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Dual bicomplex Horadam quaternions

Kübra Gül

Department of Computer Engineering, University of Kafkas
Kars, Turkey
e-mail: kubra.gul@kafkas.edu.tr

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Abstract: The aim of this work is to introduce a generalization of dual quaternions called dual bicomplex Horadam quaternions and to present some properties, the Binet's formula, Catalan's identity, Cassini's identity and the summation formula for this type of bicomplex quaternions. Furthermore, several identities for dual bicomplex Fibonacci quaternions are given.

Keywords: Bicomplex number, Dual number, Fibonacci number, Horadam number, Bicomplex quaternion, Dual quaternion.
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1 Introduction

The quaternion algebra recently has played a significant role in several areas of science [1, 2]. It is given by the Clifford algebra classifications $Cl_{0,2}(R) \cong Cl^0_{3,0}(R)$. In [17], Hamilton introduced the set of quaternions which can be represented as

$$H = \{q \mid q = q_0 + q_1 i + q_2 j + q_3 k\ ,\ q_s \in \mathbb{R},\ s = 0, 1, 2, 3\}$$

where $i^2 = j^2 = k^2 = ijk = -1$ and $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$.

The number sequences have many applications in quaternion theory. The study of the quaternions of sequences began with the earlier work of Horadam [19] where the Fibonacci quaternion was investigated. There are several studies on different quaternions and their generalizations, for example, [7, 9–13, 28].

In 1892, Segre introduced bicomplex numbers, which were similar to quaternions in many algebraic properties [27]. In [4, 25], the authors defined the bicomplex Fibonacci, Lucas and Pell numbers. And then, in [14], the author gave a generalization for bicomplex Fibonacci numbers. Later, by using the bicomplex numbers, in [3, 5], Torunbalcı Aydın studied the bicomplex Fibonacci quaternions. In [15, 16], the authors defined a new sequence with coefficients from the complex Fibonacci numbers. In [7], Catarino defined the bicomplex k-Pell quaternions and gave some properties involving these quaternions. In [33], the authors defined the bicomplex generalized k-Horadam quaternions.

In this paper, our aim is to continue the development of bicomplex numbers. Motivated by the above papers, we introduce a new generalization of dual quaternions called dual bicomplex Horadam quaternions, and we obtain Binet's formula, generating functions, and summation formula, as well as some other properties. The real part of dual bicomplex Fibonacci quaternions was previously studied in [3] and dual part was defined in [6]. The results have been confirmed once more and also have been given some properties of the dual bicomplex Fibonacci quaternions by using [6] in this study. In [16], by using Binet formula, Halici and Çürük gave some identities concerning newly defined numbers with complex Fibonacci coefficients. We will present some of these identities with Fibonacci coefficients.

In the remaining part of this section, we give a brief summary about quaternions, bicomplex numbers and dual numbers.

The set of bicomplex numbers, denoted by BC, forms a two-dimensional algebra over $\mathbb{C}$, thus the bicomplex numbers are an algebra over $\mathbb{R}$ of dimension four. A set of bicomplex numbers forms a real vector space with addition and scalar multiplication operations. Also, the vector space BC is a commutative algebra with the properties of multiplication and the product of the bicomplex numbers, and is a real associative algebra with bicomplex product. Furthermore, there are similarities in terms of some structures and properties between complex and bicomplex numbers, but there are some differences. Bicomplex numbers form a commutative ring with unity which contain the complex numbers [7, 22].

The set of bicomplex numbers is defined as follows:

$$BC = \{z_1 + z_2 j \mid z_1,\ z_2 \in \mathbb{C}\},$$

where j is an imaginary unit such that $i^2 = j^2 = -1$, $ij = ji$ and $\mathbb{C}$ is the set of complex numbers with $i = \sqrt{-1}$. Thus, the set of bicomplex numbers can be expressed by

$$BC = \{q \mid q = a_1 + a_2 i + a_3 j + a_4 ij \ \&\ a_1,\ a_2,\ a_3,\ a_4 \in \mathbb{R}\}.$$

Multiplication of basis elements of bicomplex numbers is given in the following Table 1.

$\times$	1	i	j	ij
1	1	i	j	ij
i	i	-1	ij	$-j$
j	j	ij	-1	$-i$
ij	ij	$-j$	$-i$	1

Table 1. Multiplication of imaginary units

Bicomplex numbers have three different conjugations (involutions), which are as follows:

$$\begin{aligned} q_i^* &= a_1 - a_2 i + a_3 j - a_4 ij, \\ q_j^* &= a_1 + a_2 i - a_3 j - a_4 ij, \\ q_{ij}^* &= a_1 - a_2 i - a_3 j + a_4 ij \end{aligned}$$

for $q = a_1 + a_2 i + a_3 j + a_4 ij$. The norms of the bicomplex numbers which arise from the definitions of involutions are defined as

$$\begin{aligned} N_{q_i^*} &= \|q \times q_i^*\| = \sqrt{\left|a_1^2 + a_2^2 - a_3^2 - a_4^2 + 2j\left(a_1 a_3 + a_2 a_4\right)\right|}, \\ N_{q_j^*} &= \left\|q \times q_j^*\right\| = \sqrt{\left|a_1^2 - a_2^2 + a_3^2 - a_4^2 + 2i\left(a_1 a_2 + a_3 a_4\right)\right|}, \\ N_{q_{ij}^*} &= \left\|q \times q_{ij}^*\right\| = \sqrt{\left|a_1^2 + a_2^2 + a_3^2 + a_4^2 + 2ij\left(a_1 a_4 - a_2 a_3\right)\right|}. \end{aligned}$$

Note that all of these norms are isotropic. For example, we calculate the norm $N_{q_i^*}$ for $q = 1 + ij$.

$$N_{q_i^*} = (1 + ij)(1 - ij) = 1^2 - ij + ji - (ij)^2 = 0.$$

Bicomplex numbers and quaternions are generalizations of complex numbers. But there are some differences between them. We can list them as real quaternions are non-commutative, do not have zero divisors and non-trivial idempotent elements, but bicomplex numbers are commutative, have zero divisors and non-trivial idempotent elements:

$$\begin{aligned} ij &= ji, \\ (i + j)(i - j) &= i^2 - ij + ji - j^2 = 0, \\ \left(\frac{1 + ij}{2}\right)^2 &= \frac{1 + ij}{2}. \end{aligned}$$

For details, we refer to [22, 26, 28]

Similarly, bicomplex quaternions are defined by the basis: $\{1, i, j, ij\}$

$$\mathbb{C}_2^Q = \{Q \mid Q = q_1 + q_2 i + q_3 j + q_4 ij \;\&\; q_1, q_2, q_3, q_4 \in \mathbb{R}\},$$

where $i^2 = j^2 = -1$, $ij = ji$.

For any bicomplex quaternions $Q = q_1 + q_2 i + q_3 j + q_4 ij$ and $P = p_1 + p_2 i + p_3 j + p_4 ij$, the addition and multiplication of these bicomplex quaternions are given respectively by

$$Q + P = (q_1 + p_1) + (q_2 + p_2)\, i + (q_3 + p_3)\, j + (q_4 + p_4)\, ij$$

and

$$\begin{aligned} Q \times P &= (q_1 p_1 - q_2 p_2 - q_3 p_3 - q_4 p_4) + i\,(q_1 p_2 + q_2 p_1 - q_3 p_4 - q_4 p_3) \\ &\quad + j\,(q_1 p_3 + q_3 p_1 - q_2 p_4 - q_4 p_2) + ij\,(q_1 p_4 + q_4 p_1 + q_2 p_3 + q_3 p_2) \\ &= P \times Q. \end{aligned}$$

The dual number invented by Clifford [8] has the form $A = a + \varepsilon a^*$, where a, a^* real numbers and ε is the dual unit such that $\varepsilon \neq 0$, $\varepsilon^2 = 0$.

A dual quaternion is an extension of dual numbers. The dual quaternion is a Clifford algebra that can be used to represent spatial rigid body displacements in mathematics and mechanics. Rigid motions in 3-dimensional space can be represented by dual quaternions of unit length this fact is used a theoretical kinematics and in applications to 3-dimensional computer graphics, robotics and computer vision [23, 29, 32]. The dual quaternion is represented by: $Q = q + \varepsilon q^*$, where q and q^* are quaternions and ε is the dual unit such that $\varepsilon^2 = 0$, $\varepsilon \neq 0$. The dual quaternion Q can be written as

$$Q = (q_0 + \varepsilon q_0^*) + i\,(q_1 + \varepsilon q_1^*) + j\,(q_2 + \varepsilon q_2^*) + k\,(q_3 + \varepsilon q_3^*)\,,$$

where $q = q_0 + iq_1 + jq_2 + kq_3$ and $q^* = q_0^* + iq_1^* + jq_2^* + kq_3^*$. So the dual quaternion is constructed by eight real parameters. In addition, the set of dual quaternions forms a noncommutative but associative algebra over the real numbers. Further background on dual quaternions may be seen in [17, 24].

2 Dual bicomplex Fibonacci quaternions

Our aim is to introduce a new generalization of dual quaternions and to give some properties for the dual bicomplex Fibonacci quaternion, thus we will remind the some necessary definitions and concepts.

The n-th dual Fibonacci and the n-th dual Lucas numbers are defined respectively by

$$\tilde{F}_n = F_n + \varepsilon F_{n+1}, \qquad \tilde{L}_n = L_n + \varepsilon L_{n+1},$$

where F_n and L_n are the n-th Fibonacci and the n-th Lucas numbers.

In [3], the bicomplex Fibonacci quaternions are defined

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + ijF_{n+3}$$

where $i^2 = -1$, $j^2 = -1$, $ij = ji$.

The dual bicomplex Fibonacci numbers are defined in [6] as follows

$$\widetilde{x}_n = \tilde{F}_n + i\tilde{F}_{n+1} + j\tilde{F}_{n+2} + ij\tilde{F}_{n+3}.$$

Definition 1. *The dual bicomplex Fibonacci quaternions are defined*

$$\tilde{Q}_n = Q_n + \varepsilon Q_{n+1} \tag{1}$$

where Q_n is the bicomplex Fibonacci quaternion and $\varepsilon^2 = 0, \varepsilon = (0, 1)$.

