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On vertex resolvability of a circular ladder of nonagons

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Abstract: Let $H = H(V, E)$ be a non-trivial simple connected graph with edge and vertex set $E(H)$ and $V(H)$, respectively. A subset $\mathbb{D} \subset V(H)$ with distinct vertices is said to be a vertex resolving set in H if for each pair of distinct vertices p and q in H we have $d(p, u) \neq d(q, u)$ for some vertex $u \in H$. A resolving set H with minimum possible vertices is said to be a metric basis for H. The cardinality of metric basis is called the metric dimension of H, denoted by $\dim_v(H)$. In this paper, we prove that the metric dimension is constant and equal to 3 for certain closely related families of convex polytopes.

Keywords: Resolving set, Metric dimension, Nonagonal circular ladder, Planar graph, Convex polytopes.
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1 Introduction

The concept of metric dimension for a connected graph $H = (V, E)$ is equivalent to the total number of satellites needed for Global Positioning Systems (GPS) to work perfectly. The main objective is to choose a small set of ordered vertices $\mathbb{D} \subseteq V(H)$ that can identify each vertex using just the shortest path distances to $\mathbb{D}$. The exact solution to this problem is computationally difficult, but it provides information that is important in a variety of situations. A small set of *landmarks* or *satellites* in a discrete space can be effective for assisting robots in navigating through physical space or tracing the transmission of disease between cities. Moreover, this might be used for more abstract tasks like comparing network structures, locating a source of lies and false news in a social network, numerically representing symbolic data, or categorizing the chemical structure.

Slater [20] and Harary and Melter [7] separately introduced the concept of metric dimension in the context of graphs, but the dimension of graphs had indeed been studied by Erdős et al. [6]. Both of the introductory articles [7, 20] concentrate on the metric dimension of trees and offer equivalent exact formulas for these types of graphs. The metric dimension of numerous different types of graphs, such as complete graphs, complete bipartite graphs, and cycle graphs, is briefly discussed in [7], although the metric dimension of wheel graphs was wrongly reported as 2. They also provide an algorithm for reconstructing a tree using the given distances between vertices and resolving set elements. For generic graphs, this is not possible since not all edges are guaranteed to be represented in a shortest path with a resolving set member as an endpoint.

The notions of metric dimension and resolving sets finds their application in various fields including, robot navigation [12], image processing and pattern recognition [14], pharmaceutical chemistry and connected joins in graphs [3], network discovery and verification [15], strategies for the mastermind games [5], etc. For the metric dimension of several known graph families, readers are referred to [2, 17–19]. For a given integer q and a graph G, the decision problem associated with metric dimension is to demonstrate whether or not $\dim_v(H) \leq q$. This decision problem is computationally intractable, i.e., NP-complete [8, 11].

Next, a polytope is a flat-sided geometric object (faces). The term "polytope" refers to an extensive collection of items. Convex polytopes are those polytopes that are convex sets, and that exist in the Euclidean space $\mathbb{R}^n$ (n-dimensional space). Convex polytopes have found application in the field of optimization, where linear programming studies the minima and maxima of linear functions; these minima and maxima occur on the boundary of a polytope with n-dimensions. Further, in twistor theory (theoretical physics), a polytope known as amplituhedron is used to calculate the scattering amplitudes of subatomic particles when they collide [1]. By considering some planar families of convex polytopes, in this paper, we study the concept of metric dimension for them. In last two decades, the concept of metric basis and dimension for several significant families of convex polytopes have been discussed [9, 10, 17].

The rest of this manuscript is organized in the following manner. In Section 2, some basic findings and results related to the resolving sets and metric dimension are discussed. In Sections 3–6, we investigate the minimum resolving sets for four closely related classes of convex polytopes, and determine their metric dimension. Finally, the conclusion and future work of this paper is presented.

2 Preliminaries

This section is devoted to recollect some fundamental terminologies and findings about the metric basis, as well as metric dimension of graphs.

Definition 2.1. (Adjacent vertices) *Let u and v be two vertices in a simple connected graph H. Then, u and v are said to be adjacent if they are joined by an edge in H.*

Definition 2.2. (Degree of a vertex) *Let u be a vertex in a simple connected graph H. Then, degree of u, denoted by d_u, is the totality of edges incident on u in H.*

Definition 2.3. (Independent set) *Let S be a subset of vertices in H. Then, S is said to be an independent set, if no two vertices in S are adjacent.*

Definition 2.4. (Metric dimension) *For completely distinct vertices p, w, $q \in V(H)$, if $d(p,w) \neq d(q,w)$, then the common vertex w in H is said to recognize (distinguish or resolve) the distinct pair of vertices p and q in H. Let $\mathbb{D} \subseteq V(H)$ be a subset of k distinct ordered vertices. If every pair of distinct vertices in a given simple connected graph $\mathbb{D}$ is distinguished by one (at least) member of $\mathbb{D}$, then $\mathbb{D}$ is said to be a resolving set (vertex) for H. The smallest possible cardinality of a resolving set is said to be a metric dimension of H, and is usually represented by $\dim_v(H)$ [7, 20]. The minimal cardinality resolving set $\mathbb{D}$ serves as the metric basis for H. For a subset of k distinct ordered vertices $\mathbb{D} = \{a_1, a_2, a_3, \ldots, a_k\}$, the k-length code (representation or coordinate) of vertex j in $V(H)$ is*

$$\zeta(v|\mathbb{D}) = (d(a_1,v), d(a_2,v), d(a_3,v), \ldots, d(a_k,v)).$$

Then, we say that the set $\mathbb{D}$ is a metric generator for H, if $r_v(a|\mathbb{D}) \neq r_v(w|\mathbb{D})$, for every pair $a, w \in V(H)$ of vertices with $a \neq w$.

Definition 2.5. (Independent resolving set) [4] *Let S be a subset of vertices in H. Then, S is said to be an independent resolving set if S is (1) an independent set, and (2) a resolving set.*

Further, Imran et al. [9] raised an open problem regarding some planar graphs:

Problem: *Characterize the classes of radially symmetrical plane graphs H_1 obtained from H by adding new edges in H such that* $\dim(H) = \dim(H_1)$ *and* $V(H) = V(H_1)$.

To address this problem partially, in this work a planar graph family, that is denoted by N_n (*nonagonal circular ladder*), has been constructed. Then, further by placing new edges in N_n at different positions we construct three new families of convex polytopes from N_n viz., N_n^1, N_n^2, and N_n^3 with the same vertex set. Hence, in this article, we consider a problem of metric dimension for four closely related classes of convex polytopes with the same vertex set.

In this paper, we have consider four convex polytopes for which we have $V(N_n) = V(N_n^1) = V(N_n^2) = V(N_n^3) = \{p_l, q_l, r_l, s_l, l_l : 1 \leq l \leq n\}$. We denote the sets of metric codes/coordinates for the vertices q_l, s_l, t_l, p_l, and r_l $(1 \leq l \leq n)$, respectively by $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{S}$, and $\mathbb{L}$, for the convex polytopes N_n, N_n^1, N_n^2, and N_n^3.

Khuller et al. [12] derived various results including the metric dimension of a graph with m vertices that can be estimated in polynomial time within a factor of $O(\log\ m)$, corrected proof for the characterization of metric dimension in trees, and some few properties of two-metric dimension graphs. For the graphs with metric dimension 2, the result is as follows:

Theorem 2.1. *Consider a metric basis set U in H with cardinality 2, i.e., $|U| = 2$ and let $U = \{v_1, v_2\}$. Then,*

- *There always exists a shortest unique path P_s between the two basis vertices v_1 and v_2.*
- *Degrees of basis vertices v_1 and v_2 is at most 3.*
- *Vertices other than v_1 and v_2 on P_s have a degree at most 5.*

3 Metric dimension of a nonagonal circular ladder N_n

In this section, we present an interesting family of the planar graphs, denoted by N_n. For this family, we discuss some of its basic characteristics and determine its basis set, as well as metric dimension.

