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# New Tribonacci recurrence relations and addition formulas

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**Abstract:** Only one three-term recurrence relation, namely, $W_r = 2W_{r-1} - W_{r-4}$, is known for the generalized Tribonacci numbers, $W_r$, $r \in \mathbb{Z}$, defined by $W_r = W_{r-1} + W_{r-2} + W_{r-3}$ and $W_{-r} = W_{-r+3} - W_{-r+2} - W_{-r+1}$, where $W_0$, $W_1$ and $W_2$ are given, arbitrary integers, not all zero. Also, only one four-term addition formula is known for these numbers, which is $W_{r+s} = T_{s-1}W_{r-1} + (T_{s-1} + T_{s-2})W_r + T_sW_{r+1}$, where $(T_r)_{r\in\mathbb{Z}}$ is the Tribonacci sequence, a special case of the generalized Tribonacci sequence, with $W_0 = T_0 = 0$ and $W_1 = W_2 = T_1 = T_2 = 1$. In this paper we discover three new three-term recurrence relations and two identities from which a plethora of new addition formulas for the generalized Tribonacci numbers may be discovered. We obtain a simple relation connecting the Tribonacci numbers and the Tribonacci–Lucas numbers. Finally, we derivequadratic and cubic recurrence relations for the generalized Tribonacci numbers.
**Keywords:** Tribonacci number, Tribonacci–Lucas number, Recurrence relation.
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## 1 Introduction

For $r \geq 3$, we define the generalized Tribonacci numbers $W_r$ by the third order recurrence relation:

$$W_r = W_{r-1} + W_{r-2} + W_{r-3}\,, \tag{1}$$

where $W_0$, $W_1$ and $W_2$ are arbitrary integers. By writing $W_{r-1} = W_{r-2} + W_{r-3} + W_{r-4}$ and subtracting this from relation (1), we see that $W_r$ also obeys the useful three-term recurrence

$$W_r = 2W_{r-1} - W_{r-4}\,. \tag{2}$$

Extension of the definition of the generalized Tribonacci numbers to negative subscripts is provided by writing identity (2) as $W_{r+4} = 2W_{r+3} - W_r$; so that

$$W_{-r} = 2W_{-r+3} - W_{-r+4}\,. \tag{3}$$

Well known examples of $W_r$ are the Tribonacci sequence, $(T_r)$, $r \in \mathbb{Z}$, for which $W = T$, $W_0 = T_0 = 0$, $W_1 = W_2 = T_1 = T_2 = 1$ and the Tribonacci–Lucas sequence , $(K_r)$, $r \in \mathbb{Z}$, for which $W = K$, $W_0 = K_0 = 3$, $W_1 = K_1 = 1$, $W_2 = K_2 = 3$. Table 1 shows the first few Tribonacci and Tribonacci–Lucas numbers for $-20 \le r \le 24$.

| $r$ | $-20$ | $-19$ | $-18$ | $-17$ | $-16$ | $-15$ | $-14$ | $-13$ | $-12$ |
|---|---|---|---|---|---|---|---|---|---|
| $T_r$ | $-56$ | 159 | $-103$ | 0 | 56 | $-47$ | 9 | 18 | $-20$ |
| $K_r$ | 795 | $-571$ | 47 | 271 | $-253$ | 65 | 83 | $-105$ | 43 |

| $r$ | $-11$ | $-10$ | $-9$ | $-8$ | $-7$ | $-6$ | $-5$ | $-4$ | $-3$ |
|---|---|---|---|---|---|---|---|---|---|
| $T_r$ | 7 | 5 | $-8$ | 4 | 1 | $-3$ | 2 | 0 | $-1$ |
| $K_r$ | 21 | $-41$ | 23 | 3 | $-15$ | 11 | $-1$ | $-5$ | 5 |

| $r$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|---|
| $T_r$ | 1 | 0 | 0 | 1 | 1 | 2 | 4 | 7 | 13 |
| $K_r$ | $-1$ | $-1$ | 3 | 1 | 3 | 7 | 11 | 21 | 39 |

| $r$ | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|
| $T_r$ | 24 | 44 | 81 | 149 | 274 | 504 | 927 | 1705 | 3136 |
| $K_r$ | 71 | 131 | 241 | 443 | 815 | 1499 | 2757 | 5071 | 9327 |

| $r$ | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
|---|---|---|---|---|---|---|---|---|---|
| $T_r$ | 5768 | 10609 | 19513 | 35890 | 66012 | 121415 | 223317 | 410744 | 755476 |
| $K_r$ | 17155 | 31553 | 58035 | 106743 | 196331 | 361109 | 664183 | 1221623 | 2246915 |

Table 1. The first few Tribonacci and Tribonacci–Lucas numbers.