By using dual Fibonacci numbers $\tilde{F}_n$, we get

$$\tilde{Q}_n = \tilde{F}_n + i\tilde{F}_{n+1} + j\tilde{F}_{n+2} + ij\tilde{F}_{n+3}. \tag{2}$$

From Definition 1, it is obvious that $\tilde{Q}_{n+1} = \tilde{Q}_n + \tilde{Q}_{n-1}$ with the initial conditions $\tilde{Q}_0 = \varepsilon + i\,(1+\varepsilon) + j\,(1+2\varepsilon) + ij\,(2+3\varepsilon)$ and $\tilde{Q}_1 = 1 + \varepsilon + i\,(1+2\varepsilon) + j\,(2+3\varepsilon) + ij\,(3+5\varepsilon)$.

By taking into account definition and the addition, subtraction and multiplication and product with a scalar of two dual bicomplex Fibonacci quaternions $\tilde{Q}_n$ and $\tilde{Q}_m$ are given by

$$\begin{aligned}
\tilde{Q}_n \mp \tilde{Q}_m &= Q_n \mp Q_m + \varepsilon\,(Q_{n+1} \mp Q_{m+1}) \\
&= (F_n \mp F_m) + i\,(F_{n+1} \mp F_{m+1}) + j\,(F_{n+2} \mp F_{m+2}) + ij\,(F_{n+3} \mp F_{m+3}) \\
&\quad + \varepsilon\,((F_{n+1} \mp F_{m+1}) + i\,(F_{n+2} \mp F_{m+2}) + j\,(F_{n+3} \mp F_{m+3}) + ij\,(F_{n+4} \mp F_{m+4}))\,, \\
\tilde{Q}_n\tilde{Q}_m &= Q_nQ_m + \varepsilon\,(Q_nQ_{m+1} + Q_{n+1}Q_m) = \tilde{Q}_m\tilde{Q}_n.
\end{aligned}$$

The different conjugates of dual bicomplex Fibonacci quaternion $\tilde{Q}_n$ are presented as in [6]:

$$\left(\tilde{Q}_n\right)_i^* = \tilde{F}_n - i\tilde{F}_{n+1} + j\tilde{F}_{n+2} - ij\tilde{F}_{n+3},$$
$$\left(\tilde{Q}_n\right)_j^* = \tilde{F}_n + i\tilde{F}_{n+1} - j\tilde{F}_{n+2} - ij\tilde{F}_{n+3},$$
$$\left(\tilde{Q}_n\right)_{ij}^* = \tilde{F}_n - i\tilde{F}_{n+1} - j\tilde{F}_{n+2} + ij\tilde{F}_{n+3}.$$

Note that by the equation (2) and the conjugates of $\tilde{Q}_n$, the desired results are found.

Theorem 2.1. *Let $\left(\tilde{Q}_n\right)_i^*$, $\left(\tilde{Q}_n\right)_j^*$ and $\left(\tilde{Q}_n\right)_{ij}^*$ be the conjugates of dual bicomplex Fibonacci quaternion. We can give the following relations:*

$$\tilde{Q}_n\left(\tilde{Q}_n\right)_i^* = -\tilde{L}_{2n+3} + 2j\tilde{F}_{2n+3} + \varepsilon\left(2jF_{2n+4} - L_{2n+4}\right),$$
$$\tilde{Q}_n\left(\tilde{Q}_n\right)_j^* = -\tilde{F}_{2n+3} + 2i\tilde{F}_{2n+2} - 2F_{n-1}F_{n+2} + 2iF_{n+2}^2\left(1+\varepsilon\right) + \varepsilon 2iF_{2n+5} - F_{2n+4} - 4F_{n+1}^2,$$
$$\tilde{Q}_n\left(\tilde{Q}_n\right)_{ij}^* = 3\tilde{F}_{2n+3} + 3\varepsilon F_{2n+4} + 2ij\left(-1\right)^{n+1}\left(1+\varepsilon\right).$$

Proof. Using the equations $F_nF_m + F_{n+1}F_{m+1} = F_{n+m+1}$ and $F_mF_{n+1} - F_{m+1}F_n = (-1)^{n+1}F_{m-n}$, we have

$$\begin{aligned}
\tilde{Q}_n\left(\tilde{Q}_n\right)_i^* &= \tilde{F}_n^2 + \tilde{F}_{n+1}^2 - \tilde{F}_{n+2}^2 - \tilde{F}_{n+3}^2 + 2j\left(\tilde{F}_n\tilde{F}_{n+2} + \tilde{F}_{n+1}\tilde{F}_{n+3}\right)\\
&= F_n^2 + F_{n+1}^2 - F_{n+2}^2 - F_{n+3}^2 + 2\varepsilon\left(F_nF_{n+1} + F_{n+1}F_{n+2} - F_{n+2}F_{n+3} - F_{n+3}F_{n+4}\right)\\
&\quad + 2j\left[\left(F_nF_{n+2} + F_{n+1}F_{n+3}\right) + \varepsilon\left(F_{n+1}F_{n+2} + F_nF_{n+3} + F_{n+2}F_{n+3} + F_{n+1}F_{n+4}\right)\right]\\
&= F_{2n+1} - F_{2n+5} + 2\varepsilon\left(F_{2n+2} - F_{2n+6}\right) + 2j\left(F_{2n+3} + 2\varepsilon F_{2n+4}\right)\\
&= -L_{2n+3} - 2\varepsilon L_{2n+4} + 2j\left(F_{2n+3} + 2\varepsilon F_{2n+4}\right)\\
&= -\tilde{L}_{2n+3} + 2j\tilde{F}_{2n+3} + \varepsilon\left(2jF_{2n+4} - L_{2n+4}\right)
\end{aligned}$$

$$\begin{aligned}
\tilde{Q}_n\left(\tilde{Q}_n\right)_j^* &= \tilde{F}_n^2 - \tilde{F}_{n+1}^2 + \tilde{F}_{n+2}^2 - \tilde{F}_{n+3}^2 + 2i\left(\tilde{F}_n\tilde{F}_{n+1} + \tilde{F}_{n+2}\tilde{F}_{n+3}\right)\\
&= F_n^2 - F_{n+1}^2 + F_{n+2}^2 - F_{n+3}^2 + 2\varepsilon\left(F_nF_{n+1} - F_{n+1}F_{n+2} + F_{n+2}F_{n+3} - F_{n+3}F_{n+4}\right)\\
&\quad + 2i\left[\left(F_nF_{n+1} + F_{n+2}F_{n+3}\right) + \varepsilon\left(F_{n+1}F_{n+1} + F_nF_{n+2} + F_{n+2}F_{n+4} + F_{n+3}F_{n+3}\right)\right]\\
&= -F_{2n+3} - 2F_{n-1}F_{n+2} + 2\varepsilon\left(-F_{2n+4} - 2F_{n+1}F_{n+1}\right)\\
&\quad + 2i\left(F_{2n+2} + F_{n+2}F_{n+2} + \varepsilon\left(F_{2n+3} + F_{2n+5} + F_{n+2}F_{n+2}\right)\right)\\
&= -\tilde{F}_{2n+3} + 2i\tilde{F}_{2n+2} - 2F_{n-1}F_{n+2} + 2iF_{n+2}^2\left(1+\varepsilon\right) + \varepsilon\left(2iF_{2n+5} - F_{2n+4} - 4F_{n+1}^2\right)
\end{aligned}$$

$$\begin{aligned}
\tilde{Q}_n\left(\tilde{Q}_n\right)_{ij}^* &= \tilde{F}_n^2 + \tilde{F}_{n+1}^2 + \tilde{F}_{n+2}^2 + \tilde{F}_{n+3}^2 + 2ij\left(\tilde{F}_n\tilde{F}_{n+3} - \tilde{F}_{n+1}\tilde{F}_{n+2}\right)\\
&= F_n^2 + F_{n+1}^2 + F_{n+2}^2 + F_{n+3}^2 + 2\varepsilon\left(F_nF_{n+1} + F_{n+1}F_{n+2} + F_{n+2}F_{n+3} + F_{n+3}F_{n+4}\right)\\
&\quad + 2ij\left(F_nF_{n+3} - F_{n+1}F_{n+2} + \varepsilon\left(F_nF_{n+4} - F_{n+2}F_{n+2}\right)\right)\\
&= F_{2n+1} + F_{2n+5} + 2\varepsilon\left(F_{2n+2} + F_{2n+6}\right) + 2ij\left(-1\right)^{n+1}\left(1+\varepsilon\right)\\
&= 3F_{2n+3} + 6\varepsilon F_{2n+4} + 2ij\left(-1\right)^{n+1}\left(1+\varepsilon\right)\\
&= 3\tilde{F}_{2n+3} + 3\varepsilon F_{2n+4} + 2ij\left(-1\right)^{n+1}\left(1+\varepsilon\right).
\end{aligned}$$
□

Theorem 2.2. *For the Lucas number L_n and the dual bicomplex Fibonacci quaternion with negative indices $\tilde{Q}_{-n}$, we have* $\tilde{Q}_{-n} = (-1)^{n+1}\tilde{Q}_n + (-1)^n L_n\tilde{Q}_0$.

Proof. By using the identity $F_{-n} = (-1)^{n+1} F_n$ [21], we obtain

$$\begin{aligned}
\tilde{Q}_{-n} &= \tilde{F}_{-n} + \tilde{F}_{-n+1}i + \tilde{F}_{-n+2}j + \tilde{F}_{-n+3}ij \\
&= \tilde{F}_{-n} + \tilde{F}_{-(n-1)}i + \tilde{F}_{-(n-2)}j + \tilde{F}_{-(n-3)}ij \\
&= (-1)^{n+1} F_n + (-1)^n F_{n-1}i + (-1)^{n-1} F_{n-2}j + (-1)^{n-2} F_{n-3}ij \\
&\quad + \varepsilon((-1)^n F_{n-1} + (-1)^{n-1} F_{n-2}i + (-1)^{n-2} F_{n-3}j + (-1)^{n-3} F_{n-4}ij) \\
&= (-1)^{n+1} (F_n + F_{n+1}i + F_{n+2}j + F_{n+3}ij) + (-1)^n F_{n-1}i - (-1)^{n+1} F_{n+1}i \\
&\quad + (-1)^{n-1} F_{n-2}j - (-1)^{n+1} F_{n+2}j + (-1)^{n-2} F_{n-3}ij - (-1)^{n+1} F_{n+3}ij \\
&\quad + \varepsilon\left((-1)^{n+1} (F_{n+1} + F_{n+2}i + F_{n+3}j + F_{n+4}ij)\right) + \varepsilon((-1)^n F_{n-1} - (-1)^{n+1} F_{n+1} \\
&\quad + (-1)^{n-1} F_{n-2}i - (-1)^{n+1} F_{n+2}i + (-1)^{n-2} F_{n-3}j - (-1)^{n+1} F_{n+3}j \\
&\quad + (-1)^{n-3} F_{n-4}ij - (-1)^{n+1} F_{n+4}ij) \\
&= (-1)^{n+1} (F_n + F_{n+1}i + F_{n+2}j + F_{n+3}ij) + (-1)^n ((F_{n-1} + F_{n+1})\, i + (F_{n+2} - F_{n-2})\, j \\
&\quad + (F_{n-3} + F_{n+3})\, ij) + \varepsilon\, (-1)^{n+1} (F_{n+1} + F_{n+2}i + F_{n+3}j + F_{n+4}ij) \\
&\quad + \varepsilon((-1)^n (F_{n-1} + F_{n+1}) + (-1)^n (F_{n+2} - F_{n-2})\, i \\
&\quad + (-1)^n (F_{n-3} + F_{n+3})\, j + (-1)^n (F_{n+4} - F_{n-4})\, ij) \\
&= (-1)^{n+1} \tilde{Q}_n + (-1)^n L_n (i + j + 2ij) + \varepsilon\, (-1)^n L_n (1 + i + 2j + 3ij) \\
&= (-1)^{n+1} \tilde{Q}_n + (-1)^n L_n\tilde{Q}_0
\end{aligned}$$

