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The mean value of the function $\frac{d(n)}{d^*(n)}$ in arithmetic progressions

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Abstract: Let $d(n)$ and $d^*(n)$ be, respectively, the number of divisors and the number of unitary divisors of an integer $n \geq 1$. A divisor d of an integer is to be said unitary if it is prime over $\frac{n}{d}$. In this paper, we study the mean value of the function $D(n) = \frac{d(n)}{d^*(n)}$ in the arithmetic progressions $\{l + mk \mid m \in \mathbb{N}^* \text{ and } (l, k) = 1\}$; this leads back to the study of the real function $x \mapsto S(x; k, l) = \sum\limits_{\substack{n \leq x \\ n \equiv l[k]}} D(n)$. We prove that

$$S(x; k, l) = A(k)x + \mathcal{O}_k\left(x \exp\left(-\frac{\theta}{2}\sqrt{(2\ln x)(\ln\ln x)}\right)\right)(0 < \theta < 1),$$

where $A(k) = \frac{c}{k}\prod\limits_{p|k}\left(1 + \frac{1}{2}\sum\limits_{n=2}^{+\infty}\frac{1}{p^n}\right)^{-1}\left(c = \zeta(2)\prod\limits_{p}\left(1 - \frac{1}{2p^2} + \frac{1}{2p^3}\right)\right)$.

Keywords: Divisors, Unitary divisors of integer, Riemann zeta function, Dirichlet function, Dirichlet characters modulo k.

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1 Introduction

For an integer $n \in \mathbb{N}^*$, we denote by $d(n)$ the number of divisors of n, by $d^*(n)$ the number of unitary divisors of n. We recall that a divisor d of n is unitary if it is prime with the quotient $\frac{n}{d}$,

$$d(n) = \sum_{d|n} 1, \quad d^*(n) = \sum_{\substack{d|n \\ GCD(d,\frac{n}{d})=1}} 1.$$

We put $D(n) = \dfrac{d(n)}{d^*(n)}$, it is known that the function $n \mapsto \dfrac{\ln 2}{3} \dfrac{\ln n}{\ln \ln n}$ $(n \geq 3)$ is a maximum order of the function $n \mapsto \ln(D(n))$ (see [4]), however the function $D(n)$ takes the value 1 on all prime numbers p. This erratic behavior of the function $D(n)$ motivates to study its mean value in (see [5]), the authors obtained the asymptotic formula $\sum_{n \leq x} D(n) = cx + \mathcal{O}(\sqrt{x})$ where $c = \frac{\pi^2}{6} \prod_p \left(1 - \frac{1}{2p^2} + \frac{1}{2p^3}\right) = 1.4276565...$. In this paper, we are interested in the study of the mean value of the function $D(n)$ in the arithmetic progressions $\{l + mk \mid m \in \mathbb{N}^*\}$. The latter leads to studying the real function $\sum_{\substack{n \leq x \\ n \equiv l[k]}} D(n)$. We obtained the results in the following Theorem.

Theorem 1.1. *Let l and k be two integers, such that $1 \leq l \leq k$ and $(l, k) = 1$, and let $S(x; k, l)$ be the real summation function defined by $S(x; k, l) = \sum_{\substack{n \leq x \\ n \equiv l[k]}} D(n)$. Uniformly with respect to $k \geq 2$, we have*

$$S(x; k, l) = A(k)x + \mathcal{O}_k\left(x \exp\left(-\frac{\theta}{2}\sqrt{(2\ln x)(\ln\ln x)}\right)\right) (x \to +\infty)\,(0 < \theta < 1)\,,$$

where

$$A(k) = \frac{c}{k} \prod_{p|k} \left(1 + \frac{1}{2} \sum_{n=2}^{+\infty} \frac{1}{p^n}\right)^{-1} \quad \left(c = \zeta(2) \prod_p \left(1 - \frac{1}{2p^2} + \frac{1}{2p^3}\right)\right).$$

Our study is mainly based on:

a) The use of an orthogonality relation on the Dirichlet characters modulo k which has allowed us to write $S(x; k, l)$ in the form $S(x; k, l) = \dfrac{1}{\varphi(k)} \sum_{n \leq x} a(n)$, where $a(n)$ is the arithmetic function defined by $a(n) = \sum_{\chi \in \widehat{G}(k)} \overline{\chi}(l)\chi(n)D(n)\,(n \in \mathbb{N}^*)$, where $\widehat{G}(k)$ denotes the set of Dirichlet characters modulo k.

b) The estimation of the generating function $Q(s) = \sum_{n=1}^{+\infty} \dfrac{a(n)}{n^s} \quad (\Re(s) > 1)$.

