Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 26, 2020, No. 4, 128–135
DOI: 10.7546/nntdm.2020.26.4.128-135

Derangement polynomials with a complex variable

Abdelkader Benyattou

Department of Mathematics and Informatics, University of Djelfa, Algeria
RECITS Laboratory, P. O. 32 Box 32, El Alia 16111, Algiers, Algeria
e-mail: abdelkaderbenyattou@gmail.com, a.benyattou@univ-djelfa.dz

Received: 13 June 2020 **Revised:** 20 October 2020 **Accepted:** 2 November 2020

Abstract: In this paper, we define new polynomials with a complex variable related to the derangement polynomials and we give some properties of those polynomials. We use umbral calculus to establish a new congruence concerning the derangement polynomials with a complex variable.
Keywords: Derangement polynomials, Complex variable, Congruence, Umbral calculus.
2010 Mathematics Subject Classification: 11B83, 11A07, 30C10.

1 Introduction

Polynomials with a complex variable have attracted researchers' great interest, as the application of those polynomials appear in various fields of mathematics. The polynomials with a complex variable have been studied by various researchers for example, see [3, 6].

Derangement polynomials are defined by

$$\mathcal{D}_n(x) = n! \sum_{k=0}^{n} \frac{(x-1)^k}{k!}.$$

It is clear that $\mathcal{D}_n(0)$ is the n-th derangement number, denoted by $\mathcal{D}_n$ counting the number of permutation of the set $[n]$ without a fixed point. The exponential generating function for the derangement polynomials is

$$\sum_{n=0}^{\infty} \mathcal{D}_n(x) \frac{t^n}{n!} = \frac{e^{-t}}{1-t} e^{xt}. \tag{1}$$

For more information about these numbers and polynomials one can see [7–9].

If we replace x by z or $\overline{z}$ in (1), where

$$z = x + iy, \overline{z} = x - iy, i^2 = -1,$$

we get

$$\sum_{n=0}^{\infty} \mathcal{D}_n(z) \frac{t^n}{n!} = \frac{e^{-t}}{1-t} e^{(x+iy)t} = \frac{e^{-t}}{1-t} e^{xt} (\cos(yt) + i \sin(yt))$$
$$\sum_{n=0}^{\infty} \mathcal{D}_n(\overline{z}) \frac{t^n}{n!} = \frac{e^{-t}}{1-t} e^{(x-iy)t} = \frac{e^{-t}}{1-t} e^{xt} (\cos(yt) - i \sin(yt)).$$

If we add or subtract the identities presented above, we get

$$\sum_{n=0}^{\infty} [\mathcal{D}_n(z) + \mathcal{D}_n(\overline{z})] \frac{t^n}{n!} = \frac{2e^{-t}}{1-t} e^{xt} \cos(yt)$$

$$\sum_{n=0}^{\infty} [\mathcal{D}_n(z) - \mathcal{D}_n(\overline{z})] \frac{t^n}{n!} = \frac{2ie^{-t}}{1-t} e^{xt} \sin(yt).$$

Let $\mathcal{D}_{n,1}(z) = \mathcal{D}_n(z) + \mathcal{D}_n(\overline{z})$, and $\mathcal{D}_{n,2}(z) = \mathcal{D}_n(z) - \mathcal{D}_n(\overline{z})$, then we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{D}_{n,1}(z) \frac{t^n}{n!} &= \frac{2e^{-t}}{1-t} e^{xt} \cos(yt), \\ \sum_{n=0}^{\infty} \mathcal{D}_{n,2}(z) \frac{t^n}{n!} &= \frac{2ie^{-t}}{1-t} e^{xt} \sin(yt) \end{aligned}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{D}_n(z) \frac{t^n}{n!} &= \frac{e^{(-1+iy)t}}{1-t} e^{xt}, \\ \sum_{n=0}^{\infty} \mathcal{D}_n(\overline{z}) \frac{t^n}{n!} &= \frac{e^{(-1-iy)t}}{1-t} e^{xt}. \end{aligned}$$

