Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2023, Volume 29, Number 3, 421–425
DOI: 10.7546/nntdm.2023.29.3.421-425

A note on a generalization of Riordan's combinatorial identity via a hypergeometric series approach

Dongkyu Lim

Department of Mathematics Education, Andong National University
Andong 36729, Republic of Korea
e-mail: dklim@anu.ac.kr

Received: 26 October 2022 **Revised:** 25 April 2023
Accepted: 1 June 2023 **Online First:** 6 June 2023

Abstract: In this note, an attempt has been made to generalize the well-known and useful Riordan's combinatorial identity via a hypergeometric series approach.

Keywords: Combinatorial identity, Hypergeometric series, Hypergeometric identities, Riordan identity, Reed Dawson identity, Knuth's old sum.
2020 Mathematics Subject Classification: Primary 05A10, 33C20; Secondary 40A25.

1 Introduction

We begin by recalling the definition of Gauss's hypergeometric series or simply a hypergeometric series defined as follows [5]:

$$ {}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix};z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}. \tag{1}$$

Here $(a)_n$ is the well-known Pochhammer's symbol defined for any complex number $a\ (\neq 0)$ by

$$(a)_n = \begin{cases} 1, & n = 0, \\ a(a+1)\cdots(a+n-1), & n \in \mathbb{N}. \end{cases}$$

The series defined in (1) is convergent for all values of z if $|z| < 1$ and divergent if $|z| > 1$. When $z = 1$, the series is convergent if $\Re(c - a - b) > 0$ and divergent if $\Re(c - a - b) \leq 0$. Also, when $z = -1$, the series is absolutely convergent if $\Re(c - a - b) > 0$ and is convergent but not absolutely if $-1 < \Re(c - a - b) \leq 0$ and divergent if $\Re(c - a - b) < 1$.

It should be remarked here that whenever a hypergeometric series reduces to the gamma function, the results are very important from the application point of view. Thus the classical summation theorems such as those of Gauss, Gauss second, Kummer, Bailey, and a few terminating summations play an important role.

However, in our present investigation, we shall require the following hypergeometric summation theorem and identities [3,5] each valid for $n \in \mathbb{N}_0$.

Gauss second summation theorem [5]

$$ {}_2F_1\left[\begin{array}{c} a,\ b \\ \frac{1}{2}(a+b+1) \end{array}; \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}, \tag{2} $$

Hypergeometric identities [3,5]

$$ {}_2F_1\left[\begin{array}{c} -2n,\ a \\ 2a \end{array}; 2\right] = \frac{\left(\frac{1}{2}\right)_n}{\left(a+\frac{1}{2}\right)_n}, \tag{3} $$

$$ {}_2F_1\left[\begin{array}{c} -2n-1,\ a \\ 2a \end{array}; 2\right] = 0, \tag{4} $$

$$ {}_2F_1\left[\begin{array}{c} -2n,\ a \\ 2a+1 \end{array}; 2\right] = \frac{\left(\frac{1}{2}\right)_n}{\left(a+\frac{1}{2}\right)_n}, \tag{5} $$

$$ {}_2F_1\left[\begin{array}{c} -2n-1,\ a \\ 2a+1 \end{array}; 2\right] = \frac{1}{2a+1}\frac{\left(\frac{3}{2}\right)_n}{\left(a+\frac{3}{2}\right)_n}. \tag{6} $$

On the other hand, one of the applications of the hypergeometric series is to solve binomial sums [7]. Using this method, the given sum is first converted into a hypergeometric series. This is accomplished by combining any terms that are polynomial in the summation index with binomials, expanding the binomials into factorials, and transferring the factorials into Pochhammer symbols. If successfully converted, the resulting hypergeometric series is compared with known summation theorems and a closed-form sum may be obtained when a suitable match is found.

In this context, we first mention here the following famous combinatorial sum known as Knuth's old sum viz.

$$ \sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{-k}\binom{2k}{k} = \begin{cases} 2^{-2n}\binom{2n}{n} = \dfrac{\left(\frac{1}{2}\right)_n}{(1)_n}, & n \text{ is even}, \\ 0, & n \text{ is odd}. \end{cases} \tag{7} $$

The above combinatorial identity is, in the literature, also known as Reed Dawson identity because Reed Dawson presented this identity in a *private communication* to Riordan who recorded it in his well-known book [6, p. 71].

It is not out of place to mention here that from time to time, several different proofs of the above sums have been given in the literature. In this regard, we would like to refer to a survey paper by Prodinger [4]. In 1974, Andrews [1, p. 478] established the identity given in (7) by employing Gauss second summation theorem (2). Later, in 2004, Choi et al. [2] established by utilizing the hypergeometric identities (3) and (4).

Riordan [6], by the method of inverse relations, established the following identity closely related to (7) viz.

$$\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}2^{-k}\binom{2k}{k} = \begin{cases} \frac{\left(\frac{3}{2}\right)_n}{(1)_n}, & n \text{ is even,} \\ \frac{\left(\frac{3}{2}\right)_n}{(1)_n}, & n \text{ is odd.} \end{cases} \tag{8}$$

The aim of this note is to provide a natural generalization of the Riordan identity (8) via a hypergeometric series approach by employing the hypergeometric identities (5) and (6). The same is given in the next section.