□

Theorem 2.3. *Let $\tilde{Q}_n$ be the dual bicomplex Fibonacci quaternion. Then, we have the following relations*

$$\begin{aligned}
\left(\tilde{Q}_n\right)^2 &= \tilde{F}_{2n+3} (1 - 2i + 2j + 2ij) + \varepsilon F_{2n+4} (1 - 2i - 2j + 2ij) - 4F_{n+1}F_{n+3} (1 + \varepsilon)\, j \\
&\quad - 2F_{n-1}F_{n+1} (1 + \varepsilon)\, ij - 2ij\varepsilon F_{2n},
\end{aligned}$$

$$\begin{aligned}
\left(\tilde{Q}_n\right)^2 + \left(\tilde{Q}_{n+1}\right)^2 &= -\, 2\tilde{Q}_{2n+2} + \tilde{F}_{2n+4} (3 + 4ij) + 2\tilde{F}_{2n+5} (-i + ij) \\
&\quad + 2\varepsilon F_{2n+5} (-i - j + 4ij) - 4\varepsilon F_{2n+4}i,
\end{aligned}$$

$$\begin{aligned}
\left(\tilde{Q}_{n+1}\right)^2 - \left(\tilde{Q}_{n-1}\right)^2 &= 2\tilde{Q}_{2n+1} + \left(2\tilde{F}_{2n+2} + \tilde{F}_{2n}\right) (1 - 2i - 2j) + 2ij\tilde{F}_{2n+1} \\
&\quad + \varepsilon 2F_{2n+4} (1 - 2i - j + 2ij) + \varepsilon F_{2n+1} (1 - 2i) - 2i\tilde{L}_{2n+3} - 2j\tilde{L}_{2n}.
\end{aligned}$$

Proof. From the definition of $\tilde{Q}_n$ and equation (2), we have

$$\begin{aligned}
\left(\tilde{Q}_n\right)^2 &= \left(\tilde{F}_n^2 - \tilde{F}_{n+1}^2 - \tilde{F}_{n+2}^2 + \tilde{F}_{n+3}^2\right) + 2i \left(\tilde{F}_n\tilde{F}_{n+1} - \tilde{F}_{n+2}\tilde{F}_{n+3}\right) \\
&\quad + 2j \left(\tilde{F}_n\tilde{F}_{n+2} - \tilde{F}_{n+1}\tilde{F}_{n+3}\right) + 2ij \left(\tilde{F}_n\tilde{F}_{n+3} + \tilde{F}_{n+1}\tilde{F}_{n+2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= F_n^2 - F_{n+1}^2 - F_{n+2}^2 + F_{n+3}^2 + 2\varepsilon\left(F_nF_{n+1} - F_{n+1}F_{n+2} - F_{n+2}F_{n+3} + F_{n+3}F_{n+4}\right)\\
&\quad + 2i\left(F_nF_{n+1} - F_{n+2}F_{n+3} + \varepsilon\left(F_nF_{n+2} + F_{n+1}F_{n+1} - F_{n+2}F_{n+4} - F_{n+3}F_{n+3}\right)\right)\\
&\quad + 2j\left(F_nF_{n+2} - F_{n+1}F_{n+3} + \varepsilon\left(F_nF_{n+3} + F_{n+1}F_{n+2} - F_{n+1}F_{n+4} - F_{n+2}F_{n+3}\right)\right)\\
&\quad + 2ij\left(F_nF_{n+3} + F_{n+1}F_{n+2} + \varepsilon\left(F_nF_{n+4} + F_{n+1}F_{n+3} + F_{n+1}F_{n+3} + F_{n+2}F_{n+2}\right)\right)\\
&= \tilde{F}_{2n+3} + \varepsilon F_{2n+4} - 2i\left(\tilde{F}_{2n+3} + \varepsilon F_{2n+4}\right) + 2j\left(\tilde{F}_{2n+3} - 2F_{n+1}F_{n+3}\left(1+\varepsilon\right)\right)\\
&\quad + 2ij\left(\tilde{F}_{2n+3} - F_{n-1}F_{n+1}\right) - 2j\varepsilon F_{2n+4} + 2ij\varepsilon\left(F_{2n+4} - F_{2n}\right) - 2ij\varepsilon F_{n-1}F_{n+1}\\
&= \tilde{F}_{2n+3}\left(1 - 2i + 2j + 2ij\right) + \varepsilon F_{2n+4}\left(1 - 2i - 2j + 2ij\right) - 4jF_{n+1}F_{n+3}\left(1+\varepsilon\right)\\
&\quad - 2ijF_{n-1}F_{n+1}\left(1+\varepsilon\right) - 2ij\varepsilon F_{2n},
\end{aligned}
$$

$$
\begin{aligned}
&\left(\tilde{Q}_n\right)^2 + \left(\tilde{Q}_{n+1}\right)^2\\
&= \left(\tilde{F}_n^2 - 2\tilde{F}_{n+2}^2 + \tilde{F}_{n+4}^2\right) + 2i\left(\tilde{F}_n\tilde{F}_{n+1} + \tilde{F}_{n+1}\tilde{F}_{n+2} - \tilde{F}_{n+2}\tilde{F}_{n+3} - \tilde{F}_{n+3}\tilde{F}_{n+4}\right)\\
&\quad + 2j\left(\tilde{F}_n\tilde{F}_{n+2} - \tilde{F}_{n+2}\tilde{F}_{n+4}\right)\\
&\quad + 2ij\left(\tilde{F}_n\tilde{F}_{n+3} + \tilde{F}_{n+1}\tilde{F}_{n+4} + \tilde{F}_{n+1}\tilde{F}_{n+2} + \tilde{F}_{n+2}\tilde{F}_{n+3}\right)\\
&= \tilde{F}_{2n+6} - \tilde{F}_{2n+2} + \varepsilon\left(F_{2n+7} - F_{2n+3}\right) + 2i\left(\tilde{F}_{2n+2} - \tilde{F}_{2n+6} + \varepsilon\left(F_{2n+3} - F_{2n+7}\right)\right)\\
&\quad - 2j\left(\tilde{F}_{2n+4} + \varepsilon F_{2n+5}\right) + 4ij\left(\tilde{F}_{2n+4} + \varepsilon F_{2n+5}\right)\\
&= -2\tilde{Q}_{2n+2} + \tilde{F}_{2n+4}\left(3 + 4ij\right) + 2\tilde{F}_{2n+5}\left(-i + ij\right) + \varepsilon F_{2n+5}\left(1 - 2i - 2j + 4ij\right)\\
&\quad + 2\varepsilon F_{2n+4}\left(1 - 2i\right),
\end{aligned}
$$

$$
\begin{aligned}
&\left(\tilde{Q}_{n+1}\right)^2 - \left(\tilde{Q}_{n-1}\right)^2\\
&= \left(\tilde{F}_{n+1}^2 - \tilde{F}_{n+2}^2 - \tilde{F}_{n+3}^2 + \tilde{F}_{n+4}^2\right) - \left(\tilde{F}_{n-1}^2 - \tilde{F}_n^2 - \tilde{F}_{n+1}^2 + \tilde{F}_{n+2}^2\right)\\
&\quad + 2i\left(\tilde{F}_{n+1}\tilde{F}_{n+2} - \tilde{F}_{n+3}\tilde{F}_{n+4} - \tilde{F}_{n-1}\tilde{F}_n + \tilde{F}_{n+1}\tilde{F}_{n+2}\right)\\
&\quad + 2j\left(\tilde{F}_{n+1}\tilde{F}_{n+3} - \tilde{F}_{n+2}\tilde{F}_{n+4} - \tilde{F}_{n-1}\tilde{F}_{n+1} + \tilde{F}_n\tilde{F}_{n+2}\right)\\
&\quad + 2ij\left(\tilde{F}_{n+1}\tilde{F}_{n+4} + \tilde{F}_{n+2}\tilde{F}_{n+3} - \tilde{F}_{n-1}\tilde{F}_{n+2} - \tilde{F}_n\tilde{F}_{n+1}\right)\\
&= F_{2n+5} - F_{2n+1} + 2\varepsilon\left(F_{2n+6} - F_{2n+2}\right)\\
&\quad + 2i\left(-F_{2n+5} + F_{2n+1} + 2\varepsilon\left(-F_{2n+6} + F_{2n+2}\right)\right)\\
&\quad + 2j\left(-F_{2n+3} - 2\varepsilon F_{2n+4}\right) + 2ij\left(F_{2n+4} + F_{2n+1} + 2\varepsilon\left(F_{2n+5} + F_{2n+2}\right)\right)\\
&= 2\tilde{Q}_{2n+1} + 2\tilde{F}_{2n+2} + \tilde{F}_{2n} + 2i\left(-2\tilde{F}_{2n+2} - \tilde{F}_{2n} - \tilde{F}_{2n+2} - 2\tilde{F}_{2n+1}\right)\\
&\quad + 2j\left(-2\tilde{F}_{2n+2} - \tilde{F}_{2n} - \tilde{F}_{2n+1} - \tilde{F}_{2n-1}\right) + 2ij\left(\tilde{F}_{2n+1}\right)\\
&\quad + \varepsilon\left(\left(F_{2n+6} - F_{2n+2}\right)\left(1 - 2i\right) + \varepsilon F_{2n+4}\left(-2j + 4ij\right)\right)\\
&= 2\tilde{Q}_{2n+1} + \left(2\tilde{F}_{2n+2} + \tilde{F}_{2n}\right)\left(1 - 2i - 2j\right) + 2ij\tilde{F}_{2n+1}\\
&\quad + \varepsilon 2F_{2n+4}\left(1 - 2i - j + 2ij\right) + \varepsilon F_{2n+1}\left(1 - 2i\right) - 2i\tilde{L}_{2n+3} - 2j\tilde{L}_{2n}. \qquad \square
\end{aligned}
$$