That is now

$$\cos(yt) = \frac{e^{iyt} + e^{-iyt}}{2}, \quad \sin(yt) = \frac{e^{iyt} - e^{-iyt}}{2i},$$

then

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{D}_{n,1}(z) \frac{t^n}{n!} &= \frac{e^{-t}}{1-t} e^{xt} \left(e^{iyt} + e^{-iyt}\right) \\ &= \sum_{n=0}^{\infty} \mathcal{D}_n(x) \frac{t^n}{n!} \sum_{n=0}^{\infty} \frac{[(iyt)^n + (-iyt)^n]}{n!} \\ &= \sum_{n=0}^{\infty} \mathcal{D}_n(x) \frac{t^n}{n!} \sum_{n=0}^{\infty} (iy)^n (1 + (-1)^n) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_k(x) (iy)^{n-k} \left(1 + (-1)^{n-k}\right) t^n. \end{aligned}$$

Hence

$$\begin{aligned}\mathcal{D}_{n,1}(z) &= \sum_{k=0}^{n}\binom{n}{k}\mathcal{D}_k(x)(iy)^{n-k}\left(1+(-1)^{n-k}\right),\\ \mathcal{D}_{n,2}(z) &= \sum_{k=0}^{n}\binom{n}{k}\mathcal{D}_k(x)(iy)^{n-k}\left(1-(-1)^{n-k}\right).\end{aligned}$$

The derangement polynomials with a complex variable can be defined by

$$\mathcal{D}_n(z)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{D}_k(x)(iy)^{n-k},$$

and we can write $\mathcal{D}_n(z)$ as follows

$$\mathcal{D}_n(z)=i^n\sum_{s=0}^{n}(-1)^s\binom{n}{2s}\mathcal{D}_{2s}(x)\,y^{n-2s}-i^{n+1}\sum_{s=0}^{n}(-1)^s\binom{n}{2s+1}\mathcal{D}_{2s+1}(x)\,y^{n-2s-1}.$$

The first few polynomials are:

$$\begin{aligned}\mathcal{D}_0(z) &= 1,\\ \mathcal{D}_1(z) &= x+iy,\\ \mathcal{D}_2(z) &= x^2-y^2+1+2xyi,\\ \mathcal{D}_3(z) &= x^3+3x-3xy^2+2+i\left(-y^3+3x^2y+3y\right).\end{aligned}$$

In particular, for $y=0$ or $x=y=0$, we have

$$\mathcal{D}_n(z)=\mathcal{D}_n(x),\quad \mathcal{D}_n(0)=\mathcal{D}_n.$$

2 Some properties of the derangement polynomials with a complex variable

In this section, we give some properties of the $\mathcal{D}_n(z),\mathcal{D}_{n,1}(z),\mathcal{D}_{n,2}(z)$.

Lemma 2.1. *For any non-negative integer n, we have*

$$\begin{aligned}\mathcal{D}_n(z) &= \sum_{k=0}^{n}(n)_k\left[\sum_{s=0}^{k}\frac{(x-1)^s}{s!}\right](iy)^{n-k},\\ \mathcal{D}_{n,1}(z) &= \sum_{k=0}^{n}(n)_k\left[\sum_{s=0}^{k}\frac{(x-1)^s}{s!}\right](iy)^{n-k}\left(1+(-1)^{n-k}\right),\\ \mathcal{D}_{n,2}(z) &= \sum_{k=0}^{n}(n)_k\left[\sum_{s=0}^{k}\frac{(x-1)^s}{s!}\right](iy)^{n-k}\left(1-(-1)^{n-k}\right),\end{aligned}$$

where $(n)_k$ is the falling factorial defined by

$$(n)_k=n(n-1)\cdots(n-k+1)\ \text{if}\ k\geq 1\ \text{and}\ (n)_0=1.$$

Proof. We have

$$\mathcal{D}_n(z) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_k(x)(iy)^{n-k}$$
$$= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} k! \left[\sum_{s=0}^{k} \frac{(x-1)^s}{s!}\right](iy)^{n-k}$$
$$= \sum_{k=0}^{n} (n)_k \left[\sum_{s=0}^{k} \frac{(x-1)^s}{s!}\right](iy)^{n-k}. \qquad \square$$