2 Generalization of Riordan identity

In this section, we shall establish the natural generalization of the Riordan identity (8) asserted in the following theorem.

Theorem 2.1. *For $i \in \mathbb{N}$, the following identity holds true.*

$$\sum_{k=0}^{n}(-1)^k\binom{n+i}{k+i}2^{-k}\binom{2k}{k}\frac{(i+1)_k}{(2)_k} = \begin{cases} \binom{2n+i}{i}\frac{\left(\frac{1}{2}\right)_n}{(1)_n}, & n \textit{ is even,} \\ \frac{1}{2}\binom{2n+i+1}{i}\frac{\left(\frac{3}{2}\right)_n}{(2)_n}, & n \textit{ is odd.} \end{cases} \tag{9}$$

Proof. In order to establish the result (9) asserted in the theorem, we proceed as follows. Denoting the left-hand side of (9) by S_i, we have

$$S_i = \sum_{k=0}^{n}(-1)^k\binom{n+i}{k+i}2^{-k}\binom{2k}{k}\frac{(i+1)_k}{(2)_k}.$$

Upon converting binomial coefficients into Pochhammer symbols viz.

$$\binom{2k}{k} = 2^{2k}\frac{\left(\frac{1}{2}\right)_k}{(1)_k},$$

and

$$\binom{n+i}{k+i} = \frac{(-1)^k\Gamma(n+i+1)}{\Gamma(n+1)\Gamma(i+1)}\frac{(-n)_k}{(i+1)_k}.$$

we have, after some simplification

$$S_i = \frac{\Gamma(n+i+1)}{\Gamma(n+1)\Gamma(i+1)}\sum_{k=0}^{n}(-1)^k\frac{(-n)_k\left(\frac{1}{2}\right)_k}{(2)_k k! 2^k}.$$

Finally, summing up the series with the help of the equation (1), we have

$$S_i = \frac{\Gamma(n+i+1)}{\Gamma(n+1)\Gamma(i+1)} \; {}_2F_1\left[\begin{matrix} -n, \, \frac{1}{2} \\ 2 \end{matrix} ; 2\right].$$

Now, changing n to $2n$ and $2n+1$, respectively, and using the results (5) and (6), we easily arrive at the right-hand side of (9).

This completes the proof of the generalization of the Riordan identity (9) asserted in the theorem. $\square$

3 Corollaries

In this section, we shall mention known as well as some new identities of our main findings.

Corollary 3.1. *In* (9)*, if we set $i = 1$, we at once recover the Riordan identity* (8).

Corollary 3.2. *In* (9)*, if we set $i = 2$, we get the following interesting identity.*

$$\sum_{k=0}^{n} (-1)^k \binom{n+2}{k+2} 2^{-k} \binom{2k}{k} \frac{(3)_k}{(2)_k} = \begin{cases} \binom{2n+2}{2} \frac{\left(\frac{1}{2}\right)_n}{(1)_n}, & n \text{ is even}, \\ \frac{1}{2}\binom{2n+3}{2} \frac{\left(\frac{3}{2}\right)_n}{(2)_n}, & n \text{ is odd}. \end{cases} \tag{10}$$

Corollary 3.3. *In* (9)*, if we set $i = 3$, we get the following interesting identity.*

$$\sum_{k=0}^{n} (-1)^k \binom{n+3}{k+3} 2^{-k} \binom{2k}{k} \frac{(4)_k}{(2)_k} = \begin{cases} \binom{2n+3}{3} \frac{\left(\frac{1}{2}\right)_n}{(1)_n}, & n \text{ is even}, \\ \frac{1}{2}\binom{2n+4}{3} \frac{\left(\frac{3}{2}\right)_n}{(2)_n}, & n \text{ is odd}. \end{cases} \tag{11}$$

Similarly, other results can be obtained.

Remark 3.1. *The identities* (10) *and* (11) *are closely related to the Riordan identity* (8).

4 Concluding remark

In this short note, we have provided a generalization of the well-known and useful identity due to Riordan via a hypergeometric series approach.

Acknowledgements

The work of D. Lim is supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) NRF-2021R1C1C1010902. The author is grateful to the Editor and the referee for providing constructive comments which improved the presentation of the paper in a very nice form.

References

[1] Andrews, G. E. (1974). Applications of basic hypergeometric functions. *SIAM Review*, 16(4), 441–484.

[2] Choi, J., Rathie, A. K., & Harsh, H. V. (2004). A note on Reed Dawson identities. *Korean J. Math. Sci.*, 11(2), 1–4.

[3] Kim, Y. S., Rathie, A. K., & Paris, R. B. (2018). Evaluations of some terminating series ${}_2F_1(2)$ with applications. *Turkish Journal of Mathematics*, 42(5), 2563–2575.

[4] Prodinger, H. (1994). Knuth's old sum – a survey. *EACTS Bulletin*, 52, 232–245.

[5] Rainville, E. D. (1960). *Special Functions*. Macmillan Company, New York.

[6] Riordan, J. (1971). *Combinatorial Identities*. John Wiley & Sons, Inc., New York.

[7] Roy, R. (1987). Binomial identities and hypergeometric series. *The American Mathematical Monthly*, 94(1), 36–46.