Catalan transform of the k-Pell, k-Pell–Lucas and modified k-Pell sequence

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Abstract: In this study, we present the Catalan transforms of the k-Pell sequence, the k-Pell–Lucas sequence and the Modified k-Pell sequence and examine the properties of the sequences. Then we apply the Hankel transform to the Catalan transforms of the k-Pell sequence, the Catalan transform of the k-Pell–Lucas sequence and the Catalan transform of the Modified k-Pell sequence. Also, we obtain the generating functions of the Catalan transform of the k-Pell sequence, k-Pell–Lucas sequence and Modified k-Pell sequence. Furthermore, we acquire an interesting characteristic related to the determinant of the Hankel transform of the sequences.
Keywords: k-Pell numbers, k-Pell–Lucas, Catalan numbers, Catalan transform, Hankel transform.
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1   Introduction

The known Pell numbers have some applications in many branches of mathematics such as group theory, calculus, applied mathematics, linear algebra, etc. [2, 3, 4, 6, 8, 16].

There is an extensive work in the literature concerning Fibonacci-type sequences and their applications in modern science (see e.g. [10, 11, 12] and the references therein).

There exist generalizations of the Pell numbers. This paper is an extension of the work of Falcon [7], where the author gave an application of the Catalan transform to the k-Fibonacci sequences. Özkan et al. defined a new family of Catalan k-Lucas numbers and gave some properties about the family of these numbers [14]. There are some works on polynomials of the families of k-Fibonacci numbers and k-Lucas numbers [13]. Rajkovic et al. gave some properties of the Hankel transformation of a sequence [15].

One of the latest works in this area is [17] where it is presented the Catalan transforms of the k-Jacobsthal sequence and examined the properties of the sequence. They also applied the Hankel transform to the Catalan transform of the k-Jacobsthal sequence.

In this work, we consider the Catalan transform to the k-Pell, k-Pell–Lucas and Modified k-Pell numbers and present an application of the Hankel transform to the Catalan transform of these sequences.

The paper is prepared as follows. In the following section, we recall some fundamental definitions of k-Pell numbers, k-Pell–Lucas numbers and Modified k-Pell numbers. In Section 3, the Catalan transforms of k-Pell sequence and k-Pell–Lucas sequence are given. Also, we obtain the generating functions of the Catalan transform of the k-Pell sequence, k-Pell–Lucas sequence and the Modified k-Pell sequence. Finally, we give the Hankel transforms of the new sequence obtained from k-Pell, k-Pell–Lucas and Modified k-Pell sequences.

2   Materials and methods

2.1 k-Pell, k-Pell–Lucas numbers and Modified k-Pell numbers

For any positive real number k, the k-Pell sequence, $\{P_{k,n}\}_{n\in\mathbb{N}}$, is defined by the recurrence relation

$$P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}, n \geq 1,$$

where $P_{k,0} = 0$ and $P_{k,1} = 1$, [2].

The k-Pell–Lucas sequence, $\{Q_{k,n}\}_{n\in\mathbb{N}}$, is defined as

$$Q_{k,n+1} = 2Q_{k,n} + kQ_{k,n-1}, n \geq 1,$$

where $Q_{k,0} = 2$ and $Q_{k,1} = 2$, [3].

The Modified k-Pell sequence, $\{q_{k,n}\}_{n\in\mathbb{N}}$, is defined as

$$q_{k,n+1} = 2q_{k,n} + kq_{k,n-1}, n \geq 1,$$

where $q_{k,0} = 1$ and $q_{k,1} = 1$, [2].

The Binet formula, respectively, for k-Pell sequence, k-Pell–Lucas sequence and Modified k-Pell sequence is given by:

$$P_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}, \qquad Q_{k,n} = r_1^n + r_2^n, \qquad q_{k,n} = \frac{r_1^n + r_2^n}{2},$$

where $r_1 = 1 + \sqrt{1+k}$ and $r_2 = 1 - \sqrt{1+k}$, respectively, [2]. Note that, since $k > 0$, $r_1 + r_2 = 2$, $r_1 r_2 = -k$ and $r_1 - r_2 = 2\sqrt{1+k}$.