The following references contain useful information on the properties of the Tribonacci numbers and related: [1, 3, 5, 6, 8, 9]. Various congruence properties of Tribonacci sequences are discussed in the recent paper by Atanassova [2].

Among other interesting results, we found the following three-term recurrence relations which are presumably new: $W_{r-16} = -103W_r + 56W_{r+1}\,, 2W_{r-17} = 9W_r - 103W_{r-4}$ and $W_{r-14} = 9W_{r+2} - 56W_{r-1}\,.$

We also found a simple relation linking the Tribonacci numbers and the Tribonacci–Lucas numbers:

$$K_{r-2} = 5T_{r-1} - T_{r+1}\,. \tag{4}$$

## 2 Linear recurrence relations and addition formulas

**Theorem 2.1.** *The following identity holds for integers $a$, $b$, $c$, $d$ and $e$:*

$$\begin{aligned}
&(T_{a-c}T_{c-b}T_{b-a} + T_{b-c}T_{c-a}T_{a-b})W_{d+e}\\
&= (T_{b-c}T_{c-a}T_{e-b} - T_{b-c}T_{c-b}T_{e-a} + T_{c-b}T_{b-a}T_{e-c})W_{d+a}\\
&\quad + (T_{a-c}T_{c-b}T_{e-a} - T_{a-c}T_{c-a}T_{e-b} + T_{c-a}T_{a-b}T_{e-c})W_{d+b}\\
&\quad + (T_{b-c}T_{a-b}T_{e-a} - T_{a-b}T_{b-a}T_{e-c} + T_{a-c}T_{b-a}T_{e-b})W_{d+c}
\end{aligned}$$

*Proof.* We seek to express a generalized Tribonacci number as a linear combination of three Tribonacci numbers. Let

$$W_{d+e} = f_1T_{e-a} + f_2T_{e-b} + f_3T_{e-c}\,, \tag{5}$$

where $a$, $b$, $c$, $d$ and $e$ are arbitrary integers and the coefficients $f_1$, $f_2$ and $f_3$ are to be determined. Setting $e = a$, $e = b$ and $e = c$, in turn, we obtain three simultaneous equations:

$$W_{d+a} = f_2T_{a-b} + f_3T_{a-c}\,, \quad W_{d+b} = f_1T_{b-a} + f_3T_{b-c}\,, \quad W_{d+c} = f_1T_{c-a} + f_2T_{c-b}\,.$$

The identity of Theorem 2.1 is established by solving these equations for $f_1$, $f_2$ and $f_3$ and substituting the solutions into identity (5). □

**Corollary 2.1.1.** *The following identities hold for integers $r$ and $s$:*

$$4W_{r+s} = 2T_{s-1}W_{r-4} + (T_{s+4} - 7T_s)W_r + 4T_sW_{r+1}\,, \tag{6}$$

$$4W_{r+s} = 2T_{s-4}W_{r-1} + (4T_{s+1} - 7T_s)W_r + T_sW_{r+4}\,, \tag{7}$$

$$W_{r+s} = T_{s-1}W_{r-1} + (T_{s+1} - T_s)W_r + T_sW_{r+1}\,, \tag{8}$$

$$4W_{r+s} = T_{s-4}W_{r-4} + (T_{s+4} - 11T_s)W_r + T_sW_{r+4}\,, \tag{9}$$

$$W_{r+s} = (T_{s+1} - 2T_s - T_{s-2})W_{r-1} + (T_{s+1} - 2T_s)W_r + T_sW_{r+2}\,. \tag{10}$$

