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On new arithmetic function relative to a fixed positive integer. Part 1

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Abstract: The main purpose of this note is to define a new arithmetic function relative to a fixed positive integer and to study some of its properties.
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1 Introduction

Throughout this paper, we let (a, b) denote the greatest common divisor of any two integers a and b. Let

$$n = \prod_{i=1}^{r} p_i^{e_i}$$

be the prime factorization of the positive integer $n > 1$, where $r, e_1, e_2, \ldots, e_r$ are positive integers and $p_1, p_2, \ldots, p_r$ are different primes.

In recent years, many researchers have published many papers that have been the subject of arithmetic functions (see e.g., [1–6]). In [1], Atanassov defined the following function:

$$\underline{\text{mult}}(n) = \prod_{i=1}^{r} p_i, \quad \underline{\text{mult}}(1) = 1.$$

The aim of this note is to define a new arithmetic function relative to a fixed positive integer α, that can be considered a generalization of Atanassov's function and discuss some of its properties.

2 Main results

Let α be a positive integer. Then we define f_α to be the arithmetic function such that:

$$f_\alpha(n) = \prod_{i=1}^{r} p_i^{(e_i,\alpha)}, \quad f_\alpha(1) = 1. \tag{1}$$

In particular, if $\alpha = 1$, then $(e_i, \alpha) = 1$ for all $(1 \le i \le r)$. Thus

$$f_1(n) = \underline{\text{mult}}(n), \quad \text{for all } n.$$

For examples, see Table 1.

Let m be a positive integer such that $m = \prod_{j=1}^{s} q_j^{f_j}$, where $s, f_1, f_2, \ldots, f_s$ are positive integers and $q_1, q_2, \ldots, q_s$ are different primes. If $(m, n) = 1$ i.e., ($q_j \ne p_i$ for all $1 \le i \le r$ and $1 \le j \le s$), then for all α:

$$f_\alpha(mn) = \prod_{j=1}^{s} q_j^{(f_j,\alpha)} \prod_{i=1}^{r} p_i^{(e_i,\alpha)} = f_\alpha(m) f_\alpha(n).$$

On the other hand, if p_1, p_2, and p_3 are different primes, then for all α:

$$f_\alpha(p_1 \cdot p_2^2 \cdot p_3) = p_1 \cdot p_2^{(2,\alpha)} \cdot p_3, \text{while that} f_\alpha(p_1 \cdot p_2) f_\alpha(p_2 \cdot p_3) = p_1 \cdot p_2^2 \cdot p_3.$$

Consequently, one can show that the function f_α is multiplicative but not completely multiplicative.

n	$f_2(n)$	$f_3(n)$	n	$f_2(n)$	$f_3(n)$	n	$f_2(n)$	$f_3(n)$	n	$f_2(n)$	$f_3(n)$
1	1	1	11	11	11	21	21	21	31	31	31
2	2	2	12	12	6	22	22	22	32	2	2
3	3	3	13	13	13	23	23	23	33	33	33
4	4	2	14	14	14	24	6	24	34	34	34
5	5	5	15	15	15	25	25	5	35	35	35
6	6	6	16	4	2	26	26	26	36	36	6
7	7	7	17	17	17	27	3	27	37	37	37
8	2	8	18	18	6	28	28	14	38	38	38
9	9	3	19	19	19	29	29	29	39	39	39
10	10	10	20	20	10	30	30	30	40	10	40

Table 1. The first 40 values of f_2 and f_3.

It can be easily seen that $1 < f_\alpha(n) \le n$ and $f_\alpha(n) | n$ for all $n > 1$, since $(e_i, \alpha) \le e_i$ for all α. So, as a consequence $f_\alpha(p) = p$ for all primes p. The following theorem distinguishes those numbers that satisfy the equality: $f_\alpha(n) = n$ (for all α).

Theorem 2.1. *For any integer $\alpha \ge 1$, the square-free positive integers are the only integers satisfying $f_\alpha(n) = n$.*

Proof. Clearly, if n is a square-free number, i.e., $e_1 = e_2 = \cdots = e_r = 1$, then $f_\alpha(n) = n$. Now let n be such that $f_\alpha(n) = n$ for all α. Thus

$$(e_i, \alpha) = e_i \quad \text{for all } \alpha,$$

which is true only if $e_i = 1$ for all $1 \le i \le r$, i.e., only if n is a square-free number. □

Corollary 2.1.1. *Let α be a positive integer. Then for every n:*

$$f_\alpha(\underline{\text{mult}}(n)) = \underline{\text{mult}}(n) = \underline{\text{mult}}(f_\alpha(n)).$$

Proof. It is well known that $\underline{\text{mult}}(n)$ is a square-free number for every n, so by Theorem 2.1:

$$f_\alpha(\underline{\text{mult}}(n)) = \underline{\text{mult}}(n).$$

On the other hand, we have

$$\underline{\text{mult}}(f_\alpha(n)) = \underline{\text{mult}}\left(\prod_{i=1}^{r} p_i^{(e_i,\alpha)}\right) = \prod_{i=1}^{r} p_i = \underline{\text{mult}}(n).$$ □