2.2. Catalan numbers

The Catalan numbers are defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

in [1]. The latter can be written as

$$C_n = \frac{(2n)!}{(n+1)!n!}.$$

Also, one can obtain the recurrence relation for $C(n)$ from

$$\frac{C_{n+1}}{C_n} = \frac{2(2n+1)}{n+2}$$

in [5]. The first few Catalan numbers are 1, 1, 2, 5, 14, 42, 132, 429, 1430, … [18].

3   Main results

3.1   Catalan transform of the k-Pell sequence

Now, following Falcon in [7], we introduce the Catalan transform of the k-Pell sequence $\{P_{k,n}\}$ as

$$CP_{k,n} = \sum_{i=0}^{n} \frac{i}{2n-i}\binom{2n-i}{n-i}P_{k,i}$$

for $n \geq 1$ with $CP_{k,0} = 0$. We can give some of them as follow:

- $CP_{k,1} = \sum_{1}^{1} \frac{i}{2-i}\binom{2-i}{1-i}P_{k,i} = 1,$

- $CP_{k,2} = \sum_{1}^{2} \frac{i}{4-i}\binom{4-i}{2-i}P_{k,i} = 3,$

- $CP_{k,3} = \sum_{1}^{3} \frac{i}{6-i}\binom{6-i}{3-i}P_{k,i} = 10 + k,$

- $CP_{k,4} = 35 + 7k,$

- $CP_{k,5} = 126 + 37k + k^2,$

- $CP_{k,6} = 462 + 176k + 11k^2,$

- $CP_{k,7} = 1716 + 794k + 80k^2 + k^3.$

We give the coefficient of k^η in Table 1 for $0 \leq \eta \leq n$. The following equation can be written as a product of two matrices

$$\begin{bmatrix} C.P_{k,1} \\ C.P_{k,2} \\ C.P_{k,3} \\ C.P_{k,4} \\ C.P_{k,5} \\ C.P_{k,6} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 2 & 1 & & & \\ 5 & 5 & 3 & 1 & & \\ 14 & 14 & 9 & 4 & 1 & \\ 42 & 42 & 28 & 14 & 5 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} P_{k,1} \\ P_{k,2} \\ P_{k,3} \\ P_{k,4} \\ P_{k,5} \\ P_{k,6} \\ \vdots \end{bmatrix}.$$

The entries of the matrix C verify the recurrence relation $C_{i,j} = \sum_{r=j-1}^{i-1} C_{i-1,r}$. For $i > 1$, note that the first and second columns are the Catalan numbers.

The matrix $C_{n,n-i}$ is the Catalan triangle. For $1 \leq i \leq n$, the entries of the matrix $C_{n,n-i}$ verify

$$C_{n,n-i} = \frac{(2n-i)!\,(i+1)}{(n-i)!\,(n+1)!}.$$

CP_1	1			
CP_2	3			
CP_3	1	10		
CP_4	7	35		
CP_5	1	37	126	
CP_6	11	176	462	
CP_7	1	80	794	1716
...	...	...	...	...

Table 1. Catalan triangle of the k-Pell sequence

We obtain the first few Catalan transform of the k-Pell sequence as follows:

- $CP_1 = \{0, 1, 3, 11, 42, 164, 649, \dots\}$, indexed in OEIS as A143464.
- $CP_2 = \{0, 1, 3, 12, 49, 204, 858, \dots\}$,
- $CP_3 = \{0, 1, 3, 13, 56, 246, 1089, \dots\}$,
- $CP_4 = \{0, 1, 3, 14, 63, 290, 1342, \dots\}$,
- $CP_5 = \{0, 1, 3, 15, 70, 336, 1617, \dots\}$.

3.2 Catalan transform of the k-Pell–Lucas sequence

Similarly, we introduce the Catalan transform of the k-Pell–Lucas sequence $\{Q_{k,n}\}$ as

$$CQ_{k,n} = \sum_{i=0}^{n} \frac{i}{2n-i} \binom{2n-i}{n-i} Q_{k,i}$$

for $n \geq 1$ with $CQ_{k,0} = 0$. We can give some of them as follows:

- $CQ_{k,1} = \sum_{1}^{1} \frac{i}{2-i} \binom{2-i}{1-i} Q_{k,i} = 2$,
- $CQ_{k,2} = \sum_{1}^{2} \frac{i}{4-i} \binom{4-i}{2-i} Q_{k,i} = 6 + 2k$,
- $CQ_{k,3} = \sum_{1}^{3} \frac{i}{6-i} \binom{6-i}{3-i} Q_{k,i} = 20 + 10k$,
- $CQ_{k,4} = 70 + 44k + 2k^2$,
- $CQ_{k,5} = 252 + 186k + 18k^2$,
- $CQ_{k,6} = 924 + 772k + 114k^2 + 2k^3$,
- $CQ_{k,7} = 3432 + 3172k + 624k^2 + 26k^3$.