*Proof.* To derive identity (6), set $a = r-4$, $b = r$, $c = r+1$, $d = 0$ and $e = r+s$ in the identity of Theorem 2.1. The proof of (7)—(10) is similar, (see Table 2). □

| Identity | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| (6) | $r-4$ | $r$ | $r+1$ | $0$ | $r+s$ |
| (7) | $r+4$ | $r$ | $r-1$ | $0$ | $r+s$ |
| (8) | $r-1$ | $r$ | $r+1$ | $0$ | $r+s$ |

| Identity | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| (9) | $r+4$ | $r$ | $r-4$ | $0$ | $r+s$ |
| (10) | $r+2$ | $r-1$ | $r$ | $0$ | $r+s$ |

Table 2. Appropriate substitutions in the identity of Theorem 2.1
to obtain identities (6) – (10) of Corollary 2.1.1.

Note that identity (8) can be written in the familiar form

$$W_{r+s} = T_{s-1}W_{r-1} + (T_{s-1} + T_{s-2})W_r + T_s W_{r+1}\,. \tag{11}$$

Evaluating identity (6) at $s = -3$, $s = -16$ and at $s = -17$, in turn, we find the following three term recurrence relations for the generalized Tribonacci numbers:

$$W_{r-3} = 2W_r - W_{r+1}\,, \tag{12}$$

$$W_{r-16} = -103W_r + 56W_{r+1} \tag{13}$$

and

$$2W_{r-17} = 9W_r - 103W_{r-4}\,. \tag{14}$$

Evaluating identity (10) at $s = -14$ produces yet another three-term recurrence relation for the generalized Tribonacci numbers:

$$W_{r-14} = 9W_{r+2} - 56W_{r-1}\,. \tag{15}$$

Note that by interchanging $r$ and $s$ in each case and making use of the defining recurrence relation for $W$ and $T$, the identities (6) – (10) can also be written

$$4W_{r+s} = 2W_{s-1}T_{r-4} + (W_{s+4} - 7W_s)T_r + 4W_s T_{r+1}\,, \tag{16}$$

$$4W_{r+s} = 2W_{s-4}T_{r-1} + (4W_{s+1} - 7W_s)T_r + W_s T_{r+4}\,, \tag{17}$$

$$W_{r+s} = W_{s-1}T_{r-1} + (W_{s+1} - W_s)T_r + W_s T_{r+1}\,, \tag{18}$$

$$4W_{r+s} = W_{s-4}T_{r-4} + (W_{s+4} - 11W_s)T_r + W_s T_{r+4}\,, \tag{19}$$

$$W_{r+s} = (W_{s+1} - 2W_s - W_{s-2})T_{r-1} + (W_{s+1} - 2W_s)T_r + W_s T_{r+2}\,. \tag{20}$$

Evaluating identity (18) at $s = -2$ with $W = K$ gives

$$K_{r-2} = 5T_{r-1} - T_{r+1}\,, \tag{21}$$

a three-term relation connecting the Tribonacci numbers and the Tribonacci–Lucas numbers.

Relations similar to (4) and (21) are also derived by Frontczak [4] and Komatsu [7].

**Theorem 2.2.** *The following identity holds for integers $a$, $b$, $c$, $d$ and $e$:*

$$\begin{aligned}
\{K_{b-a-1}K_{a-b-1} + K_{b-a-1}K_{c-b-1}K_{a-c-1} + K_{c-a-1}K_{b-c-1}K_{a-b-1}&\\
+ K_{c-a-1}K_{a-c-1} + K_{c-b-1}K_{b-c-1} - 1\}W_{d+e}&\\
= \{(1 - K_{c-b-1}K_{b-c-1})K_{e-a} + (K_{b-a-1} + K_{c-a-1}K_{b-c-1})K_{e-b}&\\
+ (K_{c-a-1} + K_{b-a-1}K_{c-b-1})K_{e-c}\}W_{d+a-1}&\\
+ \{(1 - K_{c-a-1}K_{a-c-1})K_{e-b} + (K_{a-b-1} + K_{c-b-1}K_{a-c-1})K_{e-a}&\\
+ (K_{c-b-1} + K_{c-a-1}K_{a-b-1})K_{e-c}\}W_{d+b-1}&\\
+ \{(1 - K_{b-a-1}K_{a-b-1})K_{e-c} + (K_{b-c-1} + K_{b-a-1}K_{a-c-1})K_{e-b}&\\
+ (K_{a-c-1} + K_{b-c-1}K_{a-b-1})K_{e-a}\}W_{d+c-1}\,.&
\end{aligned}$$