2 Dirichlet characters modulo *k*

Let $k \in \mathbb{N}^*$, we denote by $G(k)$ the multiplicative group of invertible residues modulo k $\left(G(k) = \left(\frac{\mathbb{Z}}{k\mathbb{Z}}\right)^*\right)$. The Chinese Remainder Theorem gives the canonical decomposition $G(k) \simeq \left(\frac{\mathbb{Z}}{p_1^{\alpha_1}\mathbb{Z}}\right)^* \times \cdots \times \left(\frac{\mathbb{Z}}{p_r^{\alpha_r}\mathbb{Z}}\right)^*$ $(k = p_1^{\alpha_1} \times \cdots \times p_r^{\alpha_r}$ where $r \in \mathbb{N}^*, \alpha_i \geq 1$ and p_i are primes).

We know that $\left(\frac{\mathbb{Z}}{p^{\alpha}\mathbb{Z}}\right)^{*}$ is cyclic for $p \geq 3$ and $\alpha \in \mathbb{N}^{*}$ and $\left(\frac{\mathbb{Z}}{2^{\alpha}\mathbb{Z}}\right)^{*} = \langle -1 \rangle \bigodot \langle 5 \rangle \; (\alpha \geq 3)$, where $\langle -1 \rangle \bigodot \langle 5 \rangle$ denotes the direct product of the two subgroups of $\left(\frac{\mathbb{Z}}{2^{\alpha}\mathbb{Z}}\right)^{*}$ respectively generated by $\langle -1 \rangle$ and $\langle 5 \rangle$, which allows us to explicitly determine the morphisms of $G(k)$ in $\mathbb{C}^{*}$ and hence the Dirichlet characters modulo k. The canonical decomposition of $\left(\frac{\mathbb{Z}}{k\mathbb{Z}}\right)^{*}$ implies that χ is a Dirichlet character modulo k if and only if $\chi = \chi_1 \times \cdots \times \chi_r$, where χ_i are Dirichlet characters modulo $p_i^{\alpha_i}$. So, we obtain the $\varphi(k)$ characters modulo k. The characters satisfy several properties, in particular the one we used in our study in this case: Any character χ is periodic of period k, completely multiplicative, for any $n \in \mathbb{Z}$ such that $(n, k) = 1$, we have $\chi(n)\overline{\chi}(n) = \chi(n)\overline{\chi(n)} = |\chi(n)|^2 = 1$ and the following orthogonality relations : for all $l \in \mathbb{N}$ such that $(l, k) = 1$ and $1 \leq l \leq k$

$$\sum_{\chi \in \widehat{G}(k)} \overline{\chi}(l)\chi(n) = \begin{cases} \varphi(k), & \text{if } n \equiv l[k], \\ 0, & \text{if } n \not\equiv l[k]. \end{cases}$$

3 Preparatory lemmas

Lemma 3.1. *For l and k be two integers, with $1 \leq l \leq k$ and $(l, k) = 1$, we put*

$$S(x; k, l) = \sum_{\substack{n \leq x \\ n \equiv l[k]}} D(n).$$

We have $S(x; k, l) = \frac{1}{\varphi(k)} \sum_{n \leq x} a(n)$, *where* $a(n) = \sum_{\chi \in \widehat{G}(k)} \overline{\chi}(l)\chi(n)D(n)$.

This is a consequence of the following lemma.

