Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 27, 2021, No. 1, 70–75
DOI: 10.7546/nntdm.2021.27.1.70-75

On a Diophantine equation arising in the history of mathematics

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Received: 1 November 2019 **Revised:** 31 January 2021 **Accepted:** 3 February 2021

Abstract: In the present paper, we offer the complete solution to a Diophantine equation, considered by S. Naranan [2]. For the history of the problem, see also L. E. Dickson [1].
Keywords: Diophantine equations, History of mathematics.
2010 Mathematics Subject Classification: 01A05, 01A32, 11D99.

1 Introduction

There is a legend saying that once three brothers came to the Indian Emperor Akbar (1556–1605) with the problem of sharing the wealth left by their deceased father. The father wanted the eldest son to get one half, the second son to get one third, while the youngest son to get one ninth of his entire property, which consisted of 17 elephants.

The solution to this problem was offered by the Emperor's favourite courtier Birbal, a poet, musician, and intellectual, mostly known from the Indian folk tales which focus on his wit. Birbal ordered the Royal Elephant to be lined up with the 17 elephants. Then the men took their parts of 9, 6, and 2 elephants, respectively, leaving behind the Royal Elephant.

Birbal's neat solution was possible, since

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{9}=\frac{17}{18},$$

which is smaller than 1.

Motivated by this historical problem, S. Naranan in [2] (see also [1]) considered the Diophantine equation

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{u}{u+1} \tag{1}$$

and determined the positive integer solutions with the following two conditions:

(a) x, y, z are unequal

(b) $u+1$ is the least common multiple of x, y and z.

It turned out that there are 7 such solutions.

However, the problem of determining all solutions, without any conditions, has remained open. The aim of this paper is to determine the complete solutions of Diophantine equation (1). Our method is elementary, but at some points, we are using also computer calculations.

2 All solutions of the problem

Since the equation (1) is symmetrical in x, y and z, we may assume that

$$x \leq y \leq z. \tag{2}$$

In what follows, we shall determine the solutions satisfying condition (2).

Since

$$\frac{3}{z} \leq \frac{1}{x}+\frac{1}{y}+\frac{1}{z} \leq \frac{3}{x},$$

particularly we get that

$$\frac{u}{u+1} \leq \frac{3}{x},$$

i.e., $ux \leq 3u+3$, implying:

$$u(x-3) \leq 3. \tag{3}$$

Here, clearly one has $u \geq 1$, so if $x \leq 3$, clearly (3) holds true. As $x=1$ is not possible (as the left-hand side of (1) would be greater than 1, while the right-hand side would be less than 1), we have two possibilities for x, namely:

(i) $x=2$,

(ii) $x=3$.

We will return to these cases later.

2.1 Cases $x = 4, 5, 6$

First suppose that $x \geq 4$. Remark that for $x \geq 7$, relation (3) is impossible, as then the left-hand side is at least $4u > 4$.

Therefore, the cases $x = 4, 5, 6$ should be considered only.

Case $x = 4$. Then from (3) we get $u \leq 3$, so that for u there are three possible cases: $u = 1, 2, 3$.

For $x = 4, u = 1$ we get the equation

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

As this can be written also as

$$4(y + z) = yz,$$

or

$$(y - 4)(z - 4) = 16.$$

As $4 \leq y \leq z$, we get immediately from here the solutions:

$$\begin{aligned} y &= 5, \quad z = 20 \\ y &= 6, \quad z = 12 \\ y &= 8, \quad z = 8. \end{aligned}$$

For $u = 2$ we get the equation

$$\frac{1}{y} + \frac{1}{z} = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}.$$

As $\frac{1}{y} \geq \frac{1}{z}$, we get $\frac{5}{12} \geq \frac{2}{z}$ and $\frac{5}{12} \leq \frac{2}{y}$, so $5y \leq 24$. As $y \geq 4$, we can have only $y = 4$. Then $z = 6$.

For $u = 3$, we get the equation

$$\frac{1}{y} + \frac{1}{z} = \frac{3}{4} - \frac{1}{4} = \frac{1}{2},$$

and it is immediate that one can have only $y = z = 4$.

By recapitulating, we have obtained up to now the following 5 solutions:

$$\begin{aligned} x &= 4, \quad y = 5, \quad z = 20, \quad u = 1 \\ x &= 4, \quad y = 6, \quad z = 12, \quad u = 1 \\ x &= 4, \quad y = 8, \quad z = 8, \quad u = 1 \\ x &= 4, \quad y = 4, \quad z = 6, \quad u = 2 \\ x &= 4, \quad y = 4, \quad z = 4, \quad u = 3. \end{aligned} \tag{4}$$

Case $x = 5$. Let us now turn to the case $x = 5$. From relation (3) we get $2u \leq 3$, which is possible only for $u = 1$. Then, we get the equation

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}.$$

If we would have $y \geq 7$, then we would obtain

$$\frac{1}{y} + \frac{1}{z} \leq \frac{2}{7} < \frac{3}{10},$$

that is a contradiction. As $y \geq x = 5$, we can have only $y = 5$ or $y = 6$. Then, we get the only solution:

$$x = 5, \quad y = 5, \quad z = 10, \quad u = 1. \tag{5}$$

Case $x = 6$. Finally, let $x = 6$. As from (3) we get $3u \le 3$, clearly this implies again $u = 1$. Then, we are lead to the equation

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}.$$

From $6 \le y \le z$ we get

$$\frac{1}{y} + \frac{1}{z} \le \frac{2}{6} = \frac{1}{3},$$

so the only possibility is $y = z = 6$.

