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On the bivariate Padovan polynomials matrix

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Abstract: In this paper, we intruduce the bivariate Padovan sequence we examine its various identities. We define the bivariate Padovan polynomials matrix. Then, we find the Binet formula, generating function and exponential generating function of the bivariate Padovan polynomials matrix. Also, we obtain a sum formula and its series representation.
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1 Introduction

Polynomials are an expression that consists of a certain number of independent variables and fixed numbers. On the polynomials are used the exponentiation of natural numbers, addition, subtraction and multiplication. Polynomials are common in science and mathematics. They are used to solve problems in economics to chemistry, chemistry to physics, physics to cryptology and social sciences. There are some important sequences whose elements are polynomials. These are polynomial sequences such as Fibonacci, Lucas, Pell, Jacobsthal, and Padovan. In this paper, the definition of the bivariate Padovan polynomials sequence and bivariate Padovan polynomials matrix is given and its various identities are examined. Then, we find the Binet formula, generating function and exponential generating function of these sequences. Also, we obtain a sum formula and its series representation. Firstly, the Padovan sequence is defined as follows. The Padovan sequence $\{P_n\}_{n\geq 0}$ is defined by the third order recurrence

$$P_{n+3} = P_{n+1} + P_n$$

with the initial conditions $P_0 = 1$, $P_1 = 0$ and $P_2 = 1$. The Padovan sequence appears as sequence A000931 on the On-Line Encyclopedia of Integer Sequences (OEIS) [18]. For convenience, we define $P_{-1} = P_{-2} = 0$. The first few values of this sequence are given as follows

$\boldsymbol{n}$	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	...
$\boldsymbol{P_n}$	0	0	1	0	1	1	1	2	2	3	4	5	7	9	12	16	...

The plastic number is the unique real root of the characteristic equation of Padovan sequence

$$t^3 - t - 1 = 0$$

with a value of

$$\sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{23}{3}}} + \sqrt[3]{\frac{1}{2} - \frac{1}{6}\sqrt{\frac{23}{3}}} \approx 1.324718.$$

If its roots are denoted by α, β and γ then the following equalities can be derived

$$\begin{aligned} \alpha + \beta + \gamma &= 0, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= -1, \\ \alpha\beta\gamma &= 1. \end{aligned}$$

More information is available in [19, 24] for Padovan numbers. Moreover, the Binet-like formula for the Padovan sequence is

$$P_n = a\alpha^n + b\beta^n + c\gamma^n \tag{1}$$

where

$$a = \frac{\beta\gamma + 1}{(\alpha - \beta)(\alpha - \gamma)}, \quad b = \frac{\alpha\gamma + 1}{(\beta - \alpha)(\beta - \gamma)}, \quad c = \frac{\alpha\beta + 1}{(\gamma - \alpha)(\gamma - \beta)}.$$

It is well known that from [19], the following identities are valid:

$$P_{-n-3} = P_n^2 - P_{n+1}P_{n-1}, \tag{2}$$

$$P_n = P_{m-1}P_{n-m} + P_{m+1}P_{n-m+1} + P_mP_{n-m+2}. \tag{3}$$

Also, in [19] the Padovan matrix is defined by

$$Q_P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and satisfies the equality

$$Q_P^n = \begin{bmatrix} P_{n-3} & P_{n-1} & P_{n-2} \\ P_{n-2} & P_n & P_{n-1} \\ P_{n-1} & P_{n+1} & P_n \end{bmatrix}.$$

By using (2) and (3), the determinant of the Padovan matrix gives an identity as follows

$$P_{n-3}P_{-n-3} + P_{n-1}P_{-n-2} + P_{n-2}P_{-n-1} = 1.$$

That is, the determinant of Q_P^n is 1, denoted by $|Q_P^n| = 1$.

The Padovan numbers and their some generalizations are investigated in [14, 20]. Moreover, Deveci and Shannon developed properties of recurrence sequences defined from circulant matrices obtained from the characteristic polynomial of the Pell–Padovan sequence [3].

There are many studies on special number sequences and their applications in the literature. Some of these studies were presented by the following authors: Kim et al. presented a new approach to the convolved Fibonacci numbers arising from the generating function of them and gave some new and explicit identities for the convolved Fibonacci numbers in [7]. They derived Fourier series expansions for functions related to sums of finite products of Chebyshev polynomials of the first kind and of Lucas polynomials, studied sums of finite products of Chebyshev polynomials of the first kind and Lucas polynomials and represent each of them in terms of Chebyshev polynomials of all kinds, and also considered sums of finite products of Chebyshev polynomials of the second kind and of Fibonacci polynomials and derived Fourier series expansions of functions associated with them in [6, 8, 10]. They finally defined the Whitney numbers of the first and second kind for any finite geometric lattice in [9].

2 The bivariate Padovan polynomials

Inspired by the works [1, 2, 11, 16, 17, 21–23] on the bivariate Fibonacci polynomials, we define the bivariate Padovan polynomials as follows. Recently, several studies have been carried out on some bivariate sequences [4, 5, 25, 26].