Proposition 2.2. *For any non-negative integer* n *there holds*

$$\mathcal{D}_{n+1}(z) = (n+1)\mathcal{D}_n(z) + (z-1)^{n+1}, \tag{2}$$
$$\mathcal{D}_{n+2}(z) = (n+1)[\mathcal{D}_{n+1}(z) + \mathcal{D}_n(z)] + (z-1)^{n+1} + (z-1)^{n+2}, \tag{3}$$
$$\mathcal{D}_n(\overline{z}) = \overline{\mathcal{D}_n(z)},$$
$$\mathcal{D}_{n+1,1}(z) = (n+1)\mathcal{D}_{n,1}(z) + (z-1)^n + (\overline{z}-1)^n,$$
$$\mathcal{D}_{n+1,2}(z) = (n+1)\mathcal{D}_{n,2}(z) + (z-1)^n - (\overline{z}-1)^n,$$

where $\overline{\mathcal{D}_n(z)}$ *is the complex conjugate of* $\mathcal{D}_n(z)$

Proof. For (2), we have

$$\mathcal{D}_{n+1}(z) = \sum_{k=0}^{n+1} (n+1)_k \left[\sum_{s=0}^{k} \frac{(x-1)^s}{s!}\right](iy)^{n+1-k}$$
$$= (n+1)\sum_{k=1}^{n+1} (n)_{k-1} \left[\sum_{s=0}^{k} \frac{(x-1)^s}{s!}\right](iy)^{n+1-k} + (iy)^{n+1}$$
$$= (n+1)\left[\sum_{k=0}^{n} (n)_k \left[\sum_{s=0}^{k} \frac{(x-1)^s}{s!} + \frac{(x-1)^{k+1}}{(k+1)!}\right](iy)^{n-k}\right] + (iy)^{n+1},$$

then

$$\mathcal{D}_{n+1}(z) = (n+1)\mathcal{D}_n(z) + (n+1)\left[\sum_{k=0}^{n} (n)_k \frac{(x-1)^{k+1}}{(k+1)!}(iy)^{n-k}\right] + (iy)^{n+1}.$$

On the other hand, we have

$$(n+1)\sum_{k=0}^{n} (n)_k \frac{(x-1)^{k+1}}{(k+1)!}(iy)^{n-k} + (iy)^{n+1} = \sum_{k=0}^{n} \binom{n+1}{k+1}(x-1)^{k+1}(iy)^{n-k} + (iy)^{n+1}$$
$$= \sum_{k=1}^{n+1} \binom{n+1}{k}(x-1)^k (iy)^{n+1-k} + (iy)^{n+1}$$
$$= \left[(x-1+iy)^{n+1} - (iy)^{n+1}\right] + (iy)^{n+1}$$
$$= (x-1+iy)^{n+1}.$$

Hence

$$\mathcal{D}_{n+1}(z) = (n+1)\,\mathcal{D}_n(z) + (x-1+iy)^{n+1},$$

or equivalently

$$\mathcal{D}_{n+1}(z) = (n+1)\,\mathcal{D}_n(z) + (z-1)^{n+1}.$$

For (3), we have

$$\begin{aligned}\mathcal{D}_{n+2}(z) &= (n+2)\,\mathcal{D}_{n+1}(z) + (z-1)^{n+2} \\ &= (n+1)\,\mathcal{D}_{n+1}(z) + \mathcal{D}_{n+1}(z) + (z-1)^{n+2} \\ &= (n+1)\left[\mathcal{D}_{n+1}(z) + \mathcal{D}_n(z)\right] + (z-1)^{n+1} + (z-1)^{n+2}.\end{aligned}$$ □

The first few $\mathcal{D}_n(z)$ polynomials can be written as follows:

$$\mathcal{D}_0(z) = 1, \mathcal{D}_1(z) = z, \mathcal{D}_2(z) = z^2+1, \mathcal{D}_3(z) = z^3+3z+2.$$

Note that $\mathcal{D}_n(z)$ is a polynomial with integer coefficients.