Theorem 2.2. *Let α and β be positive integers. Then for every n:*

$$f_\alpha(f_\beta(n)) = f_{(\alpha,\beta)}(n).$$

In particular, if $(\alpha, \beta) = 1$, then

$$f_\alpha(f_\beta(n)) = \underline{\text{mult}}(n).$$

Proof. For $n = 1$, the statement is true. If $n > 1$, then we have

$$\begin{aligned} f_\alpha(f_\beta(n)) &= f_\alpha\left(f_\beta\left(\prod_{i=1}^{r} p_i^{e_i}\right)\right) \\ &= f_\alpha\left(\prod_{i=1}^{r} p_i^{(e_i,\beta)}\right) \\ &= \prod_{i=1}^{r} p_i^{((e_i,\beta),\alpha)} = \prod_{i=1}^{r} p_i^{(e_i,(\beta,\alpha))} \\ &= f_{(\alpha,\beta)}(n). \end{aligned}$$

Let us suppose that $(\alpha, \beta) = 1$. Then

$$f_\alpha(f_\beta(n)) = \prod_{i=1}^{r} p_i^{(e_i,1)} = \prod_{i=1}^{r} p_i = \underline{\text{mult}}(n).$$ □

Theorem 2.3. *Let $e = \text{lcm}(e_1, e_2, \ldots, e_r)$. Then $f_\alpha(n)$ is a periodic function with period e as a function of α, in other words:*

$$f_{\alpha+e}(n) = f_\alpha(n), \quad \textit{for all } \alpha.$$

Proof. First of all, there exist r positive integers $(k_1, k_2, \ldots, k_r)$ such that $e = k_i e_i$ $(1 \leq i \leq r)$, since $e = \operatorname{lcm}(e_1, e_2, \ldots, e_r)$. This means

$$(e_i, \alpha + e) = (e_i, \alpha + k_i e_i) = (e_i, \alpha) \quad (1 \leq i \leq r),$$

from which, we can get

$$f_{\alpha+e}(n) = \prod_{i=1}^{r} p_i^{(e_i,\alpha+e)} = \prod_{i=1}^{r} p_i^{(e_i,\alpha)} = f_\alpha(n). \qquad \square$$

Theorem 2.4. *Let α and β be positive integers such that $\beta = \alpha\beta'$. If $(e_i, \beta') = 1$ for all $1 \leq i \leq r$, then*

$$f_\alpha(n) = f_\beta(n).$$

Proof. We have

$$(e_i, \beta) = (e_i, \alpha\beta') = (e_i, \alpha),$$

since $(e_i, \beta') = 1$ $(1 \leq i \leq r)$. Thus

$$f_\beta(n) = \prod_{i=1}^{r} p_i^{(e_i,\beta)} = \prod_{i=1}^{r} p_i^{(e_i,\alpha\beta')} = \prod_{i=1}^{r} p_i^{(e_i,\alpha)} = f_\alpha(n). \qquad \square$$

Many mathematicians have been studied the perfect numbers and their generalizations with the help of various arithmetic functions (see e.g., [7, 8, 11]). In [9, 10], some arithmetic functions are used in characterizing generalized Mersenne primes. These primes are then used in the study of class numbers of certain number fields (see [10]). Euler showed that all even perfect numbers (EPN) are of the form $2^{p-1}m$, where $m = 2^p - 1$ is a Mersenne prime. Also, Euler stated that an odd perfect number (OPN), if it exists, must have the form $p^e m^2$, where p is a prime with $(p, m) = 1$ and $p \equiv e \equiv 1 \pmod 4$.

The next theorem gives the values of f_2 for perfect numbers.

Theorem 2.5. *Let $N > 6$ be a perfect number. Then*

$$f_2(N) = \begin{cases} 4m & \text{if } N = 2^{p-1}m \text{ is an EPN}, \\ p \cdot \underline{\text{mult}}(m)^2 & \text{if } N = p^e m^2 \text{ is an OPN}. \end{cases}$$

Proof. Let $N = 2^{p-1}m$ be an even perfect number. Then $m = 2^p - 1$ is a Mersenne prime. But, in order for m to be a prime, p must itself be a prime. Thus $(p - 1, 2) = 2$, since $N > 6$, from which we have

$$f_2\left(2^{p-1}m\right) = f_2\left(2^{p-1}\right) f_2(m) = 2^{(p-1,2)}m = 4m.$$

Let $N = p^e m^2$ be an odd perfect number. Then $(p, m) = 1$ and $p \equiv e \equiv 1 \pmod 4$. So $(e, 2) = 1$. On the other hand, if $m = \prod_{i=1}^{r} p_i^{e_i}$, then $m^2 = \prod_{i=1}^{r} p_i^{2e_i}$ and $(2e_i, 2) = 2$ for all $(1 \leq i \leq r)$. This means

$$f_2(p^e m^2) = f_2(p^e) f_2(m^2) = p^{(e,2)} \prod_{i=1}^{r} p_i^{(2e_i,2)} = p \left(\prod_{i=1}^{r} p_i\right)^2 = p \cdot \underline{\text{mult}}(m)^2. \qquad \square$$

3 Conclusion

In conclusion, we will mention, that in the second part we will study a new arithmetic function relative to a fixed positive integer α, which will define by substituting the (e_i, α) for the $[e_i, \alpha]$ in (1), where $[e_i, \alpha]$ is the least common multiple of α and e_i.

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