Theorem 2.4. *Let $\tilde{Q}_n$ be the dual bicomplex Fibonacci quaternion. Then, we have the following identities*

$$\sum_{s=1}^{n} \tilde{Q}_s = \tilde{Q}_{n+2} - \tilde{Q}_2,$$
$$\sum_{s=1}^{n} \tilde{Q}_{2s-1} = \tilde{Q}_{2n} - \tilde{Q}_0,$$
$$\sum_{s=1}^{n} \tilde{Q}_{2s} = \tilde{Q}_{2n+1} - \tilde{Q}_1.$$

Proof. From the summation formula $\sum_{s=k}^{n} F_s = F_{n+2} - F_{k+1}$ [13], we get

$$\begin{aligned}
\sum_{s=1}^{n} \tilde{Q}_s &= \sum_{s=1}^{n} \tilde{F}_s + i\sum_{s=1}^{n} \tilde{F}_{s+1} + j\sum_{s=1}^{n} \tilde{F}_{s+2} + ij\sum_{s=1}^{n} \tilde{F}_{s+3} \\
&= \sum_{s=1}^{n} F_s + i\sum_{s=1}^{n} F_{s+1} + j\sum_{s=1}^{n} F_{s+2} + ij\sum_{s=1}^{n} F_{s+3} \\
&\quad +\varepsilon\left(\sum_{s=1}^{n} F_{s+1} + i\sum_{s=1}^{n} F_{s+2} + j\sum_{s=1}^{n} F_{s+3} + ij\sum_{s=1}^{n} F_{s+4}\right) \\
&= (F_{n+2} - F_2) + i\,(F_{n+3} - F_3) + j\,(F_{n+4} - F_4) + ij\,(F_{n+5} - F_5) \\
&\quad +\varepsilon\left[(F_{n+3} - F_3) + i\,(F_{n+4} - F_4) + j\,(F_{n+5} - F_5) + ij\,(F_{n+6} - F_6)\right] \\
&= (F_{n+2} + iF_{n+3} + jF_{n+4} + ijF_{n+5}) + \varepsilon\,(F_{n+3} + iF_{n+4} + jF_{n+5} + ijF_{n+6}) \\
&\quad -(F_2 + iF_3 + jF_4 + ijF_5) - \varepsilon\,(F_3 + iF_4 + jF_5 + ijF_6) \\
&= (F_{n+2} + \varepsilon F_{n+3}) + i\,(F_{n+3} + \varepsilon F_{n+4}) + j\,(F_{n+4} + \varepsilon F_{n+5}) + ij\,(F_{n+5} + \varepsilon F_{n+6}) \\
&\quad -\left[(F_2 + \varepsilon F_3) + i\,(F_3 + \varepsilon F_4) + j\,(F_4 + \varepsilon F_5) + ij\,(F_5 + \varepsilon F_6)\right] \\
&= \tilde{Q}_{n+2} - \tilde{Q}_2.
\end{aligned}$$

By using summation formulas $\sum_{s=1}^{n} F_{2s-1} = F_{2n} - F_0$ and $\sum_{s=1}^{n} F_{2s} = F_{2n+1} - F_1$, we get

$$\begin{aligned}
\sum_{s=1}^{n} \tilde{Q}_{2s-1} &= \sum_{s=1}^{n} \tilde{F}_{2s-1} + i\sum_{s=1}^{n} \tilde{F}_{2s} + j\sum_{s=1}^{n} \tilde{F}_{2s+1} + ij\sum_{s=1}^{n} \tilde{F}_{2s+2} \\
&= F_{2n} - F_0 + i\,(F_{2n+1} - F_1) + j\,(F_{2n+2} - F_2) + ij\,(F_{2n+3} - F_3) \\
&\quad +\varepsilon\left[(F_{2n+1} - F_1) + i\,(F_{2n+2} - F_2) + j\,(F_{2n+3} - F_3) + ij\,(F_{2n+4} - F_4)\right] \\
&= (F_{2n} + \varepsilon F_{2n+1}) + i\,(F_{2n+1} + \varepsilon F_{2n+2}) + j\,(F_{2n+2} + \varepsilon F_{2n+3}) + ij\,(F_{2n+3} + \varepsilon F_{2n+4}) \\
&\quad -\left[(F_0 + \varepsilon F_1) + i\,(F_1 + \varepsilon F_2) + j\,(F_2 + \varepsilon F_3) + ij\,(F_3 + \varepsilon F_4)\right] \\
&= \tilde{Q}_{2n} - \tilde{Q}_0.
\end{aligned}$$

Similarly, we obtain that

$$\begin{aligned}
\sum_{s=1}^{n} \tilde{Q}_{2s} &= \sum_{s=1}^{n} \tilde{F}_{2s} + i\sum_{s=1}^{n} \tilde{F}_{2s+1} + j\sum_{s=1}^{n} \tilde{F}_{2s+2} + ij\sum_{s=1}^{n} \tilde{F}_{2s+3} \\
&= F_{2n+1} - F_1 + i\,(F_{2n+2} - F_2) + j\,(F_{2n+3} - F_3) + ij\,(F_{2n+4} - F_4) \\
&\quad +\varepsilon\left[(F_{2n+2} - F_2) + i\,(F_{2n+3} - F_3) + j\,(F_{2n+4} - F_4) + ij\,(F_{2n+5} - F_5)\right] \\
&= (F_{2n+1} + \varepsilon F_{2n+2}) + i\,(F_{2n+2} + \varepsilon F_{2n+3}) + j\,(F_{2n+3} + \varepsilon F_{2n+4}) + ij\,(F_{2n+4} + \varepsilon F_{2n+5}) \\
&\quad -\left[(F_1 + \varepsilon F_2) + i\,(F_2 + \varepsilon F_3) + j\,(F_3 + \varepsilon F_4) + ij\,(F_4 + \varepsilon F_5)\right] \\
&= \tilde{Q}_{2n+1} - \tilde{Q}_1.
\end{aligned}$$
□

Theorem 2.5. *Let $\tilde{Q}_n$ and $\tilde{Q}_m$ be the dual bicomplex Fibonacci quaternions. For n, $m \geq 0$, the Honsberger identity is given as follow;*
$\tilde{Q}_n\tilde{Q}_m+\tilde{Q}_{n+1}\tilde{Q}_{m+1} = 2\tilde{Q}_{n+m+1}+2F_{n+m+4}-F_{n+m+1}-2iF_{n+m+6}-2jF_{n+m+5}+2ijF_{n+m+4}+\varepsilon\left(4F_{n+m+5}-6iF_{n+m+6}-2jF_{n+m+7}+6ijF_{n+m+5}\right).$

Proof. By using the equation $F_nF_m+F_{n+1}F_{m+1}=F_{n+m+1}$, we have

$$\begin{aligned}
&\tilde{Q}_n\tilde{Q}_m+\tilde{Q}_{n+1}\tilde{Q}_{m+1}\\
&=\left(\tilde{F}_n+i\tilde{F}_{n+1}+j\tilde{F}_{n+2}+ij\tilde{F}_{n+3}\right)\left(\tilde{F}_m+i\tilde{F}_{m+1}+j\tilde{F}_{m+2}+ij\tilde{F}_{m+3}\right)\\
&\quad+\left(\tilde{F}_{n+1}+i\tilde{F}_{n+2}+j\tilde{F}_{n+3}+ij\tilde{F}_{n+4}\right)\left(\tilde{F}_{m+1}+i\tilde{F}_{m+2}+j\tilde{F}_{m+3}+ij\tilde{F}_{m+4}\right)\\
&=\left[\left(F_n+iF_{n+1}+jF_{n+2}+ijF_{n+3}\right)+\varepsilon\left(F_{n+1}+iF_{n+2}+jF_{n+3}+ijF_{n+4}\right)\right]\\
&\quad\left[\left(F_m+iF_{m+1}+jF_{m+2}+ijF_{m+3}\right)+\varepsilon\left(F_{m+1}+iF_{m+2}+jF_{m+3}+ijF_{m+4}\right)\right]\\
&\quad+\left[\left(F_{n+1}+iF_{n+2}+jF_{n+3}+ijF_{n+4}\right)+\varepsilon\left(F_{n+2}+iF_{n+3}+jF_{n+4}+ijF_{n+5}\right)\right]\\
&\quad\left[\left(F_{m+1}+iF_{m+2}+jF_{m+3}+ijF_{m+4}\right)+\varepsilon\left(F_{m+2}+iF_{m+3}+jF_{m+4}+ijF_{m+5}\right)\right]\\
&=\left(F_n+iF_{n+1}+jF_{n+2}+ijF_{n+3}\right)\left(F_m+iF_{m+1}+jF_{m+2}+ijF_{m+3}\right)\\
&\quad+\left(F_{n+1}+iF_{n+2}+jF_{n+3}+ijF_{n+4}\right)\left(F_{m+1}+iF_{m+2}+jF_{m+3}+ijF_{m+4}\right)\\
&\quad+\varepsilon(\left(F_n+iF_{n+1}+jF_{n+2}+ijF_{n+3}\right)\left(F_{m+1}+iF_{m+2}+jF_{m+3}+ijF_{m+4}\right)\\
&\quad+\left(F_{n+1}+iF_{n+2}+jF_{n+3}+ijF_{n+4}\right)\left(F_m+iF_{m+1}+jF_{m+2}+ijF_{m+3}\right)\\
&\quad+\left(F_{n+1}+iF_{n+2}+jF_{n+3}+ijF_{n+4}\right)\left(F_{m+2}+iF_{m+3}+jF_{m+4}+ijF_{m+5}\right)\\
&\quad+\left(F_{n+2}+iF_{n+3}+jF_{n+4}+ijF_{n+5}\right)\left(F_{m+1}+iF_{m+2}+jF_{m+3}+ijF_{m+4}\right))\\
&=F_{n+m+1}-F_{n+m+3}-F_{n+m+5}+F_{n+m+7}\\
&\quad+2i\left(F_{n+m+2}-F_{n+m+6}\right)+2j\left(F_{n+m+3}-F_{n+m+5}\right)+4ijF_{n+m+4}\\
&\quad+2\varepsilon((F_{n+m+2}-F_{n+m+4}-F_{n+m+6}+F_{n+m+8})\\
&\quad+2i\left(F_{n+m+3}-F_{n+m+7}\right)+2j\left(F_{n+m+4}-F_{n+m+6}\right)+4ijF_{n+m+5})\\
&=2F_{n+m+1}+2iF_{n+m+2}+2jF_{n+m+3}+2ijF_{n+m+4}+2F_{n+m+4}\\
&\quad-F_{n+m+1}-2iF_{n+m+6}-2jF_{n+m+5}+2ijF_{n+m+4}\\
&\quad+\varepsilon\left(2F_{n+m+2}+2iF_{n+m+3}+2jF_{n+m+4}+2ijF_{n+m+5}\right)\\
&\quad+\varepsilon\left(4F_{n+m+5}-6iF_{n+m+6}-2jF_{n+m+7}+6ijF_{n+m+5}\right)\\
&=2\tilde{Q}_{n+m+1}+2F_{n+m+4}-F_{n+m+1}-2iF_{n+m+6}-2jF_{n+m+5}\\
&\quad+2ijF_{n+m+4}+\varepsilon\left(4F_{n+m+5}-6iF_{n+m+6}-2jF_{n+m+7}+6ijF_{n+m+5}\right).
\end{aligned}$$
□