The Graph of N_n: The N_n can be obtained from the *Heptagonal circular ladder* Γ_n [17] by placing n new vertices between the vertices p_l and q_l $(1 \leq l \leq n)$ in Γ_n (see Figure 1). It has a vertex set and an edge set with cardinality $5n$ and $6n$, respectively. The set of edges and vertices of the NCL N_n are depicted separately by $E(N_n)$ and $V(N_n)$, where $V(N_n) = \{p_l, q_l, r_l, s_l, t_l : 1 \leq l \leq n\}$ and $E(N_n) = \{p_lq_l, q_lr_l, r_ls_l, s_lt_l, p_lp_{l+1}, t_ls_{l+1} : 1 \leq l \leq n\}$.

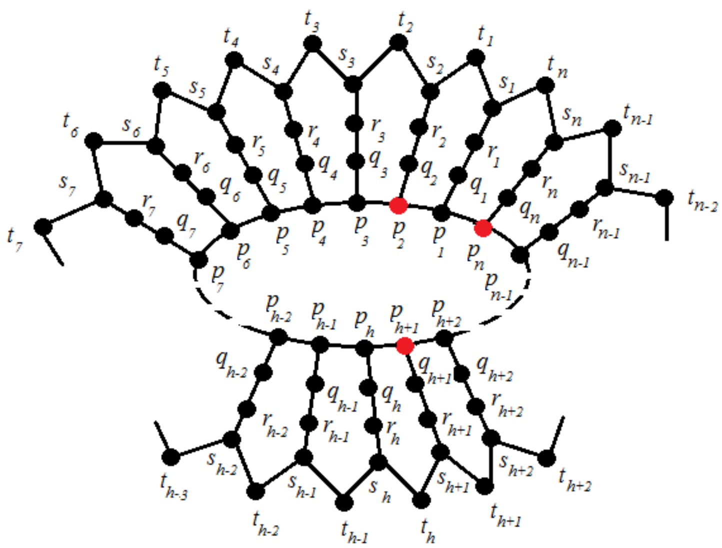

Figure 1. Nonagonal circular ladder N_n

The elements of the set $P = \{p_l : 1 \leq l \leq n\}$ in N_n, are called P-set elements, the elements of the set $Q = \{q_l : 1 \leq l \leq n\}$ in N_n, are called Q-set elements, the elements of the set $R = \{r_l : 1 \leq l \leq n\}$ in N_n, are called R-set elements, the elements of the set $ST = \{s_l, t_l : 1 \leq l \leq n\}$ in N_n, are called ST-set elements. Next, we are ready to determine the basis set, as well as the metric dimension for the planar graph N_n.

Theorem 3.1. *For the planar graph N_n with $n \geq 6$, we have* $\dim_v(N_n) = 3$.

Proof. We prove this theorem in two cases depending upon n.

Case (I) $n \equiv 0 \pmod 2$.

This means that $n = 2h$, where $h \in \mathbb{Z}^+$ and $h \geq 3$. Let $\mathbb{D} = \{p_2, p_{h+1}, p_n\}$ be a subset of $V(N_n)$ with three distinct vertices chosen from P-set elements. To complete the proof for this case, when n is even, we need to show that $\mathbb{D}$ is a basis set for the planar graph N_n. For upper bound, we give the metric coordinates for each vertex of N_n with respect to the set $\mathbb{D}$.

For the P-set elements, the metric coordinates are shown in Table 1.

Table 1. Metric coordinates for P-set elements in N_n

Vertices	**Codes**
$p_l; l = 1$	$(1, h, l)$
$p_l; 2 \leq l \leq h$	$(l - 2, h - l + 1, l)$
$p_l; l = h + 1$	$(l - 2, h - l + 1, 2h - l)$
$p_l; h + 2 \leq l \leq 2h$	$(2h - l + 2, l - h - 1, 2h - l)$

For the Q-set elements, the metric coordinates are $\zeta(q_l|\mathbb{D}) = \zeta(p_l|\mathbb{D}) + (1, 1, 1)$ for $1 \leq l \leq n$. For the R-set elements, the metric coordinates are $\zeta(r_l|\mathbb{D}) = \zeta(p_l|\mathbb{D}) + (2, 2, 2)$ for $1 \leq l \leq n$. Finally, for the ST-set elements (i.e., $ST = \{s_l, t_l : 1 \leq l \leq n\}$), the metric coordinates for vertices $\{s_l : 1 \leq l \leq n\}$ are $\zeta(s_l|\mathbb{D}) = \zeta(p_l|\mathbb{D}) + (3, 3, 3)$ for $1 \leq l \leq n$ and the metric coordinates for vertices $\{t_l : 1 \leq l \leq n\}$ are shown in Table 2.

Table 2. Metric coordinates for vertices $\{t_l : 1 \leq l \leq n\}$ in N_n

Vertices	**Codes**
$t_l; l = 1$	$(4, h + 3, 5)$
$t_l; 2 \leq l \leq h - 1$	$(l + 2, h - l + 4, l + 4)$
$t_l; l = h$	$(h + 2, 4, h + 3)$
$t_l; l = h + 1$	$(h + 3, 4, h + 2)$
$t_l; h + 2 \leq l \leq 2h - 1$	$(2h - l + 5, l - h + 3, 2h - l + 3)$
$t_l; l = 2h$	$(2h - l + 5, l - h + 3, 4)$

The above list of metric coordinates confirms that $|\mathbb{P}| = |\mathbb{Q}| = |\mathbb{R}| = |\mathbb{S}| = |\mathbb{L}| = n$ and each of $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{S}$, and $\mathbb{L}$ are pairwise disjoint. Thus, we conclude that there exist a set $\mathbb{D}$ with cardinality 3 in N_n, corresponding to which no pair of distinct vertices in N_n have the same metric coordinates in N_n, indicating that $\dim_v(N_n) \leq 3$. Next, for the reverse inequality, i.e., $\dim_v(N_n) \geq 3$, we demonstrate that no set $\mathbb{D}$ with cardinality 2 is a resolving set for N_n. To prove this, suppose on contrary that $\dim_v(N_n) = 2$. Then, for the set $\mathbb{D}$ with cardinality 2, we have the list of possibilities as follows (here we take $n \geq 14$, as for $6 \leq n \leq 13$, one can easily derive the contradictions for any set $\mathbb{D}$ with cardinality 2 in N_n:

Resolving sets	Contradictions
$\mathbb{D} = \{p_1, p_j\}$, p_j $(2 \leq j \leq n)$	$\zeta(q_1\|\mathbb{D}) = \zeta(p_n\|\mathbb{D})$, for $2 \leq j \leq h$, and $\zeta(p_2\|\mathbb{D}) = \zeta(p_n\|\mathbb{D})$, for $j = h+1$.
$\mathbb{D} = \{q_1, q_j\}$, q_j $(2 \leq j \leq n)$	$\zeta(q_n\|\mathbb{D}) = \zeta(p_{n-1}\|\mathbb{D})$, for $2 \leq j \leq h-1$; $\zeta(r_2\|\mathbb{D}) = \zeta(q_{n-1}\|\mathbb{D})$, when $j = h$, and $\zeta(p_2\|\mathbb{D}) = \zeta(p_n\|\mathbb{D})$, when $j = h+1$.
$\mathbb{D} = \{r_1, r_j\}$, r_j $(2 \leq j \leq n)$	$\zeta(q_n\|\mathbb{D}) = \zeta(p_{n-1}\|\mathbb{D})$, for $2 \leq j \leq h-1$; $\zeta(r_2\|\mathbb{D}) = \zeta(q_{n-1}\|\mathbb{D})$, when $j = h$, and $\zeta(p_2\|\mathbb{D}) = \zeta(p_n\|\mathbb{D})$, when $j = h+1$.
$\mathbb{D} = \{s_1, s_j\}$, s_j $(2 \leq j \leq n)$	$\zeta(q_{n-1}\|\mathbb{D}) = \zeta(p_{n-2}\|\mathbb{D})$, for $2 \leq j \leq h-2$; $\zeta(s_3\|\mathbb{D}) = \zeta(p_n\|\mathbb{D})$, when $j = h-1$; $\zeta(r_3\|\mathbb{D}) = \zeta(p_{n-1}\|\mathbb{D})$, when $j = h$, and $\zeta(p_2\|\mathbb{D}) = \zeta(p_n\|\mathbb{D})$, when $j = h+1$.
$\mathbb{D} = \{t_1, t_j\}$, t_j $(2 \leq j \leq n)$	$\zeta(q_{n-1}\|\mathbb{D}) = \zeta(p_{n-2}\|\mathbb{D})$, for $2 \leq j \leq h-3$; $\zeta(r_3\|\mathbb{D}) = \zeta(p_1\|\mathbb{D})$, when $h-2 \leq j \leq h$, and $\zeta(r_3\|\mathbb{D}) = \zeta(r_n\|\mathbb{D})$, when $j = h+1$.
$\mathbb{D} = \{p_1, q_j\}$, q_j $(1 \leq j \leq n)$	$\zeta(q_n\|\mathbb{D}) = \zeta(p_{n-1}\|\mathbb{D})$, for $1 \leq j \leq h-1$; $\zeta(r_2\|\mathbb{D}) = \zeta(q_{n-1}\|\mathbb{D})$, when $j = h$, and $\zeta(p_2\|\mathbb{D}) = \zeta(p_n\|\mathbb{D})$, when $j = h+1$.
$\mathbb{D} = \{p_1, r_j\}$, r_j $(1 \leq j \leq n)$	$\zeta(q_n\|\mathbb{D}) = \zeta(p_{n-1}\|\mathbb{D})$, for $1 \leq j \leq h-1$; $\zeta(p_n\|\mathbb{D}) = \zeta(q_1\|\mathbb{D})$, when $j = h$, and $\zeta(p_2\|\mathbb{D}) = \zeta(p_n\|\mathbb{D})$, when $j = h+1$.
$\mathbb{D} = \{p_1, s_j\}$, s_j $(1 \leq j \leq n)$	$\zeta(q_{n-1}\|\mathbb{D}) = \zeta(p_{n-2}\|\mathbb{D})$, for $1 \leq j \leq h-2$; $\zeta(q_1\|\mathbb{D}) = \zeta(p_n\|\mathbb{D})$, when $h-2 \leq j \leq h-1$; $\zeta(p_{n-1}\|\mathbb{D}) = \zeta(q_2\|\mathbb{D})$, when $j = h$, and $\zeta(p_2\|\mathbb{D}) = \zeta(p_n\|\mathbb{D})$, when $j = h+1$.
$\mathbb{D} = \{p_1, t_j\}$, l_j $(1 \leq j \leq n)$	$\zeta(q_{n-1}\|\mathbb{D}) = \zeta(p_{n-2}\|\mathbb{D})$, for $1 \leq j \leq h-3$; $\zeta(q_1\|\mathbb{D}) = \zeta(p_n\|\mathbb{D})$, when $h-2 \leq j \leq h-1$; $\zeta(l_1\|\mathbb{D}) = \zeta(s_n\|\mathbb{D})$, when $j = h$, and $\zeta(s_2\|\mathbb{D}) = \zeta(t_n\|\mathbb{D})$, when $j = h+1$.
$\mathbb{D} = \{q_1, r_j\}$, r_j $(1 \leq j \leq n)$	$\zeta(q_n\|\mathbb{D}) = \zeta(p_{n-1}\|\mathbb{D})$, for $1 \leq j \leq h-1$; $\zeta(q_2\|\mathbb{D}) = \zeta(p_{n-1}\|\mathbb{D})$, when $j = h$, and $\zeta(p_2\|\mathbb{D}) = \zeta(p_n\|\mathbb{D})$, when $j = h+1$.
$\mathbb{D} = \{q_1, s_j\}$, s_j $(1 \leq j \leq n)$	$\zeta(q_{n-1}\|\mathbb{D}) = \zeta(p_{n-2}\|\mathbb{D})$, for $1 \leq j \leq h-2$; $\zeta(r_2\|\mathbb{D}) = \zeta(p_{n-2}\|\mathbb{D})$, when $j = h-1$; $\zeta(q_2\|\mathbb{D}) = \zeta(p_{n-1}\|\mathbb{D})$, when $j = h$, and $\zeta(p_2\|\mathbb{D}) = \zeta(p_n\|\mathbb{D})$, when $j = h+1$.
$\mathbb{D} = \{q_1, t_j\}$, t_j $(1 \leq j \leq n)$	$\zeta(q_{n-1}\|\mathbb{D}) = \zeta(p_{n-2}\|\mathbb{D})$, for $1 \leq j \leq h-3$; $\zeta(q_n\|\mathbb{D}) = \zeta(p_{n-1}\|\mathbb{D})$, when $j = h-2$; $\zeta(q_{n-1}\|\mathbb{D}) = \zeta(s_2\|\mathbb{D})$, when $j = h-1$; $\zeta(s_2\|\mathbb{D}) = \zeta(r_n\|\mathbb{D})$, when $j = h$, and $\zeta(r_2\|\mathbb{D}) = \zeta(s_n\|\mathbb{D})$, when $j = h+1$.
$\mathbb{D} = \{r_1, s_j\}$, s_j $(1 \leq j \leq n)$	$\zeta(p_2\|\mathbb{D}) = \zeta(p_n\|\mathbb{D})$, for $j = 1, h+1$; $\zeta(q_n\|\mathbb{D}) = \zeta(p_{n-1}\|\mathbb{D})$, when $2 \leq j \leq h-1$, and $\zeta(q_2\|\mathbb{D}) = \zeta(p_{n-1}\|\mathbb{D})$, when $j = h$.
$\mathbb{D} = \{r_1, t_j\}$, t_j $(1 \leq j \leq n)$	$\zeta(q_{n-1}\|\mathbb{D}) = \zeta(p_{n-2}\|\mathbb{D})$, for $1 \leq j \leq h-3$; $\zeta(q_n\|\mathbb{D}) = \zeta(p_{n-1}\|\mathbb{D})$, when $j = h-2$; $\zeta(q_n\|\mathbb{D}) = \zeta(t_2\|\mathbb{D})$, when $j = h-1$; $\zeta(r_2\|\mathbb{D}) = \zeta(q_n\|\mathbb{D})$, when $j = h$, and $\zeta(r_n\|\mathbb{D}) = \zeta(q_2\|\mathbb{D})$, when $j = h+1$.
$\mathbb{D} = \{s_1, t_j\}$, l_j $(1 \leq j \leq n)$	$\zeta(q_{n-1}\|\mathbb{D}) = \zeta(p_{n-2}\|\mathbb{D})$, for $1 \leq j \leq h-3$; $\zeta(q_n\|\mathbb{D}) = \zeta(p_{n-1}\|\mathbb{D})$, when $j = h-2$; $\zeta(p_n\|\mathbb{D}) = \zeta(s_3\|\mathbb{D})$, when $j = h-1$; $\zeta(t_2\|\mathbb{D}) = \zeta(r_n\|\mathbb{D})$, when $j = h$, and $\zeta(r_2\|\mathbb{D}) = \zeta(t_{n-1}\|\mathbb{D})$, when $j = h+1$.

The list of contradictions as mentioned above confirms that no set $\mathbb{D}$ consisting two elements forms a resolving set for $V(N_n)$ indicating that $\dim_v(N_n) = 3$ in this case.

Case (II) $n \equiv 1 (\mathrm{mod}\ 2)$.