Definition 1. *For any integer numbers $x > 0$ and $y \neq 0$ and $27y^2 - 4x^3 = 0$, the bivariate Padovan polynomials sequence $\{P_n(x, y)\}_{n \geq 0}$ is defined by a third order recurrence,*

$$P_{n+3}(x, y) = xP_{n+1}(x, y) + yP_n(x, y), \quad n \geq 0 \tag{4}$$

with the initial conditions

$$P_0(x, y) = 1, \quad P_1(x, y) = 0, \quad P_2(x, y) = x.$$

To simplify notation, take $P_n(x, y) = \mathcal{P}_n$. The first few values of this sequence are given as follows:

n	-2	-1	0	1	2	3	4	5	6	7	8	$\dots$
$\mathcal{P}_n$	0	0	1	0	x	y	x^2	$2xy$	x^3+y^2	$3x^2y$	x^4+3xy^2	$\dots$

For $n<0$, $\mathcal{P}_n$ can be defined by rewriting the recurrence relation (4) as

$$P_n(x,y)=\frac{1}{y}P_{n+3}(x,y)-\frac{x}{y}P_{n+1}(x,y),\quad n<0 \tag{5}$$

So, we can establish the negative indexes of $\mathcal{P}_n$.

The recurrence (4) involves the characteristic equality

$$\lambda^3-x\lambda-y=0. \tag{6}$$

If its roots are denoted by $\alpha_{(x,y)}$, $\beta_{(x,y)}$ and $\gamma_{(x,y)}$ then the following equalities can be derived

$$\begin{aligned}\alpha_{(x,y)}+\beta_{(x,y)}+\gamma_{(x,y)}&=0,\\ \alpha_{(x,y)}\beta_{(x,y)}+\alpha_{(x,y)}\gamma_{(x,y)}+\beta_{(x,y)}\gamma_{(x,y)}&=-x,\\ \alpha_{(x,y)}\beta_{(x,y)}\gamma_{(x,y)}&=y.\end{aligned}$$

Using the characteristic equation (6) and the equalities above, we can derive the Binet-like formula for the bivariate Padovan polynomials sequence as follows:

Theorem 2.1. *The Binet-like formula for the bivariate Padovan polynomial sequence is*

$$\mathcal{P}_n=a_{(x,y)}\alpha^n_{(x,y)}+b_{(x,y)}\beta^n_{(x,y)}+c_{(x,y)}\gamma^n_{(x,y)} \tag{7}$$

where

$$a_{(x,y)}=\frac{\beta_{(x,y)}\gamma_{(x,y)}+x}{(\alpha_{(x,y)}-\beta_{(x,y)})(\alpha_{(x,y)}-\gamma_{(x,y)})},$$

$$b_{(x,y)}=\frac{\alpha_{(x,y)}\gamma_{(x,y)}+x}{(\beta_{(x,y)}-\alpha_{(x,y)})(\beta_{(x,y)}-\gamma_{(x,y)})},$$

$$c_{(x,y)}=\frac{\alpha_{(x,y)}\beta_{(x,y)}+x}{(\gamma_{(x,y)}-\alpha_{(x,y)})(\gamma_{(x,y)}-\beta_{(x,y)})}.$$

Proof. Assume that

$$\mathcal{P}_n=a_{(x,y)}\alpha^n_{(x,y)}+b_{(x,y)}\beta^n_{(x,y)}+c_{(x,y)}\gamma^n_{(x,y)}$$

Let us determine the coefficients $a_{(x,y)}$, $b_{(x,y)}$ and $c_{(x,y)}$. For $n=0,1,2$. We write

$$\begin{aligned}\mathcal{P}_0&=a_{(x,y)}+b_{(x,y)}+c_{(x,y)}\\ \mathcal{P}_1&=a_{(x,y)}\alpha_{(x,y)}+b_{(x,y)}\beta_{(x,y)}+c_{(x,y)}\gamma_{(x,y)}\\ \mathcal{P}_2&=a_{(x,y)}\alpha^2_{(x,y)}+b_{(x,y)}\beta^2_{(x,y)}+c_{(x,y)}\gamma^2_{(x,y)}.\end{aligned}$$

By The Cramer's rule, we can easily prove that the coefficient $a_{(x,y)}$, $b_{(x,y)}$ and $c_{(x,y)}$ are in the desired form. □

Proposition 1. *Let $\mathcal{P}_n$ be n th bivariate Padovan polynomial. Then,*

$$y^{n+1}\mathcal{P}_{-n-3}=\mathcal{P}_n^2-\mathcal{P}_{n+1}\mathcal{P}_{n-1}. \tag{8}$$

Proof. We use the principle of mathematical induction. Since

$$\begin{aligned}\mathcal{P}_0^2-\mathcal{P}_1\mathcal{P}_{-1}&=1-0=y\mathcal{P}_{-3},\\ \mathcal{P}_1^2-\mathcal{P}_2\mathcal{P}_0&=0-x=y^2\mathcal{P}_{-4},\end{aligned}$$

the result is true when $n=0,1$.