Lemma 3.2. *For l and k two integers such that $1 \leq l \leq k$ and $(l, k) = 1$, we have*

$$\sum_{\chi \in \widehat{G}(k)} \overline{\chi}(l)\chi(n) = \begin{cases} \varphi(k), & \text{if } n \equiv l[k], \\ 0, & \text{if } n \not\equiv l[k]. \end{cases}$$

Proof. If $n \equiv l[k]$, then $\chi(n) = \chi(l + mk) = \chi(l) \; (m \in \mathbb{Z})$. So we have

$$\sum_{\chi \in \widehat{G}(k)} \overline{\chi}(l)\chi(n) = \sum_{\chi \in \widehat{G}(k)} \overline{\chi}(n)\chi(n) = \sum_{\chi \in \widehat{G}(k)} |\chi(n)|^2 = \sum_{\chi \in \widehat{G}(k)} 1 = \left|\widehat{G}(k)\right| = \varphi(k).$$

If $n \not\equiv l[k]$, then $\chi(n) = 0 \left(\forall \chi \in \widehat{G}(k)\right)$. So we have $\sum_{\chi \in \widehat{G}(k)} \overline{\chi}(l)\chi(n) = 0$. ☐

Lemma 3.3 ([5], p. 556)**.** *In the half-plane $\Re(s) > \frac{1}{2}$, the following equality takes place:*

$$\prod_p \left(1 + \frac{1}{2}\sum_{n=2}^{+\infty} \frac{1}{p^{ns}}\right) = \zeta(2s) \prod_p \left(1 - \frac{1}{2p^{2s}} + \frac{1}{2p^{3s}}\right).$$

Lemma 3.4 ([5]). *The function* $s \mapsto \sum_{n=1}^{+\infty} \frac{D(n)}{n^s}$ $(\Re(s) > 1)$ *extends to the half-plane* $\Re(s) > \frac{1}{2}$ *by the formula* $\sum_{n=1}^{+\infty} \frac{D(n)}{n^s} = \zeta(s)G(s)$, *where* $G(s) = \zeta(2s) \prod_p \left(1 - \frac{1}{2p^{2s}} + \frac{1}{2p^{3s}}\right)$, *into a meromorphic function admitting a single pole at* $s = 1$ *in which the residue is*

$$G(1) = \zeta(2) \prod_p \left(1 - \frac{1}{2p^2} + \frac{1}{2p^3}\right).$$

Lemma 3.5. *For any character* $\chi \in \widehat{G}(k)$, *we put*

$$F(s,\chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)D(n)}{n^s} \quad (\Re(s) > 1).$$

The function $F(s,\chi)$ *extends to the half-plane* $\Re(s) > \frac{1}{2}$ *by the formulas*

$$F(s,\chi) = L(s,\chi)G(s,\chi), \text{ where } G(s,\chi) = \prod_p \left(1 + \frac{1}{2}\sum_{n=2}^{+\infty} \frac{(\chi(p))^n}{p^{ns}}\right) \left(\chi \in \widehat{G}(k)\right),$$

and

$$F(s,\chi_0) = \zeta(s)G(s) \prod_{p|k} \left(1 - \frac{1}{p^s}\right) \left(1 + \frac{1}{2}\sum_{n=2}^{+\infty} \frac{1}{p^{ns}}\right)^{-1},$$

where

$$G(s) = \zeta(2s) \prod_p \left(1 - \frac{1}{2p^{2s}} + \frac{1}{2p^{3s}}\right).$$

Proof. The arithmetic function $n \mapsto \frac{\chi(n)D(n)}{n^s}$ $(n \in \mathbb{N}^*)$ is multiplicative and, for all prime numbers p and $m \in \mathbb{N}$, we have

$$\frac{\chi(p^m)D(p^m)}{p^{ms}} = \begin{cases} 1, & \text{if } m = 0 \\ 0, & \text{if } m \geq 1 \text{ and } p \text{ divides } k \\ \frac{\chi(p)}{p^s}, & \text{if } m = 1 \text{ and } (p,k) = 1 \\ \frac{(m+1)(\chi(p))^m}{2p^{ms}}, & \text{if } m \geq 2 \text{ and } (p,k) = 1. \end{cases}$$