Theorem 2.6. *For n, $m \geq 0$ the d'Ocagne's identity for the dual bicomplex Fibonacci quaternions $\tilde{Q}_n$ and $\tilde{Q}_m$ is given by*

$$\tilde{Q}_m\tilde{Q}_{n+1}-\tilde{Q}_{m+1}\tilde{Q}_n=3\left(-1\right)^nF_{m-n}\left(2j+ij\right)\left(1+\varepsilon\right).$$

Proof. By using the d'Ocagne's identity for Fibonacci number [31]

$$F_mF_{n+1}-F_{m+1}F_n=\left(-1\right)^nF_{m-n},\quad F_mF_n-F_{m+r}F_{n-r}=\left(-1\right)^{n-r}F_{m+r-n}F_r,$$

we have

$$
\begin{aligned}
&\tilde{Q}_m\tilde{Q}_{n+1}-\tilde{Q}_{m+1}\tilde{Q}_n\\
&=\left(Q_m+\varepsilon Q_{m+1}\right)\left(Q_{n+1}+\varepsilon Q_{n+2}\right)-\left(Q_{m+1}+\varepsilon Q_{m+2}\right)\left(Q_n+\varepsilon Q_{n+1}\right)\\
&=Q_mQ_{n+1}-Q_{m+1}Q_n+\varepsilon\left(Q_mQ_{n+2}-Q_{m+2}Q_n\right)\\
&=\left(F_m+iF_{m+1}+jF_{m+2}+ijF_{m+3}\right)\left(F_{n+1}+iF_{n+2}+jF_{n+3}+ijF_{n+4}\right)\\
&\quad-\left(F_{m+1}+iF_{m+2}+jF_{m+3}i_2+ijF_{m+4}\right)\left(F_n+iF_{n+1}+jF_{n+2}+ijF_{n+3}\right)\\
&\quad+\varepsilon((F_m+iF_{m+1}+jF_{m+2}+ijF_{m+3})\left(F_{n+2}+iF_{n+3}+jF_{n+4}+ijF_{n+5}\right)\\
&\quad-\left(F_{m+2}+iF_{m+3}+jF_{m+4}+ijF_{m+5}\right)\left(F_n+iF_{n+1}+jF_{n+2}+ijF_{n+3}\right))\\
&=\left(F_mF_{n+1}-F_{m+1}F_n\right)-\left(F_{m+1}F_{n+2}-F_{m+2}F_{n+1}\right)\\
&\quad-\left(F_{m+2}F_{n+3}-F_{m+3}F_{n+2}\right)+\left(F_{m+3}F_{n+4}-F_{m+4}F_{n+3}\right)\\
&\quad+i((F_mF_{n+2}-F_{m+1}F_{n+1})+\left(F_{m+1}F_{n+1}-F_{m+2}F_n\right)\\
&\quad-\left(F_{m+2}F_{n+4}-F_{m+3}F_{n+3}\right)-\left(F_{m+3}F_{n+3}-F_{m+4}F_{n+2}\right))\\
&\quad+j((F_mF_{n+3}-F_{m+1}F_{n+2})+\left(F_{m+2}F_{n+1}-F_{m+3}F_n\right)\\
&\quad-\left(F_{m+1}F_{n+4}-F_{m+2}F_{n+3}\right)-\left(F_{m+3}F_{n+2}-F_{m+4}F_{n+1}\right))\\
&\quad+ij((F_mF_{n+4}-F_{m+1}F_{n+3})+\left(F_{m+3}F_{n+1}-F_{m+4}F_n\right)\\
&\quad+\left(F_{m+1}F_{n+3}-F_{m+2}F_{n+2}\right)+\left(F_{m+2}F_{n+2}-F_{m+3}F_{n+1}\right))\\
&\quad+\varepsilon((F_mF_{n+2}-F_{m+2}F_n)-\left(F_{m+1}F_{n+3}-F_{m+3}F_{n+1}\right)\\
&\quad-\left(F_{m+2}F_{n+4}-F_{m+4}F_{n+2}\right)+\left(F_{m+3}F_{n+5}-F_{m+5}F_{n+3}\right)\\
&\quad+i((F_mF_{n+3}-F_{m+2}F_{n+1})+\left(F_{m+1}F_{n+2}-F_{m+3}F_n\right)\\
&\quad-\left(F_{m+2}F_{n+5}-F_{m+4}F_{n+3}\right)-\left(F_{m+3}F_{n+4}-F_{m+5}F_{n+2}\right))\\
&\quad+j((F_mF_{n+4}-F_{m+4}F_n)-\left(F_{m+1}F_{n+5}-F_{m+5}F_{n+1}\right))\\
&\quad+ij((F_mF_{n+5}-F_{m+5}F_n)+\left(F_{m+1}F_{n+4}-F_{m+4}F_{n+1}\right)))\\
&=\left(-1\right)^n(F_{m-n}-\left(-1\right)F_{m-n}-\left(-1\right)^2F_{m-n}+\left(-1\right)^3F_{m-n}\\
&\quad+i((-1)\,F_{m-n-1}+F_{m-n+1}-\left(-1\right)^3F_{m-n-1}-\left(-1\right)^2F_{m-n+1})\\
&\quad+j((-1)^2\,F_{m-n-2}+F_{m-n+2}-\left(-1\right)^3F_{m-n-2}+F_{m-n+2})\\
&\quad+ij((-1)^3\,F_{m-n-3}+F_{m-n+3}+\left(-1\right)^2F_{m-n-1}+\left(-1\right)F_{m-n+1})\\
&\quad+\varepsilon(F_{m-n}-\left(-1\right)F_{m-n}-\left(-1\right)^2F_{m-n}+\left(-1\right)^3F_{m-n}\\
&\quad+i((-1)\,F_{m-n-1}+F_{m-n+1}-\left(-1\right)^3F_{m-n-1}-\left(-1\right)^2F_{m-n+1})\\
&\quad+j(3F_{m-n}-3\left(-1\right)F_{m-n})+ij(5F_{m-n}+2\left(-1\right)F_{m-n})))\\
&=3\left(-1\right)^nF_{m-n}\left(2j+ij\right)\left(1+\varepsilon\right). \qquad\square
\end{aligned}
$$

In [16], Halici and Çürük gave the Binet formula of the dual Fibonacci bicomplex numbers with complex coefficient. We now give this formula for bicomplex quaternion versions of Fibonacci sequences.

Theorem 2.7. *Let $\tilde{Q}_n$ be the dual bicomplex Fibonacci quaternion. For $n \geq 1$, Binet's formula is given as follows:*

$$\tilde{Q}_n=\frac{\alpha^*\alpha^n-\beta^*\beta^n}{\alpha-\beta},$$

where $\alpha^*=\hat{\alpha}\left(1+\varepsilon\alpha\right)$, $\beta^*=\hat{\beta}\left(1+\varepsilon\beta\right)$ and $\hat{\alpha}=1+i\alpha+j\alpha^2+ij\alpha^3$, $\hat{\beta}=1+i\beta+j\beta^2+ij\beta^3$.

Proof. In [3], Torunbalcı Aydın gave the Binet's formula for bicomplex Fibonacci quaternion by

$$Q_n = \frac{\hat{\alpha}\alpha^n - \hat{\beta}\beta^n}{\alpha - \beta}. \tag{3}$$

So by using the equations (1) and (3), we obtain

$$\begin{aligned} \tilde{Q}_n &= Q_n + \varepsilon Q_{n+1} \\ &= \frac{\hat{\alpha}\alpha^n - \hat{\beta}\beta^n}{\alpha - \beta} + \varepsilon \frac{\hat{\alpha}\alpha^{n+1} - \hat{\beta}\beta^{n+1}}{\alpha - \beta} \\ &= \frac{\hat{\alpha}\alpha^n \left(1 + \varepsilon\alpha\right) - \hat{\beta}\beta^n \left(1 + \varepsilon\beta\right)}{\alpha - \beta} \\ &= \frac{\alpha^* \alpha^n - \beta^* \beta^n}{\alpha - \beta}. \end{aligned}$$

□

Theorem 2.8. *Let $\tilde{Q}_n$ be the dual bicomplex Fibonacci quaternion. For $n \geq 1$, the Cassini's identity for $\tilde{Q}_n$ is given as $\tilde{Q}_{n-1}\tilde{Q}_{n+1} - \tilde{Q}_n^2 = 3\left(-1\right)^n \left(2j + ij\right)\left(1 + \varepsilon\right)$.*