Assume that the result is true for all positive integers $n \le k$ where $k \ge 2$. Then, by (4) and the hypothesis of mathematical induction, we write

$$\begin{aligned}
\mathcal{P}_{k+1}^2 - \mathcal{P}_{k+2}\mathcal{P}_k &= \left(x\mathcal{P}_{k-1} + y\mathcal{P}_{k-2}\right)^2 - \left(x\mathcal{P}_k + y\mathcal{P}_{k-1}\right)\left(x\mathcal{P}_{k-2} + y\mathcal{P}_{k-3}\right) \\
&= x^2\mathcal{P}_{k-1}^2 + 2xy\mathcal{P}_{k-1}\mathcal{P}_{k-2} + y^2\mathcal{P}_{k-2}^2 - x^2\mathcal{P}_k\mathcal{P}_{k-2} - xy\mathcal{P}_k\mathcal{P}_{k-3} \\
&\quad - xy\mathcal{P}_{k-1}\mathcal{P}_{k-2} - y^2\mathcal{P}_{k-1}\mathcal{P}_{k-3} \\
&= x^2y^k\mathcal{P}_{-k-2} - xy^k\mathcal{P}_{-k} + y^{k+1}\mathcal{P}_{-k-1} \\
&= x^2y^k\mathcal{P}_{-k-2} + xy^{k+1}\mathcal{P}_{-k-3} - xy^{k+1}\mathcal{P}_{-k-3} - xy^k\mathcal{P}_{-k} + y^{k+1}\mathcal{P}_{-k-1} \\
&= xy^k\mathcal{P}_{-k} - xy^{k+1}\mathcal{P}_{-k-3} - xy^k\mathcal{P}_{-k} + y^{k+1}\mathcal{P}_{-k-1} \\
&= y^{k+2}\mathcal{P}_{-k-4}.
\end{aligned}$$

Thus, by the strong version of the principle of mathematical induction, the formula works for all positive integers $n \ge 3$. □

Theorem 2.2. *The generating function for the bivariate Padovan polynomials is*

$$G_{\mathcal{P}}(x,y) = \sum_{n=1}^{\infty} \mathcal{P}_n t^n = \frac{xt^2 + yt^3}{1 - xt^2 - yt^3}.$$

Proof. Let

$$G_{\mathcal{P}}(x,y) = \sum_{n=1}^{\infty} \mathcal{P}_n t^n = \mathcal{P}_1 t + \mathcal{P}_2 t^2 + \mathcal{P}_3 t^3 + \cdots + \mathcal{P}_n t^n + \dots$$

Let us multiplied this equality by $-yt^3$ and $-xt^2$ such as

$$\begin{aligned}
-yt^3 G_{\mathcal{P}}(x,y) &= -y\mathcal{P}_1 t^4 - y\mathcal{P}_2 t^5 - y\mathcal{P}_3 t^6 - \cdots - y\mathcal{P}_n t^{n+3} - \cdots \\
-xt^2 G_{\mathcal{P}}(x,y) &= -x\mathcal{P}_1 t^3 - x\mathcal{P}_2 t^4 - x\mathcal{P}_3 t^5 - \cdots - x\mathcal{P}_n t^{n+2} - \cdots
\end{aligned}$$

Then, we write

$$\begin{aligned}
(1 - xt^2 - yt^3)G_{\mathcal{P}}(x,y) &= \mathcal{P}_1 t + \mathcal{P}_2 t^2 + (\mathcal{P}_3 - x\mathcal{P}_1)t^3 + (\mathcal{P}_4 - x\mathcal{P}_2 - y\mathcal{P}_1)t^4 + \dots \\
&\quad + \left(\mathcal{P}_n - x\mathcal{P}_{n-2} - y\mathcal{P}_{n-3}\right)t^n + \dots
\end{aligned}$$

Since $\mathcal{P}_1 = 0$, $\mathcal{P}_2 = x$, $\mathcal{P}_3 = y$, $\mathcal{P}_4 = x^2$ and $\mathcal{P}_n - x\mathcal{P}_{n-2} - y\mathcal{P}_{n-3} = 0$, we obtain

$$G_{\mathcal{P}}(x,y) = \frac{xt^2 + yt^3}{1 - xt^2 - yt^3}.$$

Thus, the proof is completed. □

Theorem 2.3. *The exponential generating function for the bivariate Padovan polynomials is*

$$E_{\mathcal{P}}(x,y) = \sum_{n=1}^{\infty} \frac{\mathcal{P}_n}{n!} t^n = a_{(x,y)} e^{\alpha_{(x,y)} t} + b_{(x,y)} e^{\beta_{(x,y)} t} + c_{(x,y)} e^{\gamma_{(x,y)} t} \tag{9}$$

where $a_{(x,y)}$, $b_{(x,y)}$ and $c_{(x,y)}$ are defined in Theorem 2.1, and $\alpha_{(x,y)}$, $\beta_{(x,y)}$ and $\gamma_{(x,y)}$ are the roots of the characteristic equation (4).

Proof. We know that

$$e^{\alpha_{(x,y)} t} = \sum_{n=1}^{\infty} \frac{\alpha_{(x,y)}^n}{n!} t^n, \quad e^{\beta_{(x,y)} t} = \sum_{n=1}^{\infty} \frac{\beta_{(x,y)}^n}{n!} t^n, \quad e^{\gamma_{(x,y)} t} = \sum_{n=1}^{\infty} \frac{\gamma_{(x,y)}^n}{n!} t^n.$$

Multiplying these equalities by $a_{(x,y)}$, $b_{(x,y)}$ and $c_{(x,y)}$, respectively, and summing side by side, we obtain

$$a_{(x,y)}e^{\alpha_{(x,y)}t}+b_{(x,y)}e^{\beta_{(x,y)}t}+c_{(x,y)}e^{\gamma_{(x,y)}t}=\sum_{n=1}^{\infty}\frac{a_{(x,y)}\alpha_{(x,y)}^{n}+b_{(x,y)}\beta_{(x,y)}^{n}+c_{(x,y)}\gamma_{(x,y)}^{n}}{n!}t^{n}.$$