Proposition 2.3. *Let z_0 and $z = z_0 + h$ be two points. The function $\mathcal{D}_n(z)$ is holomorphic on $\mathbb{C}$ and for any non-negative integer n, we have*

$$\mathcal{D}'_n(z) = n\mathcal{D}_{n-1}(z), \tag{4}$$

$$\mathcal{D}_n(z) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_{n-k}(z_0)(z-z_0)^k. \tag{5}$$

If $z_0 = 0$, we obtain

$$\mathcal{D}_n(z) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_{n-k} z^k, \tag{6}$$

where $\mathcal{D}'_n(z)$ is the derivative of $\mathcal{D}_n(z)$

Proof. For (4), we proceed by induction on n. Indeed, for $n = 1$, $\mathcal{D}_1(z) = z$, we have

$$\mathcal{D}'_1(z) = 1 = \mathcal{D}_0(z).$$

For $n = 2$, $\mathcal{D}_2(z) = z^2 + 1$, we have

$$\mathcal{D}'_2(z) = 2z = 2\mathcal{D}_1(z).$$

Assume for any integer $n \geq 1$, $\mathcal{D}'_n(z) = n\mathcal{D}_{n-1}(z)$. Using the relationship (2), we get

$$\begin{aligned}\mathcal{D}'_{n+1}(z) &= \left[(n+1)\,\mathcal{D}_n(z) + (z-1)^{n+1}\right]' \\ &= (n+1)\,\mathcal{D}'_n(z) + (n+1)(z-1)^n \\ &= (n+1)\,n\mathcal{D}_{n-1}(z) + (n+1)(z-1)^n \\ &= (n+1)\,\mathcal{D}_n(z).\end{aligned}$$

For (5), we have $\mathcal{D}'_n(z) = n\mathcal{D}_{n-1}(z)$, then $\mathcal{D}^{(2)}_n(z) = n(n-1)\,\mathcal{D}_{n-2}(z)$, and by induction the k-th derivative of $\mathcal{D}_n(z)$ is

$$\mathcal{D}^{(k)}_n(z) = (n)_k\,\mathcal{D}_{n-k}(z),$$

which gives

$$\frac{\mathcal{D}^{(k)}_n(z)}{k!} = \binom{n}{k} \mathcal{D}_{n-k}(z).$$

Then the Taylor's series for $\mathcal{D}_n(z)$ is to be

$$\mathcal{D}_n(z) = \sum_{k=0}^{n} \frac{\mathcal{D}_n^{(k)}(z_0)}{k!}(z - z_0)^k$$
$$= \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_{n-k}(z_0)(z - z_0)^k.$$

This completes the proof. □

3 Congruence on the derangement polynomials with a complex variable

In this section, we use the properties of the classical umbral calculus to drive new congruences involving the derangement polynomials with a complex variable. The derangement polynomials with a complex variable are defined by

$$\mathcal{D}_n(z) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_{n-k} z^k.$$

Let $\mathbf{D}$ be the derangement umbra defined by $\mathbf{D}^n = \mathcal{D}_n$, then we can define the generalized derangement umbra $\mathbf{D_z}$ as follows

$$\mathbf{D}_{\mathbf{z}}^n = \mathcal{D}_n(z) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_{n-k} z^k = (\mathbf{D} + z)^n.$$

For more information on the umbral calculus see [1, 2, 4, 5, 10, 11]. In the remainder of this section, for any polynomials f and g, with integer coefficients we denote by $f(z) \equiv g(z)$ to mean $f(z) \equiv g(z) \pmod{p\mathbb{Z}_p[z]}$ and for any numbers a and b by $a \equiv b$ we mean $a \equiv b \pmod{p}$.

Lemma 3.1. *Let f be a polynomial in $\mathbb{Z}[z]$ and s be a non-negative integer, then for any prime $p \geq 3$, there holds*

$$\left(\mathbf{D}_{\mathbf{z}}^{p^s} + 1\right) f(\mathbf{D_z}) \equiv z^{p^s} f(\mathbf{D_z}).$$