Proof. By using the d'Ocagne's identity for Fibonacci numbers which is $F_m F_{n+1} - F_{m+1} F_n = \left(-1\right)^n F_{m-n}$ and the equation (1), we obtain

$$\begin{aligned} &\tilde{Q}_{n-1}\tilde{Q}_{n+1} - \tilde{Q}_n^2 \\ &= \left(Q_{n-1} + \varepsilon Q_n\right)\left(Q_{n+1} + \varepsilon Q_{n+2}\right) - \left(Q_n + \varepsilon Q_{n+1}\right)^2 \\ &= Q_{n-1}Q_{n+1} - Q_n^2 + \varepsilon\left(Q_{n-1}Q_{n+2} - Q_n Q_{n+1}\right) \\ &= F_{n-1}F_{n+1} - F_nF_{n+2} - F_{n+1}F_{n+3} + F_{n+2}F_{n+4} - F_n^2 + F_{n+1}^2 + F_{n+2}^2 - F_{n+3}^2 \\ &\quad + i\left(F_{n-1}F_{n+2} + F_nF_{n+1} - F_{n+1}F_{n+4} - F_{n+2}F_{n+3} - 2F_nF_{n+1} + 2F_{n+2}F_{n+3}\right) \\ &\quad + j\left(F_{n-1}F_{n+3} + F_{n+1}F_{n+1} - F_nF_{n+4} - F_{n+2}F_{n+2} - 2F_nF_{n+2} + 2F_{n+1}F_{n+3}\right) \\ &\quad + ij\left(F_{n-1}F_{n+4} + F_{n+1}F_{n+2} + F_nF_{n+3} + F_{n+1}F_{n+2} - 2F_nF_{n+3} - 2F_{n+1}F_{n+2}\right) \\ &\quad + \varepsilon[(F_{n-1}F_{n+2} - F_nF_{n+1}) - (F_nF_{n+3} - F_{n+1}F_{n+2}) - (F_{n+1}F_{n+4} - F_{n+2}F_{n+3}) \\ &\quad + (F_{n+2}F_{n+5} - F_{n+3}F_{n+4}) + i((F_{n-1}F_{n+3} - F_nF_{n+2}) + (F_nF_{n+2} - F_{n+1}F_{n+1}) \\ &\quad - (F_{n+1}F_{n+5} - F_{n+2}F_{n+4}) - (F_{n+2}F_{n+4} - F_{n+3}F_{n+3})) + j((F_{n-1}F_{n+4} - F_nF_{n+3}) \\ &\quad - (F_nF_{n+5} - F_{n+1}F_{n+4}) + (F_{n+1}F_{n+2} - F_{n+2}F_{n+1}) - (F_{n+2}F_{n+3} - F_{n+3}F_{n+2})) \\ &\quad + ij((F_{n-1}F_{n+5} - F_nF_{n+4}) + (F_nF_{n+4} - F_{n+1}F_{n+3}) + (F_{n+1}F_{n+3} - F_{n+2}F_{n+2}) \\ &\quad + (F_{n+2}F_{n+2} - F_{n+3}F_{n+1}))] \\ &= \left(-1\right)^n F_{-1} - \left(-1\right)^{n+1} F_{-1} - \left(-1\right)^{n+2} F_{-1} + \left(-1\right)^{n+3} F_{-1} \\ &\quad + i((-1)^{n+1} F_{-2} - (-1)^{n+3} F_{-2}) + j((-1)^{n+2} F_{-3} \\ &\quad + (-1)^n F_1 - (-1)^{n+3} F_{-3} - (-1)^{n+1} F_1) + ij\left(-1\right)^{n+3} F_{-4} \\ &\quad \varepsilon((-1)^{n+1} F_{-2} - (-1)^{n+2} F_{-2} - (-1)^{n+3} F_{-2} + (-1)^{n+4} F_{-2} \\ &\quad + i((-1)^{n+2} F_{-3} + (-1)^{n+1} F_{-1} - (-1)^{n+4} F_{-3} - (-1)^{n+3} F_{-1}) \\ &\quad + j((-1)^{n+3} F_{-4} - (-1)^{n+4} F_{-4}) + ij((-1)^{n+4} F_{-5} \\ &\quad + (-1)^{n+3} F_{-3} + (-1)^{n+2} F_{-1} + (-1)^{n+1} F_1)) \\ &= \left(-1\right)^n \left(6j + 3ij\right) + \varepsilon\left(\left(-1\right)^n \left(6j + 3ij\right)\right) \\ &= 3\left(-1\right)^n \left(2j + ij\right)\left(1 + \varepsilon\right). \end{aligned}$$

□

Theorem 2.9. *Let $\tilde{Q}_n$ be the dual bicomplex Fibonacci quaternion. For $n \geq 1$, the Catalan's identity for $\tilde{Q}_n$ is given as:*

$$\tilde{Q}_n^2 - \tilde{Q}_{n-r}\tilde{Q}_{n+r} = 3\left(-1\right)^{n-r} F_r^2 \left(2j + ij\right)\left(1 + \varepsilon\right).$$

Proof. By using the Catalan's identity for Fibonacci numbers: $F_n^2 - F_{n+r}F_{n-r} = \left(-1\right)^{n-r} F_r^2$ and equation (2), we have

$$\begin{aligned}
&\tilde{Q}_n^2 - \tilde{Q}_{n-r}\tilde{Q}_{n+r} \\
&= (\tilde{F}_n^2 - \tilde{F}_{n-r}\tilde{F}_{n+r}) - (\tilde{F}_{n+1}^2 - \tilde{F}_{n-r+1}\tilde{F}_{n+r+1}) - (\tilde{F}_{n+2}^2 - \tilde{F}_{n-r+2}\tilde{F}_{n+r+2}) \\
&\quad + (\tilde{F}_{n+3}^2 - \tilde{F}_{n-r+3}\tilde{F}_{n+r+3}) + i(2\tilde{F}\tilde{F}_{n+1} - 2\tilde{F}_{n+2}\tilde{F}_{n+3} \\
&\quad - \tilde{F}_{n-r}\tilde{F}_{n+r+1} - \tilde{F}_{n+r}\tilde{F}_{n-r+1} + \tilde{F}_{n-r+2}\tilde{F}_{n+r+3} + \tilde{F}_{n-r+3}\tilde{F}_{n+r+2}) \\
&\quad + j(2\tilde{F}_n\tilde{F}_{n+2} - 2\tilde{F}_{n+1}\tilde{F}_{n+3} - \tilde{F}_{n-r}\tilde{F}_{n+r+2} - \tilde{F}_{n+r}\tilde{F}_{n-r+2} \\
&\quad + \tilde{F}_{n-r+1}\tilde{F}_{n+r+3} + \tilde{F}_{n-r+3}\tilde{F}_{n+r+1}) + ij(2\tilde{F}_n\tilde{F}_{n+3} + 2\tilde{F}_{n+1}\tilde{F}_{n+2} \\
&\quad - \tilde{F}_{n-r}\tilde{F}_{n+r+3} - \tilde{F}_{n+r}\tilde{F}_{n-r+3} - \tilde{F}_{n-r+1}\tilde{F}_{n+r+2} - \tilde{F}_{n-r+2}\tilde{F}_{n+r+1}) \\
&= \left(-1\right)^{n-r}\left(F_r^2 - \left(-1\right)^1 F_r^2 - \left(-1\right)^2 F_r^2 + \left(-1\right)^3 F_r^2\right) \\
&\quad + i\left(-2F_{2n+3} + F_{2n+4} - F_{2n} + F_{2n+2}\right) + 2j\left(-1\right)^{n-r}\left(F_rF_{r-2} + F_rF_{r+2}\right) \\
&\quad + ij\left(-1\right)^{n-r}\left(F_rF_{r-2} + F_rF_{r+2}\right) + \varepsilon[(2F_{2n+2} + 2F_{2n+5} - 2F_{2n+4} \\
&\quad - 2F_{2n+2} - 2F_{2n+5} + 2F_{2n+4}) + i\left(-2F_{2n+2} - 2F_{2n+5} + 2F_{2n+2} + 2F_{2n+5}\right) \\
&\quad + \left(-1\right)^{n-r}(j(F_rF_{r+3} + \left(-1\right)^3 F_rF_{r-3} + \left(-1\right)^1 F_rF_{r+1} + \left(-1\right)^2 F_rF_{r-1} \\
&\quad - \left(-1\right)^1 F_rF_{r+3} - \left(-1\right)^4 F_rF_{r-3} - \left(-1\right)^2 F_rF_{r+1} - \left(-1\right)^3 F_rF_{r-1}) \\
&\quad + ij(F_rF_{r+4} + \left(-1\right)^4 F_rF_{r-4} + \left(-1\right)^3 F_rF_{r-2} + \left(-1\right)^1 F_rF_{r+2} \\
&\quad + 2\left(-1\right)^2 F_r^2 + \left(-1\right)^3 F_rF_{r-2} + \left(-1\right)^1 F_rF_{r+2}))] \\
&= 2j\left(-1\right)^{n-r}\left(F_rF_{r-2} + F_rF_{r+2}\right) + ij\left(-1\right)^{n-r}\left(F_rF_{r-2} + F_rF_{r+2}\right) \\
&\quad + 2\varepsilon j\left(-1\right)^{n-r} F_r\left(F_{r+3} - F_{r-3} + F_{r-1} - F_{r+1}\right) \\
&\quad + \varepsilon ij\left(-1\right)^{n-r} F_r\left(F_{r+4} + F_{r-4} - 2F_{r-2} - 2F_{r+2} + 2F_r\right) \\
&= 3\left(-1\right)^{n-r} F_r^2\left(2j + ij\right) + 2\varepsilon j\left(-1\right)^{n-r} F_r\left(F_{r+2} + F_{r-2}\right) \\
&\quad + \varepsilon ij\left(-1\right)^{n-r} F_r\left(F_{r+2} + F_{r-2}\right) \\
&= 3\left(-1\right)^{n-r} F_r^2\left(2j + ij\right) + 3\varepsilon\left(-1\right)^{n-r} F_r^2\left(2j + ij\right) \\
&= 3\left(-1\right)^{n-r} F_r^2\left(2j + ij\right)\left(1 + \varepsilon\right).
\end{aligned}$$

□

Theorem 2.10. *For $n \geq 1$, we have*

$$\sum_{k=0}^{n-1} \tilde{Q}_k = \tilde{Q}_{n+1} - \tilde{Q}_1.$$

Proof.