This means that $n = 2h + 1$, where $h \in \mathbb{Z}^+$ and $h \geq 3$. Let $\mathbb{D} = \{p_2, p_{h+1}, p_n\}$ be a subset of $V(N_n)$ with three distinct vertices chosen from P-set elements. To complete the proof for this case, when n is odd, we need to show that $\mathbb{D}$ is a basis set for the planar graph N_n. For upper bound, we give the metric coordinates for each vertex of N_n with respect to the set $\mathbb{D}$. For the P-set elements, the metric coordinates are shown in Table 3.

Table 3. Metric coordinates for P-set elements in N_n

Vertices	**Codes**
$p_l; l = 1$	$(1, h, l)$
$p_l; 2 \leq l \leq h$	$(l-2, h-l+1, l)$
$p_l; l = h+1$	$(l-2, h-l+1, 2h-l+1)$
$p_l; l = h+2$	$(l-2, l-h-1, 2h-l+1)$
$p_l; h+3 \leq l \leq 2h+1$	$(2h-l+3, l-h-1, 2h-l+1)$

For the Q-set elements, the metric coordinates are $\zeta(q_l|\mathbb{D}) = \zeta(p_l|\mathbb{D}) + (1, 1, 1)$ for $1 \leq l \leq n$. For the R-set elements, the metric coordinates are $\zeta(r_l|\mathbb{D}) = \zeta(p_l|\mathbb{D}) + (2, 2, 2)$ for $1 \leq l \leq n$. Finally, for the ST-set elements (i.e., $ST = \{s_l, t_l : 1 \leq l \leq n\}$), the metric coordinates for vertices $\{s_l : 1 \leq l \leq n\}$ are $\zeta(s_l|\mathbb{D}) = \zeta(p_l|\mathbb{D}) + (3, 3, 3)$ for $1 \leq l \leq n$ and the metric coordinates for vertices $\{t_l : 1 \leq l \leq n\}$ are shown in Table 4.

Table 4. Metric coordinates for vertices $\{t_l : 1 \leq l \leq n\}$ in N_n

Vertices	**Codes**
$t_l; l = 1$	$(4, h+3, 5)$
$t_l; 2 \leq l \leq h$	$(l+2, h-l+4, l+4)$
$t_l; l = h+1$	$(h+3, 4, h+3)$
$t_l; h+2 \leq l \leq 2h$	$(2h-l+6, l-h+3, 2h-l+4)$
$t_l; l = 2h+1$	$(2h-l+6, l-h+3, 4)$

The above list of metric coordinates confirms that $|\mathbb{P}| = |\mathbb{Q}| = |\mathbb{R}| = |\mathbb{S}| = |\mathbb{L}| = n$ and each of $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{S}$, and $\mathbb{L}$ are pairwise disjoint. Thus, we conclude that there exist a set $\mathbb{D}$ with cardinality 3 in N_n, corresponding to which no pair of distinct vertices in N_n have the same metric coordinates in N_n, indicating that $\dim_v(N_n) \leq 3$. Next, for the reverse inequality, i.e., $\dim_v(N_n) \geq 3$, we demonstrate that no set $\mathbb{D}$ with cardinality 2 is a resolving set for N_n. To prove this, suppose on contrary that $\dim_v(N_n) = 2$. Then, for the set $\mathbb{D}$ with cardinality 2, we have almost the same list of possibilities and contradictions as we obtained for Case (I). Hence, we have $\dim_v(N_n) = 3$ as well in this case, which proves the theorem. $\square$

Next, in accordance with independent resolving set, we have the following corollary.

Corollary 3.1. *N_n with $n \geq 6$ has an independent resolving set $\mathbb{D}$ with cardinality 3.*

4 Metric dimension of a convex polytope N_n^1

In this section, we present an interesting family of the planar graphs, denoted by N_n^1, which is derived from N_n by adding some new edges to it. For this family, we discuss some of its basic characteristics and determine its basis set, as well as metric dimension.

The Graph of N_n^1 : The convex polytope N_n^1 is obtained from N_n by inserting n new edges in the graph N_n between the vertices s_l and s_{l+1} for $1 \leq l \leq n$. It has a vertex set and an edge set with cardinality $5n$ and $7n$, respectively. It has 8-sides faces and 3-sides faces each with cardinality n. Further, it has a face consisting of $2n$-sides and a face having n-sides (see Figure 2). The set of edges and vertices of N_n^1 are depicted separately by $E(N_n^1)$ and $V(N_n^1)$, where $V(N_n^1) = V(N_n)$ and $E(N_n^1) = V(N_n) \cup \{s_l s_{l+1} : 1 \leq l \leq n\}$.

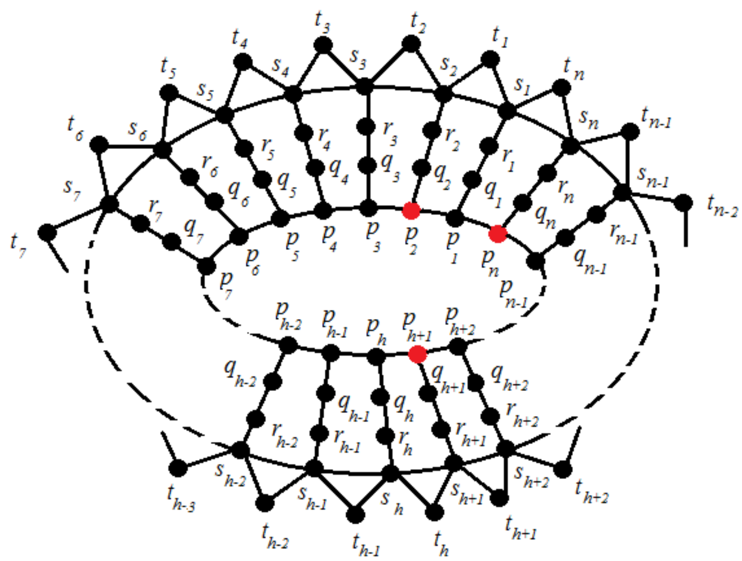

Figure 2. The graph N_n^1

The elements of the set $P = \{p_l : 1 \leq l \leq n\}$ in N_n^1, are called P-set elements; the elements of the set $Q = \{q_l : 1 \leq l \leq n\}$ in N_n^1, are called Q-set elements; the elements of the set $R = \{r_l : 1 \leq l \leq n\}$ in N_n^1, are called R-set elements; the elements of the set $S = \{s_l : 1 \leq l \leq n\}$ in N_n^1, are called S-set elements; and the elements of the set $T = \{t_l : 1 \leq l \leq n\}$ in N_n^1, are called T-set elements. Next, we are ready to determine the basis set, as well as the metric dimension for the planar graph N_n^1.

Theorem 4.1. *For the planar graph N_n^1 with $n \geq 6$, we have* $\dim_v(N_n^1) = 3$.

Proof. We prove this theorem in two cases depending upon n.

Case (I) $n \equiv 0 \pmod 2$.

This means that $n = 2h$, where $h \in \mathbb{Z}^+$ and $h \geq 3$. Let $\mathbb{D} = \{p_2, p_{h+1}, p_n\}$ be a subset of $V(N_n^1)$ with three distinct vertices chosen from P-set elements. To complete the proof for this case, when n is even, we need to show that $\mathbb{D}$ is a basis set for the planar graph N_n^1. For upper bound, we give the metric coordinates for each vertex of N_n with respect to the set $\mathbb{D}$.

For the P-set elements, the metric coordinates are shown in Table 5.