Therefore, we have obtained now the following solution:

$$x = 6, \quad y = 6, \quad z = 6, \quad u = 1. \tag{6}$$

2.2 Cases $x = 2, 3$

Let now consider the cases (i) and (ii).

Case $x = 2$. Our first remark is that in this case $u > 1$, as the left side being higger than $\frac{1}{2}$, we get $\frac{u}{u+1} > \frac{1}{2}$, so $u > 1$.

Then equation (1) becomes

$$\frac{1}{y} + \frac{1}{z} = \frac{u}{u+1} - \frac{1}{2} = \frac{u-1}{2(u+1)}.$$

Since $2 \le y \le z$, one has

$$\frac{2}{z} \le \frac{1}{y} + \frac{1}{z} \le \frac{2}{y},$$

so particularly we can write:

$$\frac{u-1}{2(u+1)} \le \frac{2}{y},$$

i.e., $uy - y \le 4u + 4$, or

$$u(y-4) \le y + 4. \tag{7}$$

Remark that for $y \ge 5$ we get from (7) that

$$u \le \frac{y+4}{y-4}.$$

Now, if $y > 12$, then it is immediate that

$$\frac{y+4}{y-4} < 2,$$

as $y + 4 < 2y - 8$ gives $y > 12$. Thus, if $y \ge 13$, we obtain $u < 2$, which is impossible, as $u > 1$ (as we have seen above).

Therefore, for y there are possible only the following cases: $y = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$. Since

$$\frac{u-1}{2(u+1)} \le \frac{1}{2},$$

the case $y = 2$ is not possible.

It is immediate that, for each value of y from the above set, there are only a finite number of possibilities for both z and u.

For example, when $y = 3$, we get $z = \frac{6(u+1)}{u-5}$.

Now, as

$$\frac{6(u+1)}{u-5} = 6 + \frac{36}{u-5},$$

clealy, $u - 5$ should be a divisor of 36, i.e., one of numbers $1, 2, 3, 4, 6, 9, 12, 18$ and 36. Since here there are quite a many solutions, a computer search could be employed. The following 25 solutions will be obtained, with the aid of a computer (in all cases, $x = 2$):

$$\begin{array}{llll}
x=2, & y=3, & z=7, & u=41 \\
x=2, & y=3, & z=8, & u=23 \\
x=2, & y=3, & z=9, & u=17 \\
x=2, & y=3, & z=10, & u=14 \\
x=2, & y=3, & z=12, & u=11 \\
x=2, & y=3, & z=15, & u=9 \\
x=2, & y=3, & z=18, & u=8 \\
x=2, & y=3, & z=24, & u=7 \\
x=2, & y=3, & z=42, & u=6 \\
x=2, & y=4, & z=5, & u=19 \\
x=2, & y=4, & z=6, & u=11 \\
x=2, & y=4, & z=8, & u=7 \\
x=2, & y=4, & z=12, & u=5 \\
x=2, & y=4, & z=20, & u=4 \\
x=2, & y=5, & z=5, & u=9 \\
x=2, & y=5, & z=10, & u=4 \\
x=2, & y=5, & z=20, & u=3 \\
x=2, & y=6, & z=6, & u=5 \\
x=2, & y=6, & z=12, & u=3 \\
x=2, & y=7, & z=42, & u=2 \\
x=2, & y=8, & z=8, & u=3 \\
x=2, & y=8, & z=24, & u=2 \\
x=2, & y=9, & z=18, & u=2 \\
x=2, & y=10, & z=15, & u=2 \\
x=2, & y=12, & z=12, & u=2.
\end{array} \tag{8}$$

Case $x = 3$. Similarly to the above, we get the equation

$$\frac{1}{y} + \frac{1}{z} = \frac{2u-1}{3(u+1)},$$

with $3 \le y \le z$.

Now, again $\frac{1}{z} \le \frac{1}{y}$, so we get

$$y \le \frac{6(u+1)}{(2u-1)}.$$

Now, remark that $\frac{6(u+1)}{(2u-1)} < 13$, as this is equivalent to $20u > 19$, which is valid. Therefore, for y one has again to consider the cases $y = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$.

With the aid of a computer, we get the following 12 solutions (in all cases, $x = 3$)

$$\begin{array}{llll}
x = 3, & y = 3, & z = 4, & u = 11 \\
x = 3, & y = 3, & z = 6, & u = 5 \\
x = 3, & y = 3, & z = 12, & u = 3 \\
x = 3, & y = 4, & z = 4, & u = 5 \\
x = 3, & y = 4, & z = 6, & u = 3 \\
x = 3, & y = 4, & z = 12, & u = 2 \\
x = 3, & y = 6, & z = 6, & u = 2 \\
x = 3, & y = 7, & z = 42, & u = 1 \\
x = 3, & y = 8, & z = 24, & u = 1 \\
x = 3, & y = 9, & z = 18, & u = 1 \\
x = 3, & y = 10, & z = 25, & u = 1 \\
x = 3, & y = 12, & z = 12, & u = 1.
\end{array} \tag{9}$$

Finally, we can state the following theorem.

Theorem. All 44 solutions of equation (1) that satisfy condition (2) are given by (4), (5), (6), (8) and (9).

References

[1] Dickson, L. E. (1952). *History of the Theory of Numbers, Vol. II*, Chelsea, New York.

[2] Naranan, S. (1973). An “Elephantine” equation. *Mathematics Magazine*, 46(5), 276–278.