$$\begin{aligned}
\sum_{k=0}^{n-1} \tilde{Q}_k &= \sum_{k=0}^{n-1} \frac{\alpha^*\alpha^k - \beta^*\beta^k}{\alpha - \beta} = \frac{\alpha^*}{\alpha - \beta}\sum_{k=0}^{n-1}\alpha^k - \frac{\beta^*}{\alpha - \beta}\sum_{k=0}^{n-1}\beta^k \\
&= \frac{\alpha^*}{\alpha - \beta}\frac{1 - \alpha^n}{1 - \alpha} - \frac{\beta^*}{\alpha - \beta}\frac{1 - \beta^n}{1 - \beta} = \frac{\alpha^*\left(1 - \beta - \alpha^n + \alpha^n\beta\right) - \beta^*\left(1 - \alpha - \beta^n + \alpha\beta^n\right)}{-\left(\alpha - \beta\right)} \\
&= -\frac{1}{\left(\alpha - \beta\right)}\left(\left(\alpha^* - \beta^*\right) - \left(\alpha^*\alpha^n - \beta^*\beta^n\right) - \alpha^*\beta + \beta^*\alpha - \alpha^*\alpha^{n-1} + \beta^*\beta^{n-1}\right)
\end{aligned}$$

$$= -\tilde{Q}_0 + \tilde{Q}_n + \tilde{Q}_{n-1} + \frac{1}{\alpha-\beta}\left(\beta\alpha^* - \alpha\beta^*\right)$$
$$= -\tilde{Q}_0 + \tilde{Q}_n + \tilde{Q}_{n-1} + \frac{\beta\left(1+i\alpha+j\alpha^2+ij\alpha^3\right)\left(1+\varepsilon\alpha\right) - \alpha\left(1+i\beta+j\beta^2+ij\beta^3\right)\left(1+\varepsilon\beta\right)}{\alpha-\beta}$$
$$= -\tilde{Q}_0 + \tilde{Q}_n + \tilde{Q}_{n-1} - (1+j+ij) - \varepsilon\left(i+j+2ij\right)$$
$$= -\tilde{Q}_0 + \tilde{Q}_n + \tilde{Q}_{n-1} - \tilde{Q}_{-1}$$
$$= \tilde{Q}_{n+1} - \tilde{Q}_1. \qquad \square$$

Theorem 2.11. *For $n \geq 1$, we have*

$$\sum_{k=0}^{n-1} \tilde{Q}_{kr+s} = \begin{cases} \frac{(-1)^r(\tilde{Q}_{r(n-1)+s} - \tilde{Q}_{s-r}) - \tilde{Q}_{nr+s} + \tilde{Q}_s}{(-1)^r - \alpha^r - \beta^r + 1} & \text{if } s > r \\ \frac{(-1)^r\tilde{Q}_{r(n-1)+s} - (-1)^s\tilde{Q}_{r-s} - \tilde{Q}_{nr+s} + \tilde{Q}_s}{(-1)^r - \alpha^r - \beta^r + 1} & \text{otherwise} \end{cases}.$$

Proof. For $s > r$,

$$\begin{aligned}
\sum_{k=0}^{n-1} \tilde{Q}_{kr+s} &= \sum_{k=0}^{n-1} \frac{\alpha^*\alpha^{kr+s} - \beta^*\beta^{kr+s}}{\alpha-\beta} = \frac{\alpha^*\alpha^s}{\alpha-\beta}\sum_{k=0}^{n-1}\alpha^{kr} - \frac{\beta^*\beta^s}{\alpha-\beta}\sum_{k=0}^{n-1}\beta^{kr} \\
&= \frac{\alpha^*\alpha^s}{\alpha-\beta}\frac{\alpha^{nr}-1}{\alpha^r-1} - \frac{\beta^*\beta^s}{\alpha-\beta}\frac{\beta^{nr}-1}{\beta^r-1} \\
&= \frac{(\alpha\beta)^r\left(\left(\alpha^*\alpha^{nr+s-r} - \beta^*\beta^{nr+s-r}\right) - \left(\alpha^*\alpha^{s-r} - \beta^*\beta^{s-r}\right)\right)}{\alpha-\beta\left((\alpha\beta)^r - \alpha^r - \beta^r + 1\right)} \\
&\quad \frac{-\left(\alpha^*\alpha^{nr+s} - \beta^*\beta^{nr+s}\right) + \left(\alpha^*\alpha^s - \beta^*\beta^s\right)}{\alpha-\beta\left((\alpha\beta)^r - \alpha^r - \beta^r + 1\right)} \\
&= \frac{(-1)^r\left(\tilde{Q}_{r(n-1)+s} - \tilde{Q}_{s-r}\right) - \tilde{Q}_{nr+s} + \tilde{Q}_s}{(-1)^r - \alpha^r - \beta^r + 1}.
\end{aligned}$$

The second case is similarly proven. $\square$

3 Dual bicomplex Horadam quaternions

In this section, we define a new generalization of the dual bicomplex Fibonacci quaternions. We present generating function, Binet formula, and some identities of these quaternions.

Horadam defined the Horadam numbers as

$$w_n = pw_{n-1} + qw_{n-2};\, n \geq 2,\, w_0 = a,\, w_1 = b$$

where a, b, p, q are integers [18, 19].

The n-th bicomplex Horadam numbers are defined by

$$BH_n = w_n + iw_{n+1} + jw_{n+2} + ijw_{n+3}$$

where w_n is the n-th Horadam number [14].

The bicomplex Horadam quaternions are defined by using the bicomplex Horadam numbers as follows: $H_n = w_n + iw_{n+1} + jw_{n+2} + ijw_{n+3}$, $i^2 = -1$, $j^2 = -1$, $ij = ji$. There is the following recursive relation among bicomplex Horadam quaternions

$$H_{n+2} = pH_{n+1} + qH_n \tag{4}$$

with the initial values

$$H_0 = a + bi + (pb + qa)\,j + \left(p^2 b + pqa + qb\right) ij$$

and

$$H_1 = b + (pb + qa)\,i + \left(p^2 b + pqa + qb\right) j + \left(p^3 b + p^2 qa + 2pqb + q^2 a\right) ij.$$

The bicomplex Horadam quaternion is the generalization of the well-known quaternions like Fibonacci and Lucas quaternions. When taken as $(a,\ b;\ p,\ q) = (0,\ 1;\ 1,\ 1)$ and $(a,\ b;\ p,\ q) = (2,\ 1;\ 1,\ 1)$ in the relation (4), the bicomplex Fibonacci and Lucas quaternions are obtained, respectively.

The n-th dual bicomplex Horadam quaternions are defined as

$$\widetilde{H}_n = H_n + \varepsilon H_{n+1} \tag{5}$$

where H_n is the n-th bicomplex Horadam quaternion. The dual bicomplex Horadam quaternion $\widetilde{H}_n$ consists of four dual elements and can be represented as

$$\widetilde{H}_n = (w_n + \varepsilon w_{n+1}) + (w_{n+1} + \varepsilon w_{n+2})\,i + (w_{n+2} + \varepsilon w_{n+3})\,j + (w_{n+3} + \varepsilon w_{n+4})\,ij.$$

By using dual Horadam numbers $\widetilde{w}_n$, we can get

$$\widetilde{H}_n = \widetilde{w}_n + i\widetilde{w}_{n+1} + j\widetilde{w}_{n+2} + ij\widetilde{w}_{n+3}.$$

There is the following recursive relation among the dual bicomplex Horadam quaternions

$$\widetilde{H}_{n+2} = p\widetilde{H}_{n+1} + q\widetilde{H}_n \tag{6}$$

For two dual bicomplex Horadam quaternions $\widetilde{H}_n$ and $\widetilde{H}_m$, addition, subtraction and multiplication with scalar are given by the following:

$$\begin{aligned}
\widetilde{H}_n \mp \widetilde{H}_m &= (w_n \mp w_m) + i\,(w_{n+1} \mp w_{m+1}) + j\,(w_{n+2} \mp w_{m+2}) + ij\,(w_{n+3} \mp w_{m+3}) \\
&\quad + \varepsilon\left((w_{n+1} \mp w_{m+1}) + i\,(w_{n+2} \mp w_{m+2}) + j\,(w_{n+3} \mp w_{m+3}) + ij\,(w_{n+4} \mp w_{m+4})\right), \\
\lambda\widetilde{H}_n &= \lambda\,(w_n + iw_{n+1} + jw_{n+2} + ijw_{n+3}) + \lambda\varepsilon\,(w_{n+1} + iw_{n+2} + jw_{n+3} + ijw_{n+4})
\end{aligned}$$

The different conjugates for the dual bicomplex Horadam quaternions are presented by the following:

$$\begin{aligned}
\left(\widetilde{H}_n\right)_i^* &= \widetilde{w}_n - i\widetilde{w}_{n+1} + j\widetilde{w}_{n+2} - ij\widetilde{w}_{n+3}, \\
\left(\widetilde{H}_n\right)_j^* &= \widetilde{w}_n + i\widetilde{w}_{n+1} - j\widetilde{w}_{n+2} - ij\widetilde{w}_{n+3}, \\
\left(\widetilde{H}_n\right)_{ij}^* &= \widetilde{w}_n - i\widetilde{w}_{n+1} - j\widetilde{w}_{n+2} + ij\widetilde{w}_{n+3}.
\end{aligned}$$

Let $\widetilde{H}_n$ be a dual bicomplex Horadam quaternion, we give the following two corollary without proof.

Corollary 3.1. $\widetilde{H}_n + \left(\widetilde{H}_n\right)_i^* = 2\left(\widetilde{w}_n + j\widetilde{w}_{n+2}\right),$
$\widetilde{H}_n + \left(\widetilde{H}_n\right)_j^* = 2\left(\widetilde{w}_n + i\widetilde{w}_{n+1}\right),$
$\widetilde{H}_n + \left(\widetilde{H}_n\right)_{ij}^* = 2\left(\widetilde{w}_n + ij\widetilde{w}_{n+3}\right).$

Corollary 3.2. $\widetilde{H}_n \otimes \left(\widetilde{H}_n\right)_i^* = \widetilde{w}_n^2 + \widetilde{w}_{n+1}^2 - \widetilde{w}_{n+2}^2 - \widetilde{w}_{n+3}^2 + 2j\left(\widetilde{w}_n\widetilde{w}_{n+2} + \widetilde{w}_{n+1}\widetilde{w}_{n+3}\right),$
$\widetilde{H}_n \otimes \left(\widetilde{H}_n\right)_j^* = \widetilde{w}_n^2 - \widetilde{w}_{n+1}^2 + \widetilde{w}_{n+2}^2 - \widetilde{w}_{n+3}^2 + 2i\left(\widetilde{w}_n\widetilde{w}_{n+1} + \widetilde{w}_{n+2}\widetilde{w}_{n+3}\right),$
$\widetilde{H}_n \otimes \left(\widetilde{H}_n\right)_{ij}^* = \widetilde{w}_n^2 + \widetilde{w}_{n+1}^2 + \widetilde{w}_{n+2}^2 + \widetilde{w}_{n+3}^2 + 2ij\left(\widetilde{w}_n\widetilde{w}_{n+3} - \widetilde{w}_{n+1}\widetilde{w}_{n+2}\right).$

In the following theorem, we give the Binet formula for dual bicomplex Horadam quaternion.