*Proof.* We wish to express a generalized Tribonacci number as a linear combination of three Tribonacci–Lucas numbers. Let

$$W_{d+e} = f_1 K_{e-a} + f_2 K_{e-b} + f_3 K_{e-c}\,, \tag{22}$$

where $a$, $b$, $c$, $d$ and $e$ are arbitrary integers and the coefficients $f_1$, $f_2$ and $f_3$ are to be determined. Setting $e = a-1$, $e = b-1$ and $e = c-1$, in turn, we obtain three simultaneous equations:

$$W_{d+a-1} = -f_1 + f_2 K_{a-b-1} + f_3 K_{a-c-1}\,, \quad W_{d+b-1} = f_1 K_{b-a-1} - f_2 + f_3 K_{b-c-1}\,,$$
$$W_{d+c-1} = f_1 K_{c-a-1} + f_2 K_{c-b-1} - f_3\,.$$

The identity of Theorem 2.2 is obtained by solving these equations for $f_1$, $f_2$ and $f_3$ and substituting the solutions into identity (22). $\square$

**Corollary 2.2.1.** *The following identities hold for integers $r$ and $s$:*

$$\begin{aligned}44W_{r+s} &= (9K_{s+3} - K_{s+5} + 2K_{s+1})W_{r-6}\\ &\quad + (9K_{s+1} + K_{s+5} + 2K_{s+3})W_{r-4} + (K_{s+3} + 6K_{s+5} - K_{s+1})W_{r-2}\,,\end{aligned} \tag{23}$$

$$\begin{aligned}88W_{r+s} &= (8K_{s+3} - 30K_{s-1} - 2K_{s-2})W_r\\ &\quad + (K_{s+3} - K_{s-1} + 8K_{s-2})W_{r-4} + (3K_{s+3} + 19K_{s-1} + 2K_{s-2})W_{r+1}\,,\end{aligned} \tag{24}$$

$$\begin{aligned}22W_{r+s} &= (5K_{s-2} + K_{s-1} + 2K_s)W_{r-1}\\ &\quad + (K_{s-2} - 2K_{s-1} + 7K_s)W_r + (2K_{s-2} + 7K_{s-1} + 3K_s)W_{r+1}\,,\end{aligned} \tag{25}$$

$$\begin{aligned}5060W_{r+s} &= (-264K_{s-3} + K_{s+9} - 1079K_{s-1})W_{r-10}\\ &\quad + (-4279K_{s-3} + 21K_{s+9} - 21394K_{s-1})W_r\\ &\quad + (6325K_{s-1} + 1265K_{s-3})W_{r+2}\end{aligned} \tag{26}$$

*and*

$$\begin{aligned}5060W_{r+s} &= (-264K_{s-11} + 1265K_{s+1} - 4279K_{s-1})W_{r-2}\\ &\quad + (-1079K_{s-11} + 6325K_{s+1} - 21394K_{s-1})W_r\\ &\quad + (21K_{s-1} + K_{s-11})W_{r+10}\,.\end{aligned} \tag{27}$$

Identities (23) – (27) can also be written as follows.

$$\begin{aligned}44W_{r+s} &= (9W_{s+3} - W_{s+5} + 2W_{s+1})K_{r-6}\\ &\quad + (9W_{s+1} + W_{s+5} + 2W_{s+3})K_{r-4} + (W_{s+3} + 6W_{s+5} - W_{s+1})K_{r-2}\,,\end{aligned} \tag{28}$$

$$\begin{aligned}88W_{r+s} &= (8W_{s+3} - 30W_{s-1} - 2W_{s-2})K_r\\ &\quad + (W_{s+3} - W_{s-1} + 8W_{s-2})K_{r-4} + (3W_{s+3} + 19W_{s-1} + 2W_{s-2})K_{r+1}\,,\end{aligned} \tag{29}$$