By the formula (7) we have the desire equality (9). □

Theorem 2.4. *The series for the bivariate Padovan polynomials is*

$$S_{\mathcal{P}}(x,y)=\sum_{n=0}^{\infty}\frac{\mathcal{P}_n}{t^{n+1}}=\frac{t^2}{t^3-xt-y}$$

Proof. Let

$$S_{\mathcal{P}}(x,y)=\sum_{n=0}^{\infty}\frac{\mathcal{P}_n}{t^{n+1}}=\frac{\mathcal{P}_0}{t}+\frac{\mathcal{P}_1}{t^2}+\frac{\mathcal{P}_2}{t^3}+\cdots+\frac{\mathcal{P}_n}{t^{n+1}}+\dots$$

Let us multiplied this equality by t^3, $-xt$ and $-y$ such as

$$\begin{aligned}
t^3S_{\mathcal{P}}(x)&=\mathcal{P}_0t^2+\mathcal{P}_1t+\mathcal{P}_2+\cdots+\frac{\mathcal{P}_n}{t^{n-2}}+\dots\\
-xtS_{\mathcal{P}}(x,y)&=-x\mathcal{P}_0-x\frac{\mathcal{P}_1}{t}-x\frac{\mathcal{P}_2}{t^2}-\cdots-x\frac{\mathcal{P}_n}{t^n}-\cdots\\
-yS_{\mathcal{P}}(x,y)&=-y\frac{\mathcal{P}_0}{t}-y\frac{\mathcal{P}_1}{t^2}-y\frac{\mathcal{P}_2}{t^3}-y\cdots-y\frac{\mathcal{P}_n}{t^{n+1}}-\cdots
\end{aligned}$$

Then, we write

$$\begin{aligned}
(t^3-xt-y)S_{\mathcal{P}}(x,y)&=\mathcal{P}_0t^2+\mathcal{P}_1t+\mathcal{P}_2-x\mathcal{P}_0+(\mathcal{P}_3-x\mathcal{P}_1-y\mathcal{P}_0)\frac{1}{t}\\
&\quad-\cdots+(\mathcal{P}_{n+2}-x\mathcal{P}_n-y\mathcal{P}_{n-1})\frac{1}{t^n}+\cdots
\end{aligned}$$

Since $\mathcal{P}_1=0$, $\mathcal{P}_2=x$, $\mathcal{P}_3=y$, $\mathcal{P}_4=x^2$ and $\mathcal{P}_{n+2}-x\mathcal{P}_n-y\mathcal{P}_{n-1}=0$, we obtain

$$S_{\mathcal{P}}(x,y)=\frac{t^2}{t^3-xt-y}.$$

Thus, the proof is completed. □

Theorem 2.5. *The sum of the first n terms of $\mathcal{P}_n$ is*

$$\sum_{i=0}^{n}\mathcal{P}_i=\frac{x+y-\mathcal{P}_{n+1}-\mathcal{P}_{n+2}-y\mathcal{P}_n}{1-x-y},\quad n\geq 0.$$

Proof. We know that

$$\mathcal{P}_n=x\mathcal{P}_{n-2}+y\mathcal{P}_{n-3}.$$

So, applying to the identity above, we deduce that

$$\begin{aligned}
\mathcal{P}_3&=x\mathcal{P}_1+y\mathcal{P}_0,\\
\mathcal{P}_4&=x\mathcal{P}_2+y\mathcal{P}_1,\\
\mathcal{P}_5&=x\mathcal{P}_3+y\mathcal{P}_2,\\
&\vdots
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}\mathcal{P}_{n-2} &= x\mathcal{P}_{n-4} + y\mathcal{P}_{n-5},\\ \mathcal{P}_{n-1} &= x\mathcal{P}_{n-3} + y\mathcal{P}_{n-4},\\ \mathcal{P}_{n} &= x\mathcal{P}_{n-2} + y\mathcal{P}_{n-3}.\end{aligned}$$

Summing the both of sides of the identities above, we obtain

$$\sum_{i=0}^{n}\mathcal{P}_i = x\sum_{i=0}^{n}\mathcal{P}_i + y\sum_{i=0}^{n}\mathcal{P}_i + y\mathcal{P}_0 + \mathcal{P}_1 + \mathcal{P}_2 - \mathcal{P}_{n+1} - \mathcal{P}_{n+2} - y\mathcal{P}_n.$$

Thus, the proof is completed. □

3 The bivariate Padovan polynomials matrix

We investigate the property of bivariate Padovan polynomials in relation with the bivariate Padovan polynomial matrices formula. So, it allow us to obtain new relations for the bivariate Padovan polynomial matrices. The bivariate Padovan polynomials matrix $Q_P(x,y)$ is generated by the matrix of order 3.

$$Q_P(x,y) = \begin{bmatrix} 0 & y & 0 \\ 0 & 0 & 1 \\ 1 & x & 0 \end{bmatrix}$$

and the n th powers of $Q_P(x,y)$ polynomials matrix is given

$$Q_P^n(x,y) = \begin{bmatrix} y\mathcal{P}_{n-3} & y\mathcal{P}_{n-1} & y\mathcal{P}_{n-2} \\ \mathcal{P}_{n-2} & \mathcal{P}_{n} & \mathcal{P}_{n-1} \\ \mathcal{P}_{n-1} & \mathcal{P}_{n+1} & \mathcal{P}_{n} \end{bmatrix}, \quad n \geq 1. \tag{10}$$

We investigate a new property of the bivariate Padovan polynomials in relation with the bivariate Padovan polynomials matrices formula.