Proof. It suffices to take $f(z) = z^n$. We proceed by induction on s. For $s = 1$ we have

$$(\mathbf{D}_{\mathbf{z}}^p + 1)\mathbf{D}_{\mathbf{z}}^n = \mathbf{D}_{\mathbf{z}}^{p+n} + \mathbf{D}_{\mathbf{z}}^n$$
$$= (\mathbf{D} + z)^p (\mathbf{D} + z)^n + \mathbf{D}_{\mathbf{z}}^n$$
$$\equiv (\mathbf{D}^p + z^p)(\mathbf{D} + z)^n + \mathbf{D}_{\mathbf{z}}^n$$
$$= \mathbf{D}_{\mathbf{z}}^n + z^p \mathbf{D}_{\mathbf{z}}^n + \sum_{k=0}^{n} \binom{n}{k} \mathbf{D}^{n-k+p} z^k,$$

and by the known congruence $\mathcal{D}_{n+p} \equiv -\mathcal{D}_n$, or equivalently $\mathbf{D}^{n+p} \equiv -\mathbf{D}^n$, see [12]. So we obtain

$$
\begin{aligned}
\left(\mathbf{D}_{\mathbf{z}}^{p}+1\right)\mathbf{D}_{\mathbf{z}}^{n} &\equiv \mathbf{D}_{\mathbf{z}}^{n}+z^{p}\mathbf{D}_{\mathbf{z}}^{n}-\sum_{k=0}^{n}\binom{n}{k}\mathbf{D}^{n-k}z^{k}\\
&=\mathbf{D}_{\mathbf{z}}^{n}+z^{p}\mathbf{D}_{\mathbf{z}}^{n}-\left(\mathbf{D}+z\right)^{n}\\
&=z^{p}\mathbf{D}_{\mathbf{z}}^{n}.
\end{aligned}
$$

Assume it is true for $s\geq 1$. Then we have

$$
\begin{aligned}
\mathbf{D}_{\mathbf{z}}^{n}\left(\mathbf{D}_{\mathbf{z}}^{p^{s+1}}+1\right) &= \mathbf{D}_{\mathbf{z}}^{n}\left(\left(\mathbf{D}_{\mathbf{z}}^{p^{s}}+1-1\right)^{p}+1\right)\\
&\equiv \mathbf{D}_{\mathbf{z}}^{n}\left(\left(\mathbf{D}_{\mathbf{z}}^{p^{s}}+1\right)^{p}+(-1)^{p}+1\right)\\
&= \mathbf{D}_{\mathbf{z}}^{n}\left(\mathbf{D}_{\mathbf{z}}^{p^{s}}+1\right)^{p}\\
&= \left[\mathbf{D}_{\mathbf{z}}^{n}\left(\mathbf{D}_{\mathbf{z}}^{p^{s}}+1\right)\right]\left(\mathbf{D}_{\mathbf{z}}^{p^{s}}+1\right)^{p-1}\\
&\equiv z^{p^{s}}\mathbf{D}_{\mathbf{z}}^{n}\left(\mathbf{D}_{\mathbf{z}}^{p^{s}}+1\right)^{p-1}\\
&= z^{p^{s}}\left[\mathbf{D}_{\mathbf{z}}^{n}\left(\mathbf{D}_{\mathbf{z}}^{p^{s}}+1\right)\right]\left(\mathbf{D}_{\mathbf{z}}^{p^{s}}+1\right)^{p-2}\\
&\equiv z^{2p^{s}}\mathbf{D}_{\mathbf{z}}^{n}\left(\mathbf{D}_{\mathbf{z}}^{p^{s}}+1\right)^{p-2}\\
&\vdots\\
&\equiv \left(z^{p^{s}}\right)^{p}\mathbf{D}_{\mathbf{z}}^{n}\\
&= z^{p^{s+1}}\mathbf{D}_{\mathbf{z}}^{n}.
\end{aligned}
$$

and the proof of the induction step is complete. □

The principal result given by the following Theorem.

Theorem 3.2. *For any integers $n,s\geq 1, m\geq 0$ and for any prime $p\geq 3$, there holds*

$$\mathcal{D}_{n+mp^{s}}(z)\equiv\left(z^{p^{s}}-1\right)^{m}\mathcal{D}_{n}(z).$$

For $y=0$ or $z=0$, we obtain

$$
\begin{aligned}
\mathcal{D}_{n+mp^{s}}(x) &\equiv \left(x^{p^{s}}-1\right)^{m}\mathcal{D}_{n}(x),\\
\mathcal{D}_{n+mp^{s}} &\equiv (-1)^{m}\mathcal{D}_{n},\\
\mathcal{D}_{n+2p} &\equiv \mathcal{D}_{n}.
\end{aligned}
$$