Then in the half-plane $\Re(s) > 1$, the Euler product formula gives

$$\begin{aligned} F(s,\chi) &= \prod_p \left(1 + \frac{\chi(p)}{p^s} + \sum_{n=2}^{+\infty} \frac{(n+1)(\chi(p))^n}{2p^{ns}}\right) \\ &= \frac{\prod_p \left(1 + \frac{\chi(p)}{p^s} + \sum_{n=2}^{+\infty} \frac{(n+1)(\chi(p))^n}{2p^{ns}}\right) \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)}{\prod_p \left(1 - \frac{\chi(p)}{p^s}\right)}. \end{aligned}$$

Evaluating the product of the numerator of the last expression gives

$$\prod_p \left(1 + \frac{1}{2}\sum_{n=2}^{+\infty} \frac{(\chi(p))^n}{p^{ns}}\right) = G(s,\chi),$$

and we know that

$$\prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} = L(s,\chi).$$

So we have

$$F(s,\chi) = L(s,\chi)\prod_p \left(1 + \frac{1}{2}\sum_{n=2}^{+\infty} \frac{(\chi(p))^n}{p^{ns}}\right).$$

For $\chi = \chi_0$ we have

$$F(s,\chi_0) = L(s,\chi_0)\prod_p \left(1 + \frac{1}{2}\sum_{n=2}^{+\infty} \frac{(\chi_0(p))^n}{p^{ns}}\right).$$

The expression of $F(s)$ announced then comes from Lemma 3.3 and the two following equalities:

$$L(s,\chi_0) \;=\; \zeta(s)\prod_{p|k}\left(1 - \frac{1}{p^s}\right),$$

and

$$\prod_p \left(1 + \frac{1}{2}\sum_{n=2}^{+\infty} \frac{(\chi_0(p))^n}{p^{ns}}\right) \;=\; \prod_p \left(1 + \frac{1}{2}\sum_{n=2}^{+\infty} \frac{1}{p^{ns}}\right)\prod_{p|k}\left(1 + \frac{1}{2}\sum_{n=2}^{+\infty} \frac{1}{p^{ns}}\right)^{-1}.$$

This completes the proof. □

Lemma 3.6 ([3]). *Let $\delta(t) = 1 - \theta\frac{\ln(\ln|t|)}{\ln|t|}$, where $|t| \geqslant T_0 \geqslant 268$ and $0 < \theta < 1$. For any complex number $s = \delta + it$ and $\delta \geq \delta(t)$, we have*

a) $G(s) = \mathcal{O}(1)$, *where* $G(s) = \zeta(2s)\prod_p\left(1 - \frac{1}{2p^{2s}} + \frac{1}{2p^{3s}}\right)$

b) $\zeta(\delta + it) = \mathcal{O}\left(\ln|t|^{1+\theta}\right).$

Proof of Theorem 1.1. We consider the generating function $Q(s) = \sum\limits_{n=1}^{+\infty}\frac{a(n)}{n^s}$ $(\Re(s) > 1)$. We have

$$Q(s) = F(s) + K(s),$$

where

$$F(s) = F(s,\chi_0) = \vartheta(s)\zeta(s)G(s) \quad \left(\vartheta(s) = \prod_{p|k}\left(1 - \frac{1}{p^s}\right)\left(1 + \frac{1}{2}\sum_{n=2}^{+\infty}\frac{1}{p^{ns}}\right)^{-1}\right),$$

and

$$K(s) = \sum_{\chi\neq\chi_0} \overline{\chi}(l)F(s,\chi)$$

$(F(s,\chi)$ from Lemma 3.5.$)$.