Table 5. Metric coordinates for P-set elements in N_n^1

Vertices	Codes
$p_l; l = 1$	$(1, h, l)$
$p_l; 2 \leq l \leq h$	$(l-2, h-l+1, l)$
$p_l; l = h+1$	$(l-2, h-l+1, 2h-l)$
$p_l; h+2 \leq l \leq 2h$	$(2h-l+2, l-h-1, 2h-l)$

For the Q-set elements, the metric coordinates are $\zeta(q_l|\mathbb{D}) = \zeta(p_l|\mathbb{D}) + (1, 1, 1)$ for $1 \leq l \leq n$. For the R-set elements, the metric coordinates are $\zeta(r_l|\mathbb{D}) = \zeta(p_l|\mathbb{D}) + (2, 2, 2)$ for $1 \leq l \leq n$. Next, for the S-set elements, the metric coordinates are $\zeta(s_l|\mathbb{D}) = \zeta(p_l|\mathbb{D}) + (3, 3, 3)$ for $1 \leq l \leq n$. Finally, for the T-set elements, the metric coordinates are shown in Table 6.

Table 6. Metric coordinates for ST-set elements in N_n^1

Vertices	Codes
$t_l; l = 1$	$(4, h+3, 5)$
$t_l; 2 \leq l \leq h-1$	$(l+2, h-l+4, l+4)$
$t_l; l = h$	$(h+2, 4, h+3)$
$t_l; l = h+1$	$(h+3, 4, h+2)$
$t_l; h+2 \leq l \leq 2h-1$	$(2h-l+5, l-h+3, 2h-l+3)$
$t_l; l = 2h$	$(2h-l+5, l-h+3, 4)$

The above list of metric coordinates confirms that $|\mathbb{P}| = |\mathbb{Q}| = |\mathbb{R}| = |\mathbb{S}| = |\mathbb{L}| = n$ and each of $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{S}$, and $\mathbb{L}$ are pairwise disjoint. Thus, we conclude that there exist a set $\mathbb{D}$ with cardinality 3 in N_n^1, corresponding to which no pair of distinct vertices in N_n^1 have the same metric coordinates in N_n^1, indicating that $\dim_v(N_n^1) \leq 3$. Next, for the reverse inequality, i.e., $\dim_v(N_n^1) \geq 3$, we demonstrate that no set $\mathbb{D}$ with cardinality 2 is a resolving set for N_n^1. To prove this, suppose on contrary that $\dim_v(N_n^1) = 2$. Then, for the set $\mathbb{D}$ with cardinality 2, we have the list of possibilities as follows (here we take $n \geq 14$, as for $6 \leq n \leq 13$, one can easily derive the contradictions for any set $\mathbb{D}$ with cardinality 2 in N_n^1):

Resolving sets	Contradictions
$\mathbb{D}=\{p_1,p_j\}, p_j\ (2\le j\le n)$	$\zeta(q_1\|\mathbb{D})=\zeta(p_n\|\mathbb{D})$, for $2\le j\le h$, and $\zeta(p_2\|\mathbb{D})=\zeta(p_n\|\mathbb{D})$, when $j=h+1$.
$\mathbb{D}=\{q_1,q_j\}, q_j\ (2\le j\le n)$	$\zeta(q_n\|\mathbb{D})=\zeta(p_{n-1}\|\mathbb{D})$, for $2\le j\le h-1$; $\zeta(r_2\|\mathbb{D})=\zeta(q_{n-1}\|\mathbb{D})$, when $j=h$, and $\zeta(p_2\|\mathbb{D})=\zeta(p_n\|\mathbb{D})$, when $j=h+1$.
$\mathbb{D}=\{r_1,r_j\}, r_j\ (2\le j\le n)$	$\zeta(l_n\|\mathbb{D})=\zeta(s_n\|\mathbb{D})$, for $2\le j\le h$, and $\zeta(s_2\|\mathbb{D})=\zeta(s_n\|\mathbb{D})$, when $j=h+1$.
$\mathbb{D}=\{t_1,t_j\}, t_j\ (2\le j\le n)$	$\zeta(r_1\|\mathbb{D})=\zeta(s_n\|\mathbb{D})$, for $2\le j\le h-1$; $\zeta(r_2\|\mathbb{D})=\zeta(s_n\|\mathbb{D})$, when $j=h$, and $\zeta(s_2\|\mathbb{D})=\zeta(s_1\|\mathbb{D})$, when $j=h+1$.
$\mathbb{D}=\{p_1,q_j\}, q_j\ (1\le j\le n)$	$\zeta(q_n\|\mathbb{D})=\zeta(p_{n-1}\|\mathbb{D})$, for $1\le j\le h-1$; $\zeta(r_2\|\mathbb{D})=\zeta(q_{n-1}\|\mathbb{D})$, when $j=h$, and $\zeta(p_2\|\mathbb{D})=\zeta(p_n\|\mathbb{D})$, when $j=h+1$.
$\mathbb{D}=\{p_1,r_j\}, r_j\ (1\le j\le n)$	$\zeta(l_n\|\mathbb{D})=\zeta(s_n\|\mathbb{D})$, for $1\le j\le h$, and $\zeta(s_2\|\mathbb{D})=\zeta(s_n\|\mathbb{D})$, when $j=h+1$.
$\mathbb{D}=\{p_1,t_j\}, l_j\ (1\le j\le n)$	$\zeta(l_n\|\mathbb{D})=\zeta(s_n\|\mathbb{D})$, for $1\le j\le h$, and $\zeta(s_2\|\mathbb{D})=\zeta(t_n\|\mathbb{D})$, when $j=h+1$.
$\mathbb{D}=\{q_1,r_j\}, r_j\ (1\le j\le n)$	$\zeta(t_n\|\mathbb{D})=\zeta(s_n\|\mathbb{D})$, for $1\le j\le h$, and $\zeta(s_2\|\mathbb{D})=\zeta(s_n\|\mathbb{D})$, when $j=h+1$.
$\mathbb{D}=\{q_1,t_j\}, t_j\ (1\le j\le n)$	$\zeta(t_n\|\mathbb{D})=\zeta(s_n\|\mathbb{D})$, for $1\le j\le h$, and $\zeta(s_2\|\mathbb{D})=\zeta(l_n\|\mathbb{D})$, when $j=h+1$.
$\mathbb{D}=\{r_1,t_j\}, l_j\ (1\le j\le n)$	$\zeta(t_n\|\mathbb{D})=\zeta(s_n\|\mathbb{D})$, for $1\le j\le h$, and $\zeta(s_2\|\mathbb{D})=\zeta(t_n\|\mathbb{D})$, when $j=h+1$.

The list of contradictions as mentioned above confirms that no set $\mathbb{D}$ consisting two elements forms a resolving set for $V(N_n^1)$ indicating that $\dim_v(N_n^1)=3$ in this case.

Case (II) $n\equiv 1(\bmod 2)$.

This means that $n=2h+1$, where $h\in\mathbb{Z}^+$ and $h\ge 3$. Let $\mathbb{D}=\{p_2,p_{h+1},p_n\}$ be a subset of $V(N_n^1)$ with three distinct vertices chosen from P-set elements. To complete the proof for this case, when n is odd, we need to show that $\mathbb{D}$ is a basis set for the planar graph N_n^1. For upper bound, we give the metric coordinates for each vertex of N_n^1 with respect to the set $\mathbb{D}$.

For the P-set elements, the metric coordinates are shown in Table 7.

Table 7. Metric coordinates for P-set elements in N_n^1

Vertices	Codes
$p_l;\ l=1$	$(1,h,l)$
$p_l;\ 2\le l\le h$	$(l-2,h-l+1,l)$
$p_l;\ l=h+1$	$(l-2,h-l+1,2h-l+1)$
$p_l;\ l=h+2$	$(l-2,l-h-1,2h-l+1)$
$p_l;\ h+3\le l\le 2h+1$	$(2h-l+3,l-h-1,2h-l+1)$

For the Q-set elements, the metric coordinates are $\zeta(q_l|\mathbb{D})=\zeta(p_l|\mathbb{D})+(1,1,1)$ for $1\le l\le n$. For the R-set elements, the metric coordinates are $\zeta(r_l|\mathbb{D})=\zeta(p_l|\mathbb{D})+(2,2,2)$ for $1\le l\le n$.