$$\begin{aligned}22W_{r+s} &= (5W_{s-2} + W_{s-1} + 2W_s)K_{r-1}\\ &\quad + (W_{s-2} - 2W_{s-1} + 7W_s)K_r + (2W_{s-2} + 7W_{s-1} + 3W_s)K_{r+1}\,,\end{aligned} \tag{30}$$

$$\begin{aligned}5060W_{r+s} &= (-264W_{s-3} + W_{s+9} - 1079W_{s-1})K_{r-10}\\ &\quad + (-4279W_{s-3} + 21W_{s+9} - 21394W_{s-1})K_r\\ &\quad + (6325W_{s-1} + 1265W_{s-3})K_{r+2}\end{aligned} \tag{31}$$

and

$$\begin{aligned}5060W_{r+s} &= (-264W_{s-11} + 1265W_{s+1} - 4279W_{s-1})K_{r-2}\\ &\quad + (-1079W_{s-11} + 6325W_{s+1} - 21394W_{s-1})K_r\\ &\quad + (21W_{s-1} + W_{s-11})K_{r+10}.\end{aligned} \tag{32}$$

# 3 Quadratic relations

Our goal in this section is to derive expressions involving only pure squares of generalized Tribonacci numbers. To achieve this we must be able to express the anticipated cross-terms such as $W_{r-1}W_r$ and $W_{r-1}W_{r-4}$ as squares of generalized Tribonacci numbers.

Rearranging identity (2) and squaring, we have

$$4W_{r-1}W_r = 4W_{r-1}^2 - W_{r-4}^2 + W_r^2\,, \tag{33}$$

$$4W_{r-1}W_{r-4} = 4W_{r-1}^2 + W_{r-4}^2 - W_r^2 \tag{34}$$

and

$$2W_rW_{r-4} = 4W_{r-1}^2 - W_{r-4}^2 - W_r^2\,. \tag{35}$$

Rearranging identity (2) and multiplying through by $4W_{r-3}$ to obtain

$$8W_{r-1}W_{r-3} = 4W_rW_{r-3} + 4W_{r-4}W_{r-3}\,, \tag{36}$$

and using identities (33) and (34) to resolve the right hand side gives

$$8W_{r-1}W_{r-3} = 4W_r^2 + 2W_{r-3}^2 - W_{r+1}^2 + 4W_{r-4}^2 - W_{r-7}^2\,. \tag{37}$$

Multiplying through identity (2) by $4W_{r-5}$ to obtain

$$4W_rW_{r-5} = 8W_{r-1}W_{r-5} - 4W_{r-4}W_{r-5}\,, \tag{38}$$

which, with the use of (33) and (35), translates to

$$4W_rW_{r-5} = 16W_{r-2}^2 - 8W_{r-5}^2 - 4W_{r-1}^2 + W_{r-8}^2 - W_{r-4}^2\,. \tag{39}$$

Rearranging identity (2), shifting index $r$ and multiplying through by $2W_r$ gives

$$2W_rW_{r-8} = 4W_rW_{r-5} - 2W_rW_{r-4}\,, \tag{40}$$

from which, using (35) and (39), we get

$$2W_rW_{r-8} = 16W_{r-2}^2 - 8W_{r-5}^2 - 8W_{r-1}^2 + W_{r-8}^2 + W_r^2\,. \tag{41}$$