Theorem 3.1. *For all $n \in \mathbb{N}$ we have*

$$Q_P^n(x,y)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & y & 0 \\ 0 & 0 & 1 \\ 1 & x & 0 \end{bmatrix}^n \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} y\mathcal{P}_{n-2} \\ \mathcal{P}_{n-1} \\ \mathcal{P}_{n} \end{bmatrix}. \tag{11}$$

Proof. Let us use the principle of mathematical induction on n. For $n = 1$, it is easy to see that

$$Q_P^1(x,y)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & y & 0 \\ 0 & 0 & 1 \\ 1 & x & 0 \end{bmatrix}^1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} y\mathcal{P}_{n-2} \\ \mathcal{P}_{n-1} \\ \mathcal{P}_{n} \end{bmatrix}.$$

Assume that it is true for all positive integer $n = k$, i.e,

$$Q_P^k(x,y)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & y & 0 \\ 0 & 0 & 1 \\ 1 & x & 0 \end{bmatrix}^k \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} y\mathcal{P}_{k-2} \\ \mathcal{P}_{k-1} \\ \mathcal{P}_{k} \end{bmatrix}.$$

We have to show that the equality is valid for $n = k+1$. By the hypothesis of induction, we can write

$$
\begin{aligned}
Q_P^{k+1}(x,y)\begin{bmatrix}0\\0\\1\end{bmatrix} &= Q_P^1(x,y)\left(Q_P^k(x,y)\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\\
&= \begin{bmatrix}0 & y & 0\\0 & 0 & 1\\1 & x & 0\end{bmatrix}\begin{bmatrix}y\mathcal{P}_{k-2}\\ \mathcal{P}_{k-1}\\ \mathcal{P}_k\end{bmatrix}\\
&= \begin{bmatrix}y\mathcal{P}_{k-1}\\ \mathcal{P}_k\\ y\mathcal{P}_{k-2}+x\mathcal{P}_{k-1}\end{bmatrix} = \begin{bmatrix}y\mathcal{P}_{k-1}\\ \mathcal{P}_k\\ \mathcal{P}_{k+1}\end{bmatrix}
\end{aligned}
$$

Therefore, the result is true for every $n \geq 1$. $\square$

Let us generalize this observation using the bivariate Padovan polynomial formula matrices.

Proposition 2. *For all integers m, n such that $1 \leq m < n$, we have the following relation:*

$$
\mathcal{P}_n = y\mathcal{P}_{m-1}\mathcal{P}_{n-m-2} + \mathcal{P}_{m+1}\mathcal{P}_{n-m-1} + \mathcal{P}_m\mathcal{P}_{n-m} \tag{12}
$$

Proof. From the laws of exponent for the square matrices, we have

$$
Q_P^n(x,y)\begin{bmatrix}0\\0\\1\end{bmatrix} = Q_P^m(x,y)\left(Q_P^{n-m}(x,y)\begin{bmatrix}0\\0\\1\end{bmatrix}\right)
$$

It follows from (10) and (11) that

$$
\begin{bmatrix}y\mathcal{P}_{n-2}\\ \mathcal{P}_{n-1}\\ \mathcal{P}_n\end{bmatrix} = \begin{bmatrix}y\mathcal{P}_{m-3} & y\mathcal{P}_{m-1} & y\mathcal{P}_{m-2}\\ \mathcal{P}_{m-2} & \mathcal{P}_m & \mathcal{P}_{m-1}\\ \mathcal{P}_{m-1} & \mathcal{P}_{m+1} & \mathcal{P}_m\end{bmatrix}\begin{bmatrix}y\mathcal{P}_{n-m-2}\\ \mathcal{P}_{n-m-1}\\ \mathcal{P}_{n-m}\end{bmatrix}
$$

By the equality of the corresponding elements in the matrix equality above, we have

$$
\mathcal{P}_n = y\mathcal{P}_{m-1}\mathcal{P}_{n-m-2} + \mathcal{P}_{m+1}\mathcal{P}_{n-m-1} + \mathcal{P}_m\mathcal{P}_{n-m}
$$

This completes the proof. $\square$

Theorem 3.2. *The eigenvalues of $Q_P^n(x,y)$ are $\alpha^n_{(x,y)}$, $\beta^n_{(x,y)}$ and $\gamma^n_{(x,y)}$.*

In particular, we have the next corollary.

Corollary 3.1. *The eigenvalues of $Q_P(x,y)$ are $\alpha_{(x,y)}$, $\beta_{(x,y)}$ and $\gamma_{(x,y)}$.*

Proof.

$$
|Q_P(x,y) - \lambda I| = \begin{vmatrix}-\lambda & y & 0\\0 & -\lambda & 1\\1 & x & -\lambda\end{vmatrix} = \lambda^3 - x\lambda - y = 0.
$$

By the roots of characteristic equation (6) we prove the desire result. $\square$

Proposition 3. *The relations are valid:*
1. $|Q_P^3(x,y)+xQ_P(x,y)+yI|=8y^3$.
2. $|Q_P^n(x,y)|=y^n$.