We give the coefficient of k^η in Table 2 for $0 \leq \eta \leq n$.

The following equation can be written as the product of two matrices $CQ = C.Q$, where $CQ = (CQ_{k,i})^t$ and $Q = (Q_{k,i})^t$.

$$\begin{bmatrix} C.Q_{k,1} \\ C.Q_{k,2} \\ C.Q_{k,3} \\ C.Q_{k,4} \\ C.Q_{k,5} \\ C.Q_{k,6} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 2 & 1 & & & \\ 5 & 5 & 3 & 1 & & \\ 14 & 14 & 9 & 4 & 1 & \\ 42 & 42 & 28 & 14 & 5 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} Q_{k,1} \\ Q_{k,2} \\ Q_{k,3} \\ Q_{k,4} \\ Q_{k,5} \\ Q_{k,6} \\ \vdots \end{bmatrix}.$$

CQ_1	2			
CQ_2	2	6		
CQ_3	10	20		
CQ_4	2	44	70	
CQ_5	18	186	252	
CQ_6	2	114	772	924
CQ_7	26	624	3172	3432
...	...	...	...	...

Table 2. Catalan triangle of the k-Pell–Lucas sequence

We obtain the first few Catalan transform of the k-Pell–Lucas sequence as follows:

- $CQ_1 = \{0, 2, 8, 34, 116, 456, 1812, \ldots\}$,
- $CQ_2 = \{0, 2, 10, 48, 166, 696, 2940, \ldots\}$,
- $CQ_3 = \{0, 2, 12, 62, 220, 972, 4320, \ldots\}$,
- $CQ_4 = \{0, 2, 14, 76, 278, 1284, 5964, \ldots\}$,
- $CQ_5 = \{0, 2, 16, 90, 340, 1632, 7884, \ldots\}$.

3.3 Catalan transform of the Modified k-Pell sequence

Now, following Falcon in [7], we introduce the Catalan transform of the Modified k-Pell sequence $\{q_{k,n}\}$ as

$$Cq_{k,n} = \sum_{i=0}^{n} \frac{i}{2n-i} \binom{2n-i}{n-i} q_{k,i}$$

for $n \geq 1$ with $Cq_{k,0} = 0$. We can give some of them as follows:

- $Cq_{k,1} = \sum_{1}^{1} \frac{i}{2-i} \binom{2-i}{1-i} q_{k,i} = 1$,
- $Cq_{k,2} = \sum_{1}^{2} \frac{i}{4-i} \binom{4-i}{2-i} q_{k,i} = 3 + k$,
- $Cq_{k,3} = \sum_{1}^{3} \frac{i}{6-i} \binom{6-i}{3-i} q_{k,i} = 10 + 5k$,
- $Cq_{k,4} = 35 + 22k + k^2$,

- $Cq_{k,5} = 126 + 93k + 9k^2$,
- $Cq_{k,6} = 462 + 386k + 57k^2 + k^3$,
- $Cq_{k,7} = 1716 + 1586k + 312k^2 + 13k^3$.

We give the coefficient of k^η in Table 3, $0 \le \eta \le n$.

The following equation can be written as the product of two matrixes

$$
\begin{bmatrix} C.q_{k,1} \\ C.q_{k,2} \\ C.q_{k,3} \\ C.q_{k,4} \\ C.q_{k,5} \\ C.q_{k,6} \\ \vdots \end{bmatrix}
=
\begin{bmatrix}
1 & & & & & \\
1 & 1 & & & & \\
2 & 2 & 1 & & & \\
5 & 5 & 3 & 1 & & \\
14 & 14 & 9 & 4 & 1 & \\
42 & 42 & 28 & 14 & 5 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\begin{bmatrix} q_{k,1} \\ q_{k,2} \\ q_{k,3} \\ q_{k,4} \\ q_{k,5} \\ q_{k,6} \\ \vdots \end{bmatrix}.
$$

Cq_1	1			
Cq_2	1	3		
Cq_3	5	10		
Cq_4	1	22	35	
Cq_5	9	93	126	
Cq_6	21	57	386	462
Cq_7	13	312	1586	1716
...	...	...	...	...