Next, for the S-set elements, the metric coordinates are $\zeta(s_l|\mathbb{D}) = \zeta(p_l|\mathbb{D}) + (3,3,3)$ for $1 \leq l \leq n$. Finally, for the T-set elements, the metric coordinates are shown in Table 8.

Table 8. Metric coordinates for T-set elements in N_n^1

Vertices	**Codes**
$t_l; l = 1$	$(4, h+3, 5)$
$t_l; 2 \leq l \leq h$	$(l+2, h-l+4, l+4)$
$t_l; l = h+1$	$(h+3, 4, h+3)$
$t_l; h+2 \leq l \leq 2h$	$(2h-l+6, l-h+3, 2h-l+4)$
$t_l; l = 2h+1$	$(2h-l+6, l-h+3, 4)$

The above list of metric coordinates confirms that $|\mathbb{P}| = |\mathbb{Q}| = |\mathbb{R}| = |\mathbb{S}| = |\mathbb{L}| = n$ and each of $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{S}$, and $\mathbb{L}$ are pairwise disjoint. Thus, we conclude that there exist a set $\mathbb{D}$ with cardinality 3 in N_n^1, corresponding to which no pair of distinct vertices in N_n^1 have the same metric coordinates in N_n^1, indicating that $\dim_v(N_n^1) \leq 3$. Next, for the reverse inequality, i.e., $\dim_v(N_n^1) \geq 3$, we demonstrate that no set $\mathbb{D}$ with cardinality 2 is a resolving set for N_n^1. To prove this, suppose on contrary that $\dim_v(N_n^1) = 2$. Then, for the set $\mathbb{D}$ with cardinality 2, we have almost the same list of possibilities and contradictions as we obtained for Case (I).

Hence, we have $\dim_v(N_n^1) = 3$ as well in this case, which proves the theorem. □

Next, in accordance with independent resolving set, we have the following corollary.

Corollary 4.1. *N_n^1 with $n \geq 6$ has an independent resolving set $\mathbb{D}$ with cardinality* 3.

5 Metric dimension of a convex polytope N_n^2

In this section, we again present an interesting family of the planar graphs, denoted by N_n^2, which is derived from N_n by adding some new edges to it. For this family, we discuss some of its basic characteristics and determine its basis set, as well as metric dimension.

The Graph of N_n^2 : The convex polytope N_n^2 is obtained from the NCL N_n by inserting n new edges in the graph N_n between the vertices s_l and p_{l+1} for $1 \leq l \leq n$. It has a vertex set and an edge set with cardinality $5n$ and $7n$, respectively. It has 5-sides faces and 6-sides faces each with cardinality n. Further, it has a face consisting of $2n$-sides and a face having n-sides (see Figure 3). The set of edges and vertices of N_n^2 are depicted separately by $E(N_n^2)$ and $V(N_n^2)$, where $V(N_n^2) = V(N_n)$ and $E(N_n^2) = E(N_n) \cup \{s_l p_{l+1} : 1 \leq l \leq n\}$.

The elements of the set $P = \{p_l : 1 \leq l \leq n\}$ in N_n^2, are called P-set elements, the elements of the set $Q = \{q_l : 1 \leq l \leq n\}$ in N_n^2 are called Q-set elements, the elements of the set $R = \{r_l : 1 \leq l \leq n\}$ in N_n^2 are called R-set elements, the elements of the set $S = \{s_l : 1 \leq l \leq n\}$ in N_n^2 are called S-set elements, and the elements of the set $T = \{s_l : 1 \leq l \leq n\}$ in N_n^2 are called T-set elements. Next, we are ready to determine the basis set as well metric dimension for the planar graph N_n^2.

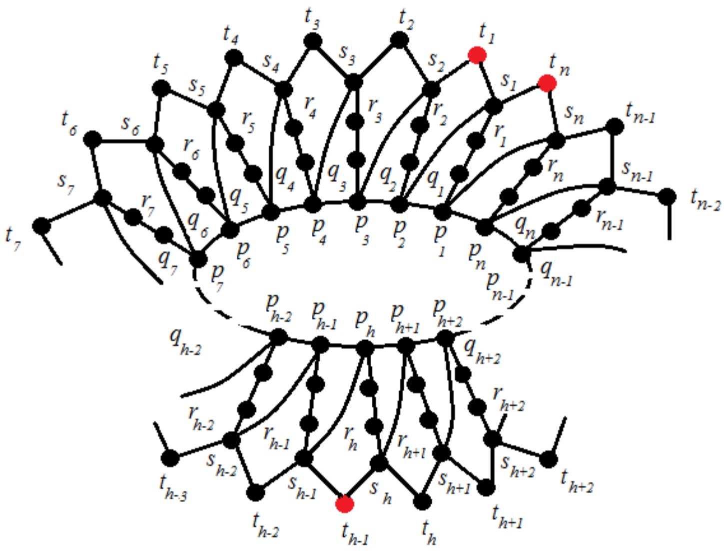

Figure 3. The graph N_n^2

Theorem 5.1. *For the planar graph N_n^2 with $n \geq 6$, we have $\dim_v(N_n^2) = 3$.*

Proof. For $6 \leq n \leq 10$, it can be easily verified that the metric dimension of N_n^2 is 3 (using the resolving set $\mathbb{D} = \{l_1, l_{h+1}, l_n\}$). Now, for $n \geq 11$, we prove this theorem in two cases depending upon n.

Case (I) $n \equiv 0 \pmod 2$.

This means that $n = 2h$, where $h \in \mathbb{Z}^+$ and $h \geq 3$. Let $\mathbb{D} = \{t_1, t_{h+1}, t_n\}$ be a subset of $V(N_n^2)$ with three distinct vertices chosen from T-set elements. To complete the proof for this case, when n is even, we need to show that $\mathbb{D}$ is a basis set for the planar graph N_n^2. For upper bound, we give the metric coordinates for each vertex of N_n with respect to the set $\mathbb{D}$.

For the P-set elements, the metric coordinates are shown in Table 9. For the Q-set elements, the metric coordinates are shown in Table 10. For the R-set elements, the metric coordinates are shown in Table 11. For the S-set elements, the metric coordinates are shown in Table 12. Finally, for the T-set elements, the metric coordinates are shown in Table 13.

Table 9. Metric coordinates for P-set elements in N_n^2

Vertices	**Codes**
$p_l;\, l = 1$	$(3, h, 2)$
$p_l;\, l = 2$	$(2, h+1, 2)$
$p_l;\, 3 \leq l \leq h+1$	$(l-1, h-l+4, l)$
$p_l;\, l = h+2$	$(h+1, 2, h+1)$
$p_l;\, h+3 \leq l \leq 2h$	$(2h-l+4, l-h-1, 2h-l+3)$

Table 10. Metric coordinates for Q-set elements in N_n^2

Vertices	**Codes**
$q_l; l = 1$	$(3, h+1, 3)$
$q_l; l = 2$	$(3, h+2, 3)$
$q_l; 3 \leq l \leq h$	$(l, h-l+5, l+1)$
$q_l; l = h+1$	$(h+1, 3, h+2)$
$q_l; l = h+2$	$(h+2, 3, h+2)$
$q_l; h+3 \leq l \leq 2h-1$	$(2h-l+5, l-h, 2h-l+4)$
$q_l; l = 2h$	$(2h-l+5, l-h, 3)$

Table 11. Metric coordinates for R-set elements in N_n^2

Vertices	**Codes**
$r_l; l = 1$	$(2, h+2, 2)$
$r_l; l = 2$	$(2, h+3, 4)$
$r_l; 3 \leq l \leq h-1$	$(l+1, h-l+5, l+3)$
$r_l; l = h$	$(h+1, 4, h+3)$
$r_l; l = h+1$	$(h+2, 2, h+3)$
$r_l; l = h+2$	$(h+3, 2, h+2)$
$r_l; h+3 \leq l \leq 2h-2$	$(2h-l+5, l-h+1, 2h-l+4)$
$r_l; l = 2h-1$	$(2h-l+5, l-h+1, 4)$
$r_l; l = 2h$	$(2h-l+5, l-h+1, 2)$