Theorem 3.1. *The Binet formula for the dual bicomplex Horadam quaternion is*

$$\widetilde{H}_n = \frac{A\alpha^*\alpha^n - B\beta^*\beta^n}{\alpha - \beta}$$

where $A = b - a\beta$, $B = b - a\alpha$, $\alpha^* = \hat{\alpha}\left(1 + \varepsilon\alpha\right)$, $\beta^* = \hat{\beta}\left(1 + \varepsilon\beta\right)$ *and* $\hat{\alpha} = 1 + i\alpha + j\alpha^2 + ij\alpha^3$, $\hat{\beta} = 1 + i\beta + j\beta^2 + ij\beta^3$.

Proof. Considering [20, 30], the roots of the characteristic equation $t^2 - pt - q = 0$ related to Horadam numbers are α and β,

$$\alpha = \frac{p + \sqrt{p^2 + 4q}}{2}, \beta = \frac{p - \sqrt{p^2 + 4q}}{2}.$$

In [14], Halıcı gave the Binet formula for the bicomplex Horadam numbers by

$$BH_n = \frac{A\hat{\alpha}\alpha^n - B\hat{\beta}\beta^n}{\alpha - \beta}. \tag{7}$$

By using the equations (5) and (7), we have

$$\widetilde{H}_n = H_n + \varepsilon H_{n+1} = \frac{A\hat{\alpha}\alpha^n - B\hat{\beta}\beta^n}{\alpha - \beta} + \varepsilon\frac{A\hat{\alpha}\alpha^{n+1} - B\hat{\beta}\beta^{n+1}}{\alpha - \beta}$$

$$= \frac{A\hat{\alpha}\left(1 + \varepsilon\alpha\right)\alpha^n - B\hat{\beta}\left(1 + \varepsilon\beta\right)\beta^n}{\alpha - \beta} = \frac{A\alpha^*\alpha^n - B\beta^*\beta^n}{\alpha - \beta}. \qquad \square$$

Theorem 3.2. *The generating function for the dual bicomplex Horadam quaternions is*

$$GF_{\widetilde{H}_n}(t) = \frac{\widetilde{H}_0 + \widetilde{H}_1 t - p\widetilde{H}_0 t}{1 - pt - qt^2}. \tag{8}$$

Proof. Let $GF_{\widetilde{H}_n}$ be the generating function for the dual bicomplex Horadam quaternions. That is

$$GF_{\widetilde{H}_n}(t) = \widetilde{H}_0 + \widetilde{H}_1 t + \widetilde{H}_2 t^2 + \cdots + \widetilde{H}_n t^n + \cdots \tag{9}$$

Multiplying both sides of (9) by $-pt$ and $-qt^2$, we have

$$-ptGF_{\widetilde{H}_n}(t) = -p\left(\widetilde{H}_0 t + \widetilde{H}_1 t^2 + \widetilde{H}_2 t^3 + \cdots + \widetilde{H}_n t^{n+1} + \cdots\right)$$
$$-qt^2 GF_{\widetilde{H}_n}(t) = -q\left(\widetilde{H}_0 t^2 + \widetilde{H}_1 t^3 + \widetilde{H}_2 t^4 + \cdots + \widetilde{H}_n t^{n+2} + \cdots\right).$$

Using the relation (6) and making the necessary operations, we have the following desired result.

$GF_{\widetilde{H}_n}(t) = \frac{\widetilde{H}_0 + \widetilde{H}_1 t - p\widetilde{H}_0 t}{1-pt-qt^2}$. □

In the following Remark, we note some special cases of generating functions given in (8).

Remark 1. *The special cases of generating functions in Theorem 3.2 are listed as follows.*

dual bicomplex Fibonacci quaternions: $\frac{t+\varepsilon+i(1+\varepsilon(1+t))+j(1+t+\varepsilon(2+t))+ij(2+t+\varepsilon(3+2t))}{1-t-t^2}$

dual bicomplex Lucas quaternions: $\frac{2-t+\varepsilon(1+2t)+i(1+2t+\varepsilon(3+t))+j(3+t+\varepsilon(4+3t))+ij(4+3t+\varepsilon(7+4t))}{1-t-t^2}$

We give some important identities for the dual bicomplex Horadam quaternions by using the Binet formula.

Theorem 3.3. *For $m \geq n$, the d'Ocagne's identity for the dual bicomplex Horadam quaternions is*

$$\widetilde{H}_m \widetilde{H}_{n+1} - \widetilde{H}_{m+1}\widetilde{H}_n = (-q)^n AB\alpha^*\beta^* F_{m-n}.$$

Proof. By using the Binet formula for the dual bicomplex Horadam quaternions, we obtain the d'Ocagne's identity as follows:

$$\begin{aligned}
\widetilde{H}_m \widetilde{H}_{n+1} - \widetilde{H}_{m+1}\widetilde{H}_n &= \frac{\left(A\alpha^*\alpha^m - B\beta^*\beta^m\right)\left(A\alpha^*\alpha^{n+1} - B\beta^*\beta^{n+1}\right)}{(\alpha-\beta)^2} \\
&\quad - \frac{\left(A\alpha^*\alpha^{m+1} - B\beta^*\beta^{m+1}\right)\left(A\alpha^*\alpha^n - B\beta^*\beta^n\right)}{(\alpha-\beta)^2} \\
&= \frac{(AB\alpha^*\beta^*)}{(\alpha-\beta)^2}\left(-\alpha^m\beta^{n+1} - \beta^m\alpha^{n+1} + \alpha^{m+1}\beta^n + \alpha^n\beta^{m+1}\right) \\
&= \frac{1}{(\alpha-\beta)^2}\left(AB\alpha^*\beta^*(\alpha\beta)^n(\alpha-\beta)\left(\alpha^{m-n} - \beta^{m-n}\right)\right) \\
&= (-q)^n AB\alpha^*\beta^* F_{m-n}.
\end{aligned}$$
□

Theorem 3.4. *Let $\widetilde{H}_n$ be the dual bicomplex Horadam quaternion. For $n \geq r$, the Catalan's identity for $\widetilde{H}_n$ is given as:*

$$\widetilde{H}_n^2 - \widetilde{H}_{n-r}\widetilde{H}_{n+r} = AB\alpha^*\beta^*(-q)^{n-r} F_r^2.$$

Proof. By using the Binet formula for the dual bicomplex Horadam quaternions, we obtain

$$\begin{aligned}
&\widetilde{H}_n^2 - \widetilde{H}_{n-r}\widetilde{H}_{n+r} \\
&= \frac{1}{(\alpha-\beta)^2}\left(\left(A\alpha^*\alpha^n - B\beta^*\beta^n\right)^2 - \left(A\alpha^*\alpha^{n-r} - B\beta^*\beta^{n-r}\right)\left(A\alpha^*\alpha^{n+r} - B\beta^*\beta^{n+r}\right)\right) \\
&= \frac{1}{(\alpha-\beta)^2}\left(AB\alpha^*\beta^*\left(-2\alpha^n\beta^n + \alpha^{n-r}\beta^{n+r} + \alpha^{n+r}\beta^{n-r}\right)\right) \\
&= \frac{1}{(\alpha-\beta)^2} AB\alpha^*\beta^*(\alpha\beta)^{n-r}\left(\alpha^r - \beta^r\right)^2 \\
&= AB\alpha^*\beta^*(-q)^{n-r} F_r^2.
\end{aligned}$$
□

Theorem 3.5. *Let $\widetilde{H}_n$ be the dual bicomplex Horadam quaternion. For $n \geq 1$, the Cassini's identity for $\widetilde{H}_n$ are given as;*

$$\widetilde{H}_n^2 - \widetilde{H}_{n-1}\widetilde{H}_{n+1} = AB\alpha^*\beta^*(-q)^{n-1}.$$

Proof. Taking $r = 1$ as a special case of Catalan's identity, the proof of this theorem can be easily done. □

When we take as $(a, b; p, q) = (0, 1; 1, 1)$, we have obtained some identities given for the dual bicomplex Fibonacci quaternions in the previous section.

In the following theorem, we write the formula which gives the summation of the first n dual bicomplex Horadam quaternion.

Theorem 3.6. *For $n \geq 1$, the summation formula for the dual bicomplex Horadam quaternions is as follows:*

$$\sum_{k=0}^{n-1} \widetilde{H}_k = \frac{1}{(p+q-1)}\left(\widetilde{H}_n + q\widetilde{H}_{n-1} - \widetilde{H}_1 - \widetilde{H}_0 + \frac{p(A\alpha^* - B\beta^*)}{\sqrt{p^2+4q}}\right)$$

where $A = b - a\beta$, $B = b - a\alpha$.

Proof. We can write the following equation by using the Binet formula and the definition of dual bicomplex Horadam quaternions.

$$\begin{aligned}
\sum_{k=0}^{n-1} \widetilde{H}_k &= \sum_{k=0}^{n-1} \frac{A\alpha^*\alpha^k - B\beta^*\beta^k}{\alpha-\beta} = \frac{A\alpha^*}{\alpha-\beta}\sum_{k=0}^{n-1}\alpha^k - \frac{B\beta^*}{\alpha-\beta}\sum_{k=0}^{n-1}\beta^k \\
&= \frac{A\alpha^*}{\alpha-\beta}\frac{1-\alpha^n}{1-\alpha} - \frac{B\beta^*}{\alpha-\beta}\frac{1-\beta^n}{1-\beta} \\
&= \frac{A\alpha^*(1-\beta-\alpha^n+\alpha^n\beta) - B\beta^*(1-\alpha-\beta^n+\alpha\beta^n)}{(\alpha-\beta)(1-p-q)} \\
&= \frac{1}{(\alpha-\beta)(1-p-q)}((A\alpha^*-B\beta^*) - (A\alpha^*\alpha^n - B\beta^*\beta^n) \\
&\quad - A\alpha^*\beta + B\beta^*\alpha - q\left(A\alpha^*\alpha^{n-1} - B\beta^*\beta^{n-1}\right)) \\
&= \frac{1}{(1-p-q)}(\widetilde{H}_0 - \widetilde{H}_n - q\widetilde{H}_{n-1} - \frac{A\alpha^*(p-\alpha) - B\beta^*(p-\beta)}{(\alpha-\beta)}) \\
&= \frac{1}{(p+q-1)}(\widetilde{H}_n + q\widetilde{H}_{n-1} - \widetilde{H}_1 - \widetilde{H}_0 + \frac{p(A\alpha^*-B\beta^*)}{\sqrt{p^2+4q}}).
\end{aligned}$$

□

4 Conclusion

In this work, we define dual bicomplex Horadam quaternions. We give many identities that take an important place in the literature for dual bicomplex Horadam quaternions. We also obtain some well-known important identities for dual bicomplex Fibonacci quaternions.

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