Table 3. Catalan triangle of the Modified k-Pell sequence

We obtain the first few Catalan transform of the Modified k-Pell sequence as follow:

- $Cq_1 = \{0, 1, 4, 17, 58, 228, 906, \dots\}$,
- $Cq_2 = \{0, 1, 5, 24, 83, 348, 1470, \dots\}$,
- $Cq_3 = \{0, 1, 6, 31, 110, 486, 2160, \dots\}$,
- $Cq_4 = \{0, 1, 7, 38, 139, 642, 2982, \dots\}$,
- $Cq_5 = \{0, 1, 8, 45, 170, 816, 3942, \dots\}$.

There is the following relation between the Catalan of Modified k-Pell sequence $Cq_{k,n}$ and the Catalan of k-Pell–Lucas sequences $Q_{k,n}$:

$$2Cq_{k,n} = CQ_{k,n}, \quad (k, n \in \mathbb{N}).$$

We obtain the first few Catalan transform of the Modified k-Pell sequence as follow:

- $Cq_1 = \{0, 1, 4, 17, 58, 228, 906, \dots\}$,
- $Cq_2 = \{0, 1, 5, 24, 83, 348, 1470, \dots\}$,
- $Cq_3 = \{0, 1, 6, 31, 110, 486, 2160, \dots\}$,
- $Cq_4 = \{0, 1, 7, 38, 139, 642, 2982, \dots\}$,
- $Cq_5 = \{0, 1, 8, 45, 170, 816, 3942, \dots\}$.

3.4 Generating functions of Catalan transforms for these sequences

We know that the generating function of the k-Pell numbers, k-Pell–Lucas numbers, Modified k-Pell numbers and Catalan numbers, respectively, are

$$f(x) = \frac{-x}{1-2x-kx^2},$$

$$g(x) = \frac{2-2x}{1-2x-kx^2},$$

$$h(x) = \frac{1-x}{1-2x-kx^2},$$

and

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}$$

in [2]. Let $c(x)$ and $A(x)$, respectively, be the generating function of the sequence of the Catalan numbers $\{C_n\}$ and any sequence $\{a_n\}$. It was proved in [1] that $A(c(x))$ is the generating function of the Catalan transform of this last sequence. Consequently, from the composition of functions, we obtain that the generating function of the Catalan transform of the k-Pell sequence, k-Pell–Lucas sequence, Modified k-Pell sequence are, respectively,

$$CP_k(x) = P_k(x*c(x)) = f(c(x)) = \frac{x\left(-1+\sqrt{1-4x}\right)}{2x(2x-1)+\sqrt{1-4x}(2x+k)+k(2x-1)},$$

$$CQ_k(x) = q_k(x*c(x)) = g(c(x)) = \frac{2x\left(2x-1+\sqrt{1-4x}\right)}{2x(2x-1)+\sqrt{1-4x}(2x+k)+k(2x-1)}$$

and

$$Cq_k(x) = q_k(x*c(x)) = h(c(x)) = \frac{x\left(2x-1+\sqrt{1-4x}\right)}{2x(2x-1)+\sqrt{1-4x}(2x+k)+k(2x-1)}.$$

3.5 Hankel transform

Let $R = \{r_0, r_1, r_2 \ldots\}$ be a sequence of real numbers. The Hankel transform (see [5, 9]) of R is the sequence of determinants $H_n = \det\left[r_{i+j-2}\right]$, [8]. That is

$$H_n = \begin{vmatrix} r_0 & r_1 & r_2 & r_3 & \cdots \\ r_1 & r_2 & r_3 & r_4 & \cdots \\ r_2 & r_3 & r_4 & r_5 & \cdots \\ r_3 & r_4 & r_5 & r_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

