar manner, but here we can obtain a more direct argument.

Namely, remark that by (25) one has

$$\pi(3x) \geq 2\pi(x). \tag{37}$$

Relation (37) implies $\pi(\pi(3x)) \geq \pi(2\pi(x))$. Now, by Corollary 1 one has $\pi(2\pi(x)) \geq \frac{3}{2}\,\pi(x)$, thus the left side of (34) follows. The cases of equality can be done with elementary verifications.

Relation (17) offered a relation for $\pi(2p_n)$. Now we will consider the sequence $(\pi(p_n^2))$. It is an old and famous conjecture that between p_n^2 and p_{n+1}^2 there are at least 4 distinct primes, due to Brocard (see e.g. [7]), i.e.,

$$\pi(p_{n+1}^2) - \pi(p_n^2) \geq 4. \tag{38}$$

□

In our opinion, even with 1 in place of (4) we have a difficult open problem. We have the following res

Theorem 5.

$$\pi(p_{n+2}^2) \leq 2\pi(p_n^2) < \pi(p_n^2) + \pi(p_{n+1}^2), \quad n \geq 4. \tag{39}$$

Proof. First we prove the following auxiliary result.

Lemma 3.

$$p_{n+2}^2 < 2p_n^2 \ \text{ for } \ n \geq 9. \tag{40}$$

Indeed, R. E. Dressler et al. (see [12]) proved that $p_{n+1}^2 \leq 2p_n^2$ for $n > 4$. A similar argument can be applied for the proof of (40). This is based on the Rosser–Schoenfeld inequalities $p_n < n(\log n + \log\log n - \frac{1}{2})$ for $n \geq 20$ and $p_n > n(\log n + \log\log n - \frac{3}{2})$ for $n \geq 2$. Then, to prove $p_{n+2} < \sqrt{2}\cdot p_n$, we have to prove an inequality

$$(n+2)\left[\log(n+2) + \log\log(n+2) - \frac{1}{2}\right] < \sqrt{2}\cdot n\left(\log n + \log\log n - \frac{3}{2}\right).$$

By considering the function

$$\begin{aligned} f(x) = \sqrt{2}x\log x - (x+2)\log(x+2) + \sqrt{2}x\log\log x - (x+2)\log\log(x+2) \\ - 1.63\cdot x + 1 > 0, \end{aligned}$$

and using the derivative of $f(x)$, and remarking that $\sqrt{2}\log x > \log(x+2) + 1.22$ for $x \geq 25$, we can easily deduce (we omit the details) that $f(x) > 0$ for $x \geq 24$. Thus (40) is true for $n \geq 24$. For $x \leq 23$ a direct verification can be done, and we get that $1 \leq n \leq 8$, excepting $n = 7$, inequality (40) is not true.

Now, for the proof of (39) remark that by Landau's inequality (28) and by (40) we can write

$$\pi(p_{n+2}^2) \leq \pi(2p_n^2) \leq 2\pi(p_n^2) = \pi(p_n^2) + \pi(p_n^2) \leq \pi(p_n^2) + \pi(p_{n+1}^2)$$

as $\pi(p_n^2) \leq \pi(p_{n+1}^2)$ for $n \geq 9$. For $1 \leq n \leq 8$ a direct verification shows that (39) is true for any $n \geq 4$. □

Remark 4. *The weaker inequality of* (39) *was an Open Problem* [4].

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