Rearranging identity (13) and squaring, we have

$$112W_{r-17}W_r = W_{r-17}^2 + 3136W_r^2 - 10609W_{r-1}^2 \tag{42}$$

$$11536W_{r-1}W_r = -W_{r-17}^2 + 3136W_r^2 + 10609W_{r-1}^2\,, \tag{43}$$

and

$$206W_{r-17}W_{r-1} = -W_{r-17}^2 - 10609W_{r-1}^2 + 3136W_r^2\,. \tag{44}$$

Rearranging and squaring identity (14), we have

$$36W_{r-17}W_r = 4W_{r-17}^2 + 81W_r^2 - 10609W_{r-4}^2\,, \tag{45}$$

$$1854W_rW_{r-4} = 81W_r^2 + 10609W_{r-4}^2 - 4W_{r-17}^2 \tag{46}$$

and

$$412W_{r-17}W_{r-4} = -4W_{r-17}^2 - 10609W_{r-4}^2 + 81W_r^2\,. \tag{47}$$

Finally, squaring and rearranging identity (15) produces

$$18W_{r-17}W_{r-1} = W_{r-17}^2 + 81W_{r-1}^2 - 3136W_{r-4}^2\,, \tag{48}$$

$$1008W_{r-1}W_{r-4} = 81W_{r-1}^2 + 3136W_{r-4}^2 - W_{r-17}^2 \tag{49}$$

and

$$112W_{r-17}W_{r-4} = -W_{r-17}^2 - 3136W_{r-4}^2 + 81W_{r-1}^2\,. \tag{50}$$

**Theorem 3.1.** *The following identities hold for any integer $r$:*

$$252W_r^2 - 927W_{r-1}^2 + 2884W_{r-4}^2 - W_{r-17}^2 = 0\,, \tag{51}$$

$$W_r^2 - 2W_{r-1}^2 - 3W_{r-2}^2 - 6W_{r-3}^2 + W_{r-4}^2 + W_{r-6}^2 = 0\,. \tag{52}$$

*Proof.* Eliminating $W_{r-1}W_r$ between identities (33) and (43) proves identity (51). To prove identity (52), write $W_r - W_{r-1} = W_{r-2} + W_{r-3}$, square both sides and use the identity (33) to resolve the cross-products $W_rW_{r-1}$ and $W_{r-2}W_{r-3}$. □

Substituting for $W_{r-17}^2$ from the identity of Theorem 3.1 into identities (42), (44) and (50), we have the following simpler versions of these identities:

$$4W_{r-17}W_r = -412W_{r-1}^2 + 103W_{r-4}^2 + 121W_r^2\,, \tag{53}$$

$$W_{r-17}W_{r-1} = -47W_{r-1}^2 - 14W_{r-4}^2 + 14W_r^2 \tag{54}$$

and

$$4W_{r-17}W_{r-4} = 36W_{r-1}^2 - 215W_{r-4}^2 - 9W_r^2\,. \tag{55}$$

Next we show how to express the square of a Tribonacci–Lucas number in terms of squares of Tribonacci numbers.

**Theorem 3.2.** *The following identity holds for any integer $r$:*

$$4K_r^2 = 5T_{r+5}^2 - 20T_{r+4}^2 + 4T_{r+3}^2 + 90T_{r+1}^2 - 20T_r^2 + 5T_{r-3}^2\,. \tag{56}$$

*Proof.* Square identity (21) and use identity (37) to eliminate the cross-term. □

**Theorem 3.3.** *The following identities hold for integers $r$ and $s$:*

$$\begin{aligned}16W_{r+s}^2 &= -(T_s+T_{s+4})(-T_{s+4}+2T_{s-1}+7T_s)W_r^2+4T_{s-1}T_sW_{r-7}^2\\&\quad+2T_{s-1}(-T_{s+4}-9T_s+2T_{s-1})W_{r-4}^2-2T_s(T_{s+4}-7T_s+2T_{s-1})W_{r-3}^2\\&\quad+8T_{s-1}(T_s+T_{s+4})W_{r-1}^2+2T_s(T_s+T_{s+4})W_{r+1}^2\,,\end{aligned}\tag{57}$$

$$\begin{aligned}16W_{r+s}^2 &= 4(2T_s-T_{s+1})(7T_s-T_{s-4}-4T_{s+1})W_r^2+4T_{s-4}(2T_s-T_{s+1})W_{r-4}^2\\&\quad-4T_{s-4}(-T_{s-4}-4T_{s+1}+9T_s)W_{r-1}^2+16T_{s-4}T_sW_{r+2}^2\\&\quad-4T_s(7T_s+T_{s-4}-4T_{s+1})W_{r+3}^2+4T_s(2T_s-T_{s+1})W_{r+4}^2\,,\end{aligned}\tag{58}$$