Table 12. Metric coordinates for S-set elements in N_n^2

Vertices	**Codes**
$s_l; l = 1$	$(1, h+3, 1)$
$s_l; l = 2$	$(1, h+2, 3)$
$s_l; l = 3$	$(3, h+1, 5)$
$s_l; 4 \leq l \leq h-1$	$(l+1, h-l+4, l+2)$
$s_l; l = h$	$(h+1, 3, h+2)$
$s_l; l = h+1$	$(h+2, 1, h+2)$
$s_l; l = h+2$	$(h+2, 1, h+1)$
$s_l; h+3 \leq l \leq 2h-2$	$(2h-l+4, l-h+1, 2h-l+3)$
$s_l; l = 2h-1$	$(2h-l+4, l-h+1, 3)$
$s_l; l = 2h$	$(3, l-h+1, 1)$

Table 13. Metric coordinates for T-set elements in N_n^2

Vertices	**Codes**
$t_l;\, l=2$	$(2,h+2,4)$
$t_l;\, l=3$	$(4,h+1,6)$
$t_l;\, 4\le l\le h-2$	$(l+2,h-l+4,l+3)$
$t_l;\, l=h-1$	$(h+1,4,h+2)$
$t_l;\, l=h$	$(h+2,2,h+3)$
$t_l;\, l=h+2$	$(h+2,2,h+1)$
$t_l;\, l=h+3$	$(h+1,4,h)$
$t_l;\, h+4\le l\le 2h-3$	$(2h-l+4,l-h+2,2h-l+3)$
$t_l;\, l=2h-2$	$(2h-l+4,l-h+2,4)$
$t_l;\, l=2h-1$	$(4,l-h+2,2))$

The above list of metric coordinates confirms that $|\mathbb{P}| = |\mathbb{Q}| = |\mathbb{R}| = |\mathbb{S}| = |\mathbb{L}| = n$ and each of $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{S}$, and $\mathbb{L}$ are pairwise disjoint. Thus, we conclude that there exist a set $\mathbb{D}$ with cardinality 3 in N_n^2, corresponding to which no pair of distinct vertices in N_n^2 have the same metric coordinates in N_n^2, indicating that $\dim_v(N_n^2) \le 3$. Next, for the reverse inequality, i.e., $\dim_v(N_n^2) \ge 3$, we demonstrate that no set $\mathbb{D}$ with cardinality 2 is a resolving set for N_n^2. To prove this, suppose on contrary that $\dim_v(N_n^2) = 2$. Then, for the set $\mathbb{D}$ with cardinality 2, we have the list of possibilities as follows (here we take $n \ge 14$, as for $6 \le n \le 13$, one can easily derive the contradictions for any set $\mathbb{D}$ with cardinality 2 in N_n^2):

Resolving sets	**Contradictions**
$\mathbb{D}=\{q_1,q_j\}$, q_j $(2\le j\le n)$	$\zeta(s_n\|\mathbb{D}) = \zeta(p_n\|\mathbb{D})$, for $2\le j\le h$, and $\zeta(p_2\|\mathbb{D}) = \zeta(p_n\|\mathbb{D})$, when $j=h+1$.
$\mathbb{D}=\{r_1,r_j\}$, r_j $(2\le j\le n)$	$\zeta(s_n\|\mathbb{D}) = \zeta(p_n\|\mathbb{D})$, for $2\le j\le h$, and $\zeta(p_2\|\mathbb{D}) = \zeta(p_n\|\mathbb{D})$, when $j=h+1$.
$\mathbb{D}=\{t_1,t_j\}$, t_j $(2\le j\le n)$	$\zeta(r_1\|\mathbb{D}) = \zeta(l_n\|\mathbb{D})$, for $2\le j\le h$, and $\zeta(s_n\|\mathbb{D}) = \zeta(q_1\|\mathbb{D})$, when $j=h+1$.
$\mathbb{D}=\{q_1,r_j\}$, r_j $(1\le j\le n)$	$\zeta(s_n\|\mathbb{D}) = \zeta(p_n\|\mathbb{D})$, for $1\le j\le h$, and $\zeta(p_2\|\mathbb{D}) = \zeta(p_n\|\mathbb{D})$, when $j=h+1$.
$\mathbb{D}=\{q_1,s_j\}$, s_j $(1\le j\le n)$	$\zeta(r_n\|\mathbb{D}) = \zeta(l_{n-1}\|\mathbb{D})$, for $j=1$; $\zeta(s_n\|\mathbb{D}) = \zeta(p_n\|\mathbb{D})$, when $2\le j\le h-2$, and $\zeta(s_{n-1}\|\mathbb{D}) = \zeta(q_n\|\mathbb{D})$, when $h-1\le j\le h+1$.
$\mathbb{D}=\{r_1,s_j\}$, s_j $(1\le j\le n)$	$\zeta(p_2\|\mathbb{D}) = \zeta(l_n\|\mathbb{D})$, for $j=1$; $\zeta(s_n\|\mathbb{D}) = \zeta(p_n\|\mathbb{D})$, when $2\le j\le h-2$, and $\zeta(s_{n-1}\|\mathbb{D}) = \zeta(q_n\|\mathbb{D})$, when $h-1\le j\le h+1$.

The list of contradictions as mentioned above confirms that no set $\mathbb{D}$ consisting two elements forms a resolving set for $V(N_n^2)$ indicating that $\dim_v(N_n^2) = 3$ in this case.

Case (II) $n \equiv 1 \pmod 2$.

This means that $n = 2h + 1$, where $h \in \mathbb{Z}^+$ and $h \geq 3$. Let $\mathbb{D} = \{t_1, t_{h+1}, t_n\}$ be a subset of $V(N_n^2)$ with three distinct vertices chosen from P-set elements. To complete the proof for this case, when n is odd, we need to show that $\mathbb{D}$ is a basis set for the planar graph N_n^2. For upper bound, we give the metric coordinates for each vertex of N_n^2 with respect to the set $\mathbb{D}$. For the P-set elements, the metric coordinates are shown in Table 14.

Table 14. Metric coordinates for P-set elements in N_n^2

Vertices	**Codes**
$p_l; l = 1$	$(3, h, 2)$
$p_l; l = 2$	$(2, h+1, 2)$
$p_l; 3 \leq l \leq h+1$	$(l-1, h-l+4, l)$
$p_l; l = h+2$	$(h+1, 2, h+2)$
$p_l; h+3 \leq l \leq 2h+1$	$(2h-l+5, l-h-1, 2h-l+4)$

For the Q-set elements, the metric coordinates are shown in Table 15.

Table 15. Metric coordinates for Q-set elements in N_n^2

Vertices	**Codes**
$q_l; l = 1$	$(3, h+1, 3)$
$q_l; l = 2$	$(3, h+2, 3)$
$q_l; 3 \leq l \leq h$	$(l, h-l+5, l+1)$
$q_l; l = h+1$	$(h+1, 3, h+2)$
$q_l; l = h+2$	$(h+2, 3, h+3)$
$q_l; h+3 \leq l \leq 2h$	$(2h-l+6, l-h, 2h-l+5)$
$q_l; l = 2h+1$	$(2h-l+6, l-h, 3)$

For the R-set elements, the metric coordinates are shown in Table 16.