Proof. For $m=1$ just take $f(z)=z^{n}$ in Lemma 3.1 and for $m>1$,we have

$$
\begin{aligned}
\mathcal{D}_{n+mp^{s}}(z) &= \mathcal{D}_{n+(m-1)p^{s}+p^{s}}(z)\\
&\equiv \left(z^{p^{s}}-1\right)\mathcal{D}_{n+(m-1)p^{s}}(z)\\
&= \left(z^{p^{s}}-1\right)\mathcal{D}_{n+(m-2)p^{s}+p^{s}}(z)\\
&\equiv \left(z^{p^{s}}-1\right)^{2}\mathcal{D}_{n+(m-2)p^{s}}(z)\\
&= \left(z^{p^{s}}-1\right)^{2}\mathcal{D}_{n+(m-3)p^{s}+p^{s}}(z)\\
&\equiv \left(z^{p^{s}}-1\right)^{3}\mathcal{D}_{n}(z)\\
&\vdots\\
&\equiv \left(z^{p^{s}}-1\right)^{m}\mathcal{D}_{n}(z).
\end{aligned}
$$

Hence the proof is complete. □

Corollary 3.2.1. *For any prime number* $p \geq 3$ *and any integers* $s \geq 1, m_0, ..., m_s \in \{0, ..., p-1\}$, *there holds*

$$\mathcal{D}_{m_0+m_1p+\cdots+m_sp^s}(z) \equiv (z^p-1)^{m_1}\left(z^{p^2}-1\right)^{m_2}\cdots\left(z^{p^s}-1\right)^{m_s}\mathcal{D}_{m_0}(z).$$

In particular, we have

$$\mathcal{D}_{m_1p+\cdots+m_sp^s}(z) \equiv -(z^p-1)^{m_1}\left(z^{p^2}-1\right)^{m_2}\cdots\left(z^{p^s}-1\right)^{m_s},$$
$$\mathcal{D}_{m_1p+\cdots+m_sp^s}(k) \equiv -(k-1)^{m_1+m_2+\cdots+m_s}.$$

References

[1] Benyattou, A., & Mihoubi, M. (2018). Curious congruences related to the Bell polynomials, *Quaest. Math.*, 41(3), 437–448.

[2] Benyattou, A., & Mihoubi, M. (2019). Real-rooted polynomials via generalized Bell umbra. *Notes on Number Theory and Discrete Mathematics*, 25(2), 136–144.

[3] Darus, M., & Ibrahim, R. (2010). On generalisation of polynomials in complex plane, *Advances in Decision Sciences*, 2010, (2010), 9 pages.

[4] Gertsch, A., & Robert, A. M. (1996). Some congruences concerning the Bell numbers, *Bull. Belg. Math. Soc. Simon Stevin*, 3, 467–475.

[5] Gessel, I. M. (2003). Applications of the classical umbral calculus, *Algebra Universalis*, 49, 397–434.

[6] Kim, D. S., Kim, T., & Lee, H. (2019). A note on Degenerate Euler and Bernoulli polynomials, *Symmetry*, 11, 1168.

[7] Kim, T., & Kim, D. S. (2018). Some identities on derangement and degenerate derangement polynomials, *Advances in Mathematical Inequalities and Applications*, 265–277, Trends Math., Birkhauser/Springer, Singapore.

[8] Kim, T., Kim, D. S., Dolgy, D. V., & Kwon, J. (2018). Some identities of derangement numbers. *Proc. Jangjeon Math. Soc.*, 21(1), 125–141.

[9] Kim, T., Kim, D. S., Kwon, H.-I., & Jang, L.-C. (2018). Fourier series of sums of products of r-derangement functions, *J. Nonlinear Sci. Appl.*, 11(4), 575–590.

[10] Roman, S. (1984). *The Umbral Calculus,* Academic Press, Orlando, FL.

[11] Rota, G. C., & Taylor, B. D. (1994). The classical umbral calculus, *SIAM J. Math. Anal.*, 25, 694–711.

[12] Sun, Z.-W., & Zagier, D. (2011). On a curious property of Bell numbers, *Bull. Aust. Math. Soc.*, 84, 153–158.