By Lemma 3.5, the function $s \mapsto Q(s)$ is meromorphic in the half-plane $\Re(s) > \frac{1}{2}$ and except the simple pole at the point $s = 1$, where the residue

$$c(k) = \zeta(2) \prod_p \left(1 - \frac{1}{2p^2} + \frac{1}{2p^3}\right) \prod_{p|k} \left(1 - \frac{1}{p}\right) \left(1 + \frac{1}{2} \sum_{n=2}^{+\infty} \frac{1}{p^n}\right)^{-1}.$$

For $k \geq 2$, we have

$$\varphi(k) = k \prod_{p|k} \left(1 - \frac{1}{p}\right).$$

With this last equality and Lemma 3.4, we can write

$$c(k) = c\left(\frac{\varphi(k)}{k}\right) \prod_{p|k} \left(1 + \frac{1}{2} \sum_{n=2}^{+\infty} \frac{1}{p^n}\right)^{-1}.$$

Thus $Q(s) - \frac{c(k)}{s-1}$ is analytic in the half-plane $\Re(s) > \frac{1}{2}$. According to Ikehara's theorem in [6], p. 332, we have

$$\sum_{n \leq x} a(n) \sim c(k)x \quad (x \to +\infty),$$

we obtain

$$S(x; k, l) \sim A(k)x \quad (x \to +\infty) \left(A(k) = \frac{c(k)}{\varphi(k)}\right).$$

We put

$$\Phi(x; k, l) = \sum_{n \leq x} a(n), \quad \text{where} \quad \left(a(n) = \sum_{\chi \in \widehat{G}(k)} \overline{\chi}(l)\chi(n)D(n)\ (n \in \mathbb{N}^*)\right).$$

Applying the Perron formula (see [2], p. 242), we have for $b > 1$ and $\Re(s) > \frac{1}{2}$

$$\int_1^x \Phi(u; k, l)du = \frac{1}{2\pi i} \lim_{T \to +\infty} \left(\int_{b-iT}^{b+iT} x^{s+1} \frac{\vartheta(s)\zeta(s)G(s)}{s(s+1)} ds + \int_{b-iT}^{b+iT} x^{s+1} \frac{K(s)}{s(s+1)} ds\right). \quad (*)$$

For $\chi \neq \chi_0$ we have

$$K(s) = \mathcal{O}(1),$$

and, for any complex number $s = \delta + it$ with $\delta \geq \delta(t) > \frac{1}{2}$, we have $\vartheta(s) = \mathcal{O}(1)$. Thus, from Lemma 3.6, we obtain the estimate

$$\frac{\vartheta(s)\zeta(s)G(s) + K(s)}{s(s+1)} = \mathcal{O}\left(\frac{1}{|t|^{(1-\theta)/2}}\right).$$

We put

$$\delta(t) = 1 - \theta \frac{\ln(\ln(\max(|t|, T_0)))}{\ln \max(|t|, T_0)} + it, \quad \text{where } T_0 \geqslant 268 \text{ and } 0 < \theta < 1.$$

For $|t| > T_0$, we move the line of integration from $(*)$ to the closed contour $\mathcal{C}_{T_0,\theta} = \bigcup_{i=1}^{i=6} (\gamma_i)$ (see Figure 1), where

$$\begin{aligned} (\gamma_1) &= [b - iT, b + iT], \\ (\gamma_2) : \delta \longmapsto \gamma_2(t) &= \delta + iT\,(b \geq \delta \geq \delta(T)), \\ (\gamma_3) : t \longmapsto \gamma_3(t) &= \delta(t) + it\,(T \geq t \geq T_0), \\ (\gamma_4) : t \longmapsto \gamma_4(t) &= \delta_0 + it\,(T_0 \geq t \geq -T_0 \text{ and } \delta_0 = \delta(T_0)), \\ (\gamma_5) : t \longmapsto \gamma_5(t) &= \delta(t) + it\,(-T_0 \geq t \geq -T), \\ (\gamma_6) : \delta \longmapsto \gamma_6(t) &= \delta - iT\,(\delta(T) \geq \delta \geq \delta(T)). \end{aligned}$$