Proof. 1.

$$Q_P^3(x,y)+xQ_P(x,y)+yI=\begin{bmatrix} y & xy & 0\\ 0 & y & x\\ x & x^2 & y\end{bmatrix}+\begin{bmatrix} 0 & xy & 0\\ 0 & 0 & x\\ x & x^2 & 0\end{bmatrix}+\begin{bmatrix} y & 0 & 0\\ 0 & y & 0\\ 0 & 0 & y\end{bmatrix}$$
$$=\begin{bmatrix} 2y & 2xy & 0\\ 0 & 2y & 2x\\ 2x & 2x^2 & 2y\end{bmatrix}=8\begin{bmatrix} y & xy & 0\\ 0 & y & x\\ x & x^2 & y\end{bmatrix}.$$

So,

$$|Q_P^3(x,y)+xQ_P(x,y)+yI|=8y^3$$

2. Using (8) and (12), we have

$$\begin{vmatrix} y\mathcal{P}_{n-3} & y\mathcal{P}_{n-1} & y\mathcal{P}_{n-2}\\ \mathcal{P}_{n-2} & \mathcal{P}_{n} & \mathcal{P}_{n-1}\\ \mathcal{P}_{n-1} & \mathcal{P}_{n+1} & \mathcal{P}_{n}\end{vmatrix}=y^{n+1}\left(y\mathcal{P}_{n-3}\mathcal{P}_{-n-3}+\mathcal{P}_{n-1}\mathcal{P}_{-n-2}+\mathcal{P}_{n-2}\mathcal{P}_{-n-1}\right)$$
$$=y^{n+1}P_{-3}=y^n. \qquad \square$$

We note that similar results for the Fibonacci matrices are given in [12, 13] the books of T. Koshy.

Theorem 3.3. *The Binet-like formulas for the bivariate Padovan polynomials matrix is*

$$Q_P^n(x,y)=a_{(x,y)}\underline{\alpha_{(x,y)}}\alpha_{(x,y)}^n+b_{(x,y)}\underline{\beta_{(x,y)}}\beta_{(x,y)}^n+c_{(x,y)}\underline{\gamma_{(x,y)}}\gamma_{(x,y)}^n,\quad n\geqslant 0 \tag{13}$$

where

$$\underline{\alpha_{(x,y)}}=\begin{bmatrix} \alpha_{(x,y)}^{-3} & \alpha_{(x,y)}^{-1} & \alpha_{(x,y)}^{-2}\\ \alpha_{(x,y)}^{-2} & 1 & \alpha_{(x,y)}^{-3}\\ \alpha_{(x,y)}^{-1} & \alpha_{(x,y)} & 1\end{bmatrix}$$
$$\underline{\beta_{(x,y)}}=\begin{bmatrix} \beta_{(x,y)}^{-3} & \beta_{(x,y)}^{-1} & \beta_{(x,y)}^{-2}\\ \beta_{(x,y)}^{-2} & 1 & \beta_{(x,y)}^{-3}\\ \beta_{(x,y)}^{-1} & \beta_{(x,y)} & 1\end{bmatrix}$$
$$\underline{\gamma_{(x,y)}}=\begin{bmatrix} \gamma_{(x,y)}^{-3} & \gamma_{(x,y)}^{-1} & \gamma_{(x,y)}^{-2}\\ \gamma_{(x,y)}^{-2} & 1 & \gamma_{(x,y)}^{-3}\\ \gamma_{(x,y)}^{-1} & \gamma_{(x,y)} & 1\end{bmatrix}.$$

Proof. From the definition of the bivariate Padovan polynomials matrix $Q_P^n(x,y)$ in (10) and Binet-like formula for the bivariate Padovan polynomials $\mathcal{P}_n$ in (7), we write

$$
\begin{aligned}
Q_P^n(x,y) &= \begin{bmatrix} \mathcal{P}_{n-3} & \mathcal{P}_{n-1} & \mathcal{P}_{n-2} \\ \mathcal{P}_{n-2} & \mathcal{P}_{n} & \mathcal{P}_{n-1} \\ \mathcal{P}_{n-1} & \mathcal{P}_{n+1} & \mathcal{P}_{n} \end{bmatrix} \\
&= \quad a_{(x,y)} \begin{bmatrix} \alpha_{(x,y)}^{-3} & \alpha_{(x,y)}^{-1} & \alpha_{(x,y)}^{-2} \\ \alpha_{(x,y)}^{-2} & 1 & \alpha_{(x,y)}^{-1} \\ \alpha_{(x,y)}^{-1} & \alpha_{(x,y)} & 1 \end{bmatrix} \alpha_{(x,y)}^{n} + b_{(x,y)} \begin{bmatrix} \beta_{(x,y)}^{-3} & \beta_{(x,y)}^{-1} & \beta_{(x,y)}^{-2} \\ \beta_{(x,y)}^{-2} & 1 & \beta_{(x,y)}^{-3} \\ \beta_{(x,y)}^{-1} & \beta_{(x,y)} & 1 \end{bmatrix} \beta_{(x,y)}^{n} \\
&\quad + c_{(x,y)} \begin{bmatrix} \gamma_{(x,y)}^{-3} & \gamma_{(x,y)}^{-1} & \gamma_{(x,y)}^{-2} \\ \gamma_{(x,y)}^{-2} & 1 & \gamma_{(x,y)}^{-3} \\ \gamma_{(x,y)}^{-1} & \gamma_{(x,y)} & 1 \end{bmatrix} \gamma_{(x,y)}^{n} \\
&= a_{(x,y)} \underline{\alpha_{(x,y)}} \alpha_{(x,y)}^{n} + b_{(x,y)} \underline{\beta_{(x,y)}} \beta_{(x,y)}^{n} + c_{(x,y)} \underline{\gamma_{(x,y)}} \gamma_{(x,y)}^{n}
\end{aligned}
$$