$$\begin{aligned}4W_{r+s}^2 &= -2(-T_{s+1}+T_s)(2T_{s+2}-T_{s-1})W_r^2-T_{s-1}T_sW_{r-5}^2\\&\quad+2T_{s-1}(-T_{s+1}+T_s)W_{r-4}^2+2T_s(-T_{s+1}+T_s)W_{r-3}^2\\&\quad+4T_{s-1}T_sW_{r-2}^2+2T_{s-1}(2T_{s-1}+4T_{s+1}-3T_s)W_{r-1}^2\\&\quad+2T_s(T_s+T_{s+1})W_{r+1}^2-T_{s-1}T_sW_{r+3}^2+4T_{s-1}T_sW_{r+2}^2\,.\end{aligned}\tag{59}$$

# 4 Cubic recurrence relations

**Theorem 4.1.** *The following identity holds for integer $r$:*

$$\begin{aligned}&W_r^3-4W_{r-1}^3-9W_{r-2}^3-34W_{r-3}^3+24W_{r-4}^3-2W_{r-5}^3\\&\quad+40W_{r-6}^3-14W_{r-7}^3-W_{r-8}^3-2W_{r-9}^3+W_{r-10}^3=0\,.\end{aligned}$$

*Proof.* Setting $a=r-8$, $b=r$, $c=r-10$, $d=0$ and $e=r-s$ with $s\in\{1,2,3,4,5,6,7,9\}$ in the identity of Theorem 2.1, the following linear combinations are formed:

$$\begin{aligned}W_{r-1}&=\frac{37}{68}W_r-\frac{5}{68}W_{r-8}+\frac{1}{17}W_{r-10}, & W_{r-2}&=\frac{5}{17}W_r+\frac{3}{17}W_{r-8}+\frac{1}{17}W_{r-10}\,,\\W_{r-3}&=\frac{11}{68}W_r-\frac{7}{68}W_{r-8}-\frac{2}{17}W_{r-10}, & W_{r-4}&=\frac{3}{34}W_r-\frac{5}{34}W_{r-8}+\frac{2}{17}W_{r-10}\,,\\W_{r-5}&=\frac{3}{68}W_r+\frac{29}{68}W_{r-8}+\frac{1}{17}W_{r-10}, & W_{r-6}&=\frac{1}{34}W_r-\frac{13}{34}W_{r-8}-\frac{5}{17}W_{r-10}\,,\\W_{r-7}&=\frac{1}{68}W_r-\frac{13}{68}W_{r-8}+\frac{6}{17}W_{r-10}, & W_{r-9}&=\frac{1}{68}W_r-\frac{81}{68}W_{r-8}-\frac{11}{17}W_{r-10}\,.\end{aligned}$$

The above $W_{r-i}, i\in\{1,2,3,4,5,6,7,9\}$ verify the identity of Theorem 4.1. □

Taking the cube in identities (12) and (13) and solving two simultaneous equations, we find

$$155736W_rW_{r-1}^2=-W_{r-17}^3+199305W_{r-1}^3-161504W_{r-4}^3+14112W_r^3\tag{60}$$

and

$$77868W_r^2W_{r-1}=-148526W_{r-4}^3+27090W_r^3-W_{r-17}^3+95481W_{r-1}^3\,.\tag{61}$$

**Theorem 4.2.** *The following identity holds for integer $r$:*

$$\begin{aligned}&11844W_r^3+3W_{r-19}^3+W_{r-17}^3+458556W_{r-6}^3\\&\quad+135548W_{r-4}^3-442179W_{r-3}^3-120204W_{r-2}^3-43569W_{r-1}^3=0\,.\end{aligned}\tag{62}$$

*Proof.* Write the defining recurrence relation of the generalized Tribonacci numbers as $W_r-W_{r-1}=W_{r-2}+W_{r-3}$, take the cube of both sides and use identities (60) and (61) to remove cross-terms. □

# 5 Conclusion

We have derived many new recurrence relations for the generalized Tribonacci sequence. We discovered three new three-term recurrence relations and two identities from which numerous new addition formulas for generalized Tribonacci numbers may be discovered. We obtained a simple relation connecting the Tribonacci numbers and the Tribonacci–Lucas numbers. Finally, we derived quadratic and cubic recurrence relations for the generalized Tribonacci sequence.


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