The Hankel determinant of order n of R is the upper-left $n \times n$ subdeterminant of $\underline{H_n}$. We obtain that the Hankel transform of the Catalan transform of the k-Pell numbers:

$$HCP_1 = \det[1] = 1,$$

$$HCP_2 = \begin{vmatrix} 1 & 3 \\ 3 & 10+k \end{vmatrix} = 1+k,$$

$$HCP_3 = \begin{vmatrix} 1 & 3 & 10+k \\ 3 & 10+k & 35+7k \\ 10+k & 35+7k & 126+37k+k^2 \end{vmatrix} = 1+3k+k^2,$$

$$HCP_4 = \begin{vmatrix} 1 & 3 & 10+k & 35+7k \\ 3 & 10+k & 35+7k & 126+37k+k^2 \\ 10+k & 35+7k & 126+37k+k^2 & 462+386k+57k^2+k^3 \\ 35+7k & 126+37k+k^2 & 462+386k+57k^2+k^3 & 1716+794k+80k^2+k^3 \end{vmatrix}$$

$$= 1+6k+5k^2+k^3.$$

The Hankel transform of the Catalan transform of the k-Pell–Lucas numbers:

$$HCQ_1 = \det[2] = 2,$$

$$HCQ_2 = \begin{vmatrix} 2 & 6+2k \\ 6+2k & 20+14k \end{vmatrix} = 4+4k-4k^2,$$

$$HCQ_3 = \begin{vmatrix} 2 & 6+2k & 20+14k \\ 6+2k & 20+14k & 70+44k+2k^2 \\ 20+14k & 70+44k+2k^2 & 252+186k+18k^2 \end{vmatrix} = 8+552k-1072k^2-808k^3+32k^4,$$

$$HCQ_4 = \begin{vmatrix} 2 & 6+2k & 20+14k & 70+44k+2k^2 \\ 6+2k & 20+14k & 70+44k+2k^2 & 252+186k+18k^2 \\ 20+14k & 70+44k+2k^2 & 252+186k+18k^2 & 924+772k+114k^2+2k^3 \\ 70+44k+2k^2 & 252+186k+18k^2 & 924+772k+114k^2+2k^3 & 3432+3172k+624k^2+26k^3 \end{vmatrix}$$

$$= 16+10464k-216688k^2-397296k^3-249408k^4-25088k^5+22192k^6+368k^7+144k^8.$$

The Hankel transform of the Catalan transform of the Hankel Modified k-Pell sequence:

$$HCq_1 = \det[1] = 1,$$

$$HCq_2 = \begin{vmatrix} 1 & 3+k \\ 3+k & 10+5k \end{vmatrix} = 1-k-k^2,$$

$$HCq_3 = \begin{vmatrix} 1 & 3+k & 10+5k \\ 3+k & 10+5k & 35+22k+k^2 \\ 10+5k & 35+22k+k^2 & 126+93k+9k^2 \end{vmatrix} = 1-3k-4k^2-k^3$$

$$HCq_4 = \begin{vmatrix} 1 & 3+k & 10+5k & 35+22k+k^2 \\ 3+k & 10+5k & 35+22k+k^2 & 126+93k+9k^2 \\ 10+5k & 35+22k+k^2 & 126+93k+9k^2 & 462+176k+11k^2 \\ 35+22k+k^2 & 126+93k+9k^2 & 462+176k+11k^2 & 1716+1586k+312k^2+13k^3 \end{vmatrix}$$

$$= 1 - 3k - 4k^2 - k^3.$$

4   Conclusion

In the present paper, we define the Catalan k-Pell sequence, the Catalan k-Pell–Lucas sequence and the Catalan Modified k-Pell sequence and give some identities between the k-Pell, and the Catalan numbers, the k-Pell–Lucas and the Catalan numbers and Modified k-Pell–Lucas and the Catalan numbers. Also, we present some properties of the Catalan k-Pell sequence, the Catalan k-Pell–Lucas sequence and the Catalan Modified k-Pell sequence. This enables us to give in a straightforward way several formulas for the sums of such sequences. Also, we obtain the generating functions of the Catalan transform of the k-Pell sequence, k-Pell–Lucas sequence and the Modified k-Pell sequence. We put in for the Hankel transform to the Catalan transform of the k-Pell sequence, the Catalan k-Pell–Lucas sequence, the Catalan Modified k-Pell sequence and get some unknown properties. These identities can be used to develop new identities of polynomials.

Acknowledgements

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