Thus, the proof is completed. □

Theorem 3.4. *The generating function for the bivariate Padovan polynomials matrix is*

$$
G_{Q_P}(x,y) = \sum_{n=1}^{\infty} Q_P^n(x,y) t^n = \frac{1}{1 - xt^2 - yt^3} \begin{bmatrix} yt^3 & yt & yt^2 \\ t^2 & xt^2 + yt^3 & t \\ t & xt + yt^2 & xt^2 + yt^3 \end{bmatrix}.
$$

Proof. Let

$$
G_{Q_P}(x,y) = \sum_{n=1}^{\infty} Q_P^n(x,y) t^n = Q_P(x,y)t + Q_P^2(x,y)t^2 + Q_P^3(x,y)t^3 + \cdots + Q_P^n(x,y)t^n + \cdots
$$

be the generating function of the bivariate Padovan polynomials matrix. Multiplying the equality by $-xt^2$ and $-yt^3$, respectively, such as

$$
\begin{aligned}
-xt^2 G_{Q_P}(x,y) &= -xQ_P(x,y)t^3 - xQ_P^2(x,y)t^4 - xQ_P^3(x,y)t^5 - \cdots - xQ_P^n(x,y)t^{n+2} - \cdots \\
-yt^3 G_{Q_P}(x,y) &= -yQ_P(x,y)t^4 - yQ_P^2(x,y)t^5 - yQ_P^3(x,y)t^6 - \cdots - yQ_P^n(x,y)t^{n+3} - \cdots
\end{aligned}
$$

Then, we write

$$
\begin{aligned}
(1 - xt^2 - yt^3) G_{Q_P}(x,y) &= Q_P(x,y)t + Q_P^2(x,y)t^2 + (Q_P^3(x,y) - xQ_P(x,y))t^3 \\
&\quad + (Q_P^4(x,y) - xQ_P^2(x,y) - yQ_P(x,y))t^4 + \cdots \\
&\quad + (Q_P^{n+3}(x,y) - xQ_P^{n+1}(x,y) - yQ_P^n(x,y))t^{n+3} + \cdots
\end{aligned}
$$

Using (10), we obtain $Q_P^{n+3}(x) - xQ_P^{n+1}(x) - yQ_P^n(x) = 0$. So,

$$
G_{Q_P}(x,y) = \frac{1}{1 - xt^2 - yt^3} \begin{bmatrix} yt^3 & yt & yt^2 \\ t^2 & xt^2 + yt^3 & t \\ t & xt + yt^2 & xt^2 + yt^3 \end{bmatrix}.
$$

Thus, the proof is completed. □

Theorem 3.5. *The exponential generating function for the bivariate Padovan polynomials matrix is*

$$E_{Q_P}(x,y)=\sum_{n=1}^{\infty}\frac{Q_P^n(x,y)}{n!}t^n=a_{(x,y)}\underline{\alpha_{(x,y)}}e^{\alpha_{(x,y)}t}+b_{(x,y)}\underline{\beta_{(x,y)}}e^{\beta_{(x,y)}t}+c_{(x,y)}\underline{\gamma_{(x,y)}}e^{\gamma_{(x,y)}t}.$$

Proof. We know that

$$e^{\alpha_{(x,y)}t}=\sum_{n=1}^{\infty}\frac{\alpha_{(x,y)}^n}{n!}t^n,\; e^{\beta_{(x,y)}t}=\sum_{n=1}^{\infty}\frac{\beta_{(x,y)}^n}{n!}t^n,\; e^{\gamma_{(x,y)}t}=\sum_{n=1}^{\infty}\frac{\gamma_{(x,y)}^n}{n!}t^n.$$

Let us multiply each side of the first equality by $a_{(x,y)}\underline{\alpha_{(x,y)}}$, the second equality by $b_{(x,y)}\underline{\beta_{(x,y)}}$ and the third equality by $c_{(x,y)}\underline{\gamma_{(x,y)}}$, and summing all of them, we obtain that

$$\begin{aligned}
E_{Q_P}(x,y)&=a_{(x,y)}\underline{\alpha_{(x,y)}}e^{\alpha_{(x,y)}t}+b_{(x,y)}\underline{\beta_{(x,y)}}e^{\beta_{(x,y)}t}+c_{(x,y)}\underline{\gamma_{(x,y)}}e^{\gamma_{(x,y)}t}\\
&=\sum_{n=1}^{\infty}\frac{a_{(x,y)}\underline{\alpha_{(x,y)}}\alpha_{(x,y)}^n+b_{(x,y)}\underline{\beta_{(x,y)}}\beta_{(x,y)}^n+c_x\underline{\gamma_x}\gamma_{(x,y)}^n}{n!}t^n\\
&=\sum_{n=1}^{\infty}\frac{Q_P^n(x,y)}{n!}t^n
\end{aligned}$$