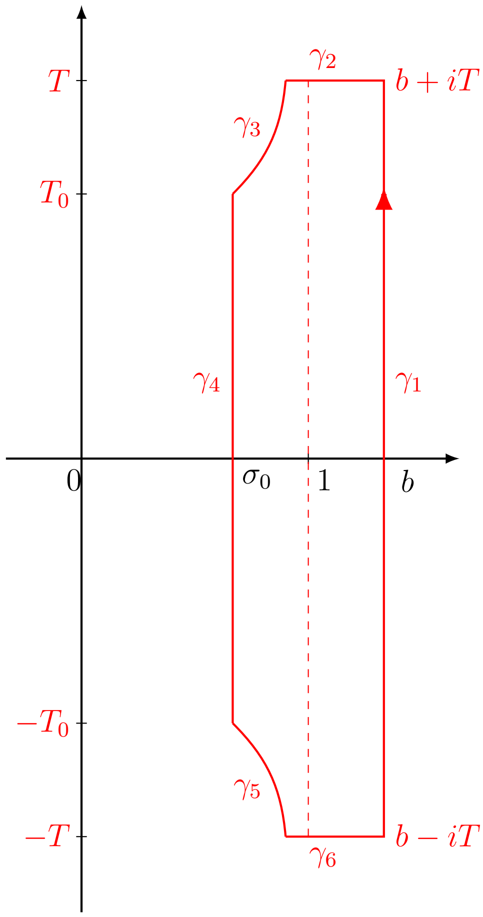

Figure 1. The closed contour $\mathcal{C}_{T_0,\theta}$

Knowing that the function $K(s)$ is analytic in the half-plane $\Re(s) > \frac{1}{2}$ for every $\chi \neq \chi_0$ modulo k, therfore using the Cauchy theorem gives us

$$\int_{\mathcal{C}_{T_0,\theta}} x^{s+1} K(s) \frac{ds}{s(s+1)} = 0.$$

The residue theorem then gives us,

$$\begin{aligned}\int_1^x \Phi(u;k,l)du - \frac{c(k)x^2}{2} &= \frac{1}{2\pi i}\int_{\mathcal{C}_{T_0,\theta}} x^{s+1}F(s)\frac{ds}{s(s+1)} \\ &= \frac{1}{2\pi}\int_{-T_0}^{T_0} \hbar(\delta_0+it)dt + \frac{1}{2\pi}\lim_{T\to+\infty}\int_{T_0}^{T} \hbar(\delta(t)+it)(\delta'(t)+i)dt \\ &\quad - \frac{1}{2\pi}\lim_{T\to+\infty}\int_{T_0}^{T} \hbar(\delta(t)-it)(\delta'(t)-i)dt\end{aligned}$$

$\left(\delta_0 = 1-\theta\frac{\ln(\ln T_0)}{\ln T_0}\right)$, where

$$\hbar(s) = \frac{F(s)}{s(s+1)} = \frac{\vartheta(s)\zeta(s)G(s)}{s(s+1)}.$$

The integrals on horizontal lines (γ_2) and (γ_6) are zero when T tends to $+\infty$.
Then we have

$$\left|\int_1^x \Phi(u;k,l)du - \frac{c(k)x^2}{2}\right| \leq \frac{1}{2\pi}\left|\int_{T_0}^{T_0} \hbar(\delta_0+it)dt\right| + \frac{\sqrt{1+(\delta'(T_0))^2}}{\pi}\int_{T_0}^{+\infty}|\hbar(\delta(t)+it)dt|.$$

Since $\hbar(s) = \vartheta(s)H(s)$, where $H(s)$ is defined in [3] by $H(s) = \frac{\zeta(s)G(s)}{s(s+1)}$, and $\vartheta(s) = \mathcal{O}(1)$, then $\hbar(s) = \mathcal{O}(H(s))$.

Applying results from [3] gives

$$\Phi(x;k,l) = c(k)x + \mathcal{O}\left(x\exp\left(-\frac{\theta}{2}\sqrt{(2\ln x)(\ln\ln x)}\right)\right)$$

and we have the estimation of the theorem, i.e.

$$S(x;k,l) = A(k)x + \mathcal{O}\left(\frac{x}{\varphi(k)}\exp\left(-\frac{\theta}{2}\sqrt{(2\ln x)(\ln\ln x)}\right)\right).$$

The improvement of the last estimation of the remainder term to $\mathcal{O}(\sqrt{x})$ will be one of the main targets of our future research.

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