Table 16. Metric coordinates for R-set elements in N_n^2

Vertices	**Codes**
$r_l; l = 1$	$(2, h+2, 2)$
$r_l; l = 2$	$(2, h+3, 4)$
$r_l; 3 \leq l \leq h-1$	$(l+1, h-l+5, l+3)$
$r_l; l = h$	$(h+1, 4, h+3)$
$r_l; l = h+1$	$(h+2, 2, h+3)$
$r_l; l = h+2$	$(h+3, 2, h+4)$
$r_l; h+3 \leq l \leq 2h-1$	$(2h-l+7, l-h+1, 2h-l+5)$
$r_l; l = 2h$	$(6, l-h+1, 4)$
$r_l; l = 2h+1$	$(4, l-h+1, 2)$

For the S-set elements, the metric coordinates are shown in Table 17.

Table 17. Metric coordinates for S-set elements in N_n^2

Vertices	Codes
$s_l; l = 1$	$(1, h + 3, 1)$
$s_l; l = 2$	$(1, h + 2, 3)$
$s_l; l = 3$	$(3, h + 1, 5)$
$s_l; 4 \leq l \leq h - 1$	$(l + 1, h - l + 4, l + 2)$
$s_l; l = h$	$(h + 1, 3, h + 3)$
$s_l; l = h + 1$	$(h + 2, 1, h + 3)$
$s_l; l = h + 2$	$(h + 3, 1, h + 2)$
$s_l; h + 3 \leq l \leq 2h - 1$	$(2h - l + 5, l - h + 1, 2h - l + 4)$
$s_l; l = 2h$	$(2h - l + 5, l - h + 1, 3)$
$s_l; l = 2h + 1$	$(3, l - h + 1, 1)$

Finally, for the T-set elements, the metric coordinates are shown in Table 18.

Table 18. Metric coordinates for T-set elements in N_n^2

Vertices	Codes
$t_l; l = 2$	$(2, h + 2, 4)$
$t_l; l = 3$	$(4, h + 1, 6)$
$t_l; 4 \leq l \leq h - 2$	$(l + 2, h - l + 4, l + 3)$
$t_l; l = h - 1$	$(h + 1, 4, h + 2)$
$t_l; l = h$	$(h + 2, 2, h + 3)$
$t_l; l = h + 2$	$(h + 3, 2, h + 2)$
$t_l; l = h + 3$	$(h + 2, 4, h + 1)$
$t_l; h + 4 \leq l \leq 2h - 2$	$(2h - l + 5, l - h + 2, 2h - l + 4)$
$t_l; l = 2h - 1$	$(2h - l + 5, l - h + 2, 4)$
$t_l; l = 2h$	$(4, l - h + 2, 2)$

The above list of metric coordinates confirms that $|\mathbb{P}| = |\mathbb{Q}| = |\mathbb{R}| = |\mathbb{S}| = |\mathbb{L}| = n$ and each of $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{S}$, and $\mathbb{L}$ are pairwise disjoint. Thus, we conclude that there exist a set $\mathbb{D}$ with cardinality 3 in N_n^2, corresponding to which no pair of distinct vertices in N_n^2 have the same metric coordinates in N_n^2, indicating that $\dim_v(N_n^2) \leq 3$. Next, for the reverse inequality, i.e., $\dim_v(N_n^2) \geq 3$, we demonstrate that no set $\mathbb{D}$ with cardinality 2 is a resolving set for N_n^2. To prove this, suppose on contrary that $\dim_v(N_n^2) = 2$. Then, for the set $\mathbb{D}$ with cardinality 2, we have almost the same list of possibilities and contradictions as we obtained for Case (I). Hence, we have $\dim_v(N_n^2) = 3$ as well in this case, which proves the theorem. $\square$

Next, in accordance with independent resolving set, we have the following corollary.

Corollary 5.1. *N_n^2 with $n \geq 6$ has an independent resolving set $\mathbb{D}$ with cardinality 3.*

6 Metric dimension of a convex polytope N_n^3

In this section, we again present an interesting family of the planar graphs, denoted by N_n^2, which is derived from N_n by adding some new edges to it. For this family, we discuss some of its basic characteristics and determine its basis set, as well as metric dimension.

The Graph of N_n^3: The convex polytope N_n^3 is obtained from the NCL N_n by inserting n new edges in the graph N_n between the vertices q_l and p_{l+1} for $1 \leq l \leq n$. It has a vertex set and an edge set with cardinality $5n$ and $7n$ respectively. It has 3-sides faces and 8-sides faces each with cardinality n. Further, it has a face consisting of $2n$-sides and a face having n-sides (see Figure 4). The set of edges and vertices of N_n^3 are depicted separately by $E(N_n^3)$ and $V(N_n^3)$, where $V(N_n^3) = V(N_n)$ and $E(N_n^3) = E(N_n) \cup \{q_l p_{l+1} : 1 \leq l \leq n\}$.

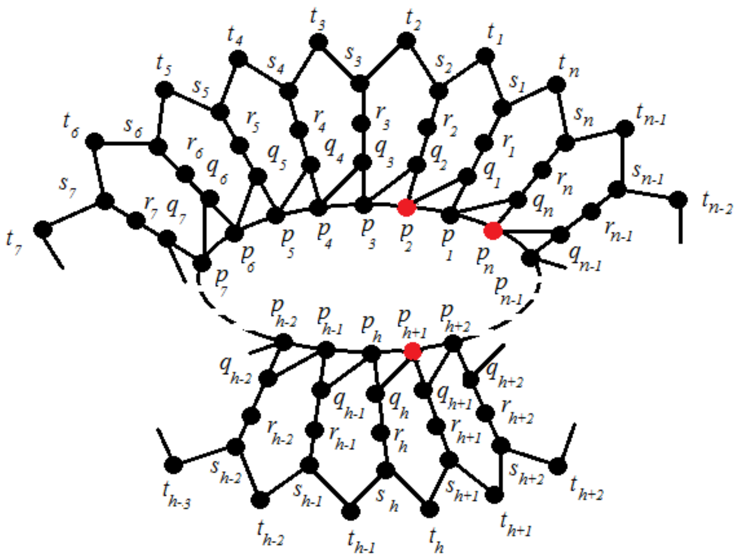

Figure 4. The graph N_n^3

The elements of the set $P = \{p_l : 1 \leq l \leq n\}$ in N_n^3, are called P-set elements, the elements of the set $Q = \{q_l : 1 \leq l \leq n\}$ in N_n^3, are called Q-set elements, the elements of the set $R = \{r_l : 1 \leq l \leq n\}$ in N_n^3, are called R-set elements, the elements of the set $S = \{s_l : 1 \leq l \leq n\}$ in N_n^3, are called S-set elements, and the elements of the set $T = \{t_l : 1 \leq l \leq n\}$ in N_n^3, are called T-set elements. Next, we are ready to determine the basis set as well metric dimension for the planar graph N_n^3.

Theorem 6.1. *For the planar graph N_n^3 with $n \geq 6$, we have $\dim_v(N_n^3) = 3$.*

Proof. Let $\mathbb{D} = \{p_2, p_{h+1}, p_n\} \subset \mathbb{V}(N_n^3)$. Then, by using a similar argument as utilized in Theorem 5.1, we prove that the set $\mathbb{D}$ is the minimum resolving set for N_n^3, and in this way $\dim_v(N_n^3) = 3$, which proves the theorem. $\square$

Next, in accordance with independent resolving set, we have the following corollary.

Corollary 6.1. *N_n^3 with $n \geq 6$ has an independent resolving set $\mathbb{D}$ with cardinality 3.*

7 Conclusion and discussion

In this article, the metric dimension of a planar graph N_n and of three classes of convex polytopes obtained from N_n have been studied. For these classes of convex polytopes, we proved that $V(N_n) = V(N_n^1) = V(N_n^2) = V(N_n^3)$ and $\dim(N_n) = \dim(N_n^1) = \dim(N_n^2) = \dim(N_n^3) = 3$ (a partial answer to the problem raised in [9]). We also proved that the resolving sets for all of these convex polytopes are independent. Future work will expand on these results to determine the edge metric dimension, mixed metric dimension, and other variants of metric dimension for each of these planar graphs.

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