This completes the proof. □

Theorem 3.6. *The series for the bivariate Padovan polynomials matrix is*

$$S_{Q_P}(x,y)=\sum_{n=1}^{\infty}\frac{Q_P^n(x,y)}{t^{n+1}}=\frac{1}{t^3-xt-y}\begin{bmatrix}\frac{y}{t} & yt & y\\ 1 & x+\frac{y}{t} & t\\ t & y+xt & x+\frac{y}{t}\end{bmatrix}.$$

Proof. Let

$$S_{Q_P}(x,y)=\sum_{n=1}^{\infty}\frac{Q_P^n(x,y)}{t^{n+1}}=\frac{Q_P(x,y)}{t^2}+\frac{Q_P^2(x,y)}{t^3}+\frac{Q_P^3(x,y)}{t^4}+\cdots+\frac{Q_P^n(x,y)}{t^{n+1}}+\ldots$$

be series of the bivariate Padovan polynomials matrix. Let us multiplying the equality by t^3, $-xt$ and $-y$, respectively, such as

$$\begin{aligned}
t^3S_{Q_P}(x,y)&=Q_P(x,y)t+Q_P^2(x,y)+\frac{Q_P^3(x,y)}{t}+\cdots+\frac{Q_P^n(x,y)}{t^{n-2}}+\cdots\\
-xtS_{Q_P}(x,y)&=-x\frac{Q_P(x,y)}{t}-x\frac{Q_P^2(x,y)}{t^2}-x\frac{Q_P^3(x,y)}{t^3}-\cdots-x\frac{Q_P^n(x,y)}{t^n}-\cdots\\
-yS_{Q_P}(x,y)&=-y\frac{Q_P(x,y)}{t^2}-y\frac{Q_P^2(x,y)}{t^3}-y\frac{Q_P^3(x,y)}{t^4}-\cdots-y\frac{Q_P^n(x,y)}{t^{n+1}}-\cdots
\end{aligned}$$

Then, we write

$$\begin{aligned}
(t^3-xt-y)S_{Q_P}(x,y)&=Q_P(x,y)t+Q_P^2(x,y)+\left(Q_P^3(x,y)-xQ_P(x,y)\right)\frac{1}{t}\\
&+\left(Q_P^4(x,y)-xQ_P^2(x,y)-yQ_P(x,y)\right)\frac{1}{t^2}+\cdots\\
&+\left(Q_P^{n+2}(x,y)-xQ_P^n(x,y)-yQ_P^{n-1}(x,y)\right)\frac{1}{t^n}+\cdots
\end{aligned}$$

Using (10), we obtain $Q_P^{n+2}(x,y) - xQ_P^n(x,y) - yQ_P^{n-1}(x,y) = 0$. So,

$$S_{Q_P}(x) = \frac{1}{t^3 - xt - y}\begin{bmatrix} \frac{y}{t} & yt & y \\ 1 & x + \frac{y}{t} & t \\ t & y + xt & x + \frac{y}{t} \end{bmatrix}.$$

Thus, the proof is completed. □

Theorem 3.7. *The sum of the first n terms of $Q_P^n(x,y)$ is*

$$\sum_{i=1}^{n} Q_P^i(x,y) = \frac{1}{x+y-1}\left(Q_P^{n+2}(x,y) + Q_P^{n+1}(x,y) + yQ_P^n(x,y) - \begin{bmatrix} y & y & y \\ 1 & x+y & 1 \\ 1 & x+y & x+y \end{bmatrix}\right).$$

Proof. We know that

$$Q_P^n(x,y) = xQ_P^{n-2}(x,y) + yQ_P^{n-3}(x,y).$$

So, applying to the identity above, we deduce that

$$\begin{aligned} Q_P^3(x,y) &= xQ_P^1(x,y) + yQ_P^0(x,y), \\ Q_P^4(x,y) &= xQ_P^2(x,y) + yQ_P^1(x,y), \\ Q_P^5(x,y) &= xQ_P^3(x,y) + yQ_P^2(x,y), \\ &\vdots \\ Q_P^{n-2}(x,y) &= xQ_P^{n-4}(x,y) + yQ_P^{n-5}(x,y), \\ Q_P^{n-1}(x,y) &= xQ_P^{n-3}(x,y) + yQ_P^{n-4}(x,y), \\ Q_P^n(x,y) &= xQ_P^{n-2}(x,y) + yQ_P^{n-3}(x,y). \end{aligned}$$

Summing of both of sides of the identities above, we obtain

$$\begin{aligned} -Q_P^1(x,y) - Q_P^2(x,y) + \sum_{i=1}^{n} Q_P^i(x,y) &= -Q_P^{n+2}(x,y) - Q_P^{n+1}(x,y) - yQ_P^n(x,y) + yQ_P^0(x,y) \\ &\quad + x\sum_{i=1}^{n} Q_P^i(x,y) + y\sum_{i=1}^{n} Q_P^i(x,y). \end{aligned}$$

Hence, we get the desired result. □

4 Conclusion

In the present work, the bivariate Padovan polynomials are defined. The Binet formula, generating function and exponential generating function of these polynomials are given. Then, the bivariate Padovan polynomials matrices are defined. The Binet formula, generating function and exponential generating function of these polynomials matrices is given. Also, a sum formula and its series representation of these polynomials are obtained.

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