Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 26, 2020, No. 4, 93–102
DOI: 10.7546/nntdm.2020.26.4.93-102

Explicit formula of a new class of q-Hermite-based Apostol-type polynomials and generalization

Mouloud Goubi

Department of Mathematics, University of UMMTO
Tizi-Ouzou 15000, Algeria
Algebra and Number Theory Laboratory, USTHB
Algiers, Algeria
e-mail: mouloud.goubi@ummto.dz

Received: 4 June 2020 **Revised:** 31 October 2020 Accepted: 4 November 2020

Abstract: The present article deals with a recent study of a new class of q-Hermite-based Apostol-type polynomials introduced by Waseem A. Khan and Divesh Srivastava. We give their explicit formula and study a generalized class depending in any q- analog generating function.
Keywords: q-Hermite-based Apostol-type polynomials, q-analog Cauchy product, f_q-Hermite-based Apostol-type polynomials and numbers.
2010 Mathematics Subject Classification: 05A10, 05A15, 11B68, 16B65.

1 Introduction

Throughout this work, $\mathbb{C}$ designates the field of complex numbers, $\mathbb{N}$ indicates the set of positive integers and $\mathbb{N}^\star = \mathbb{N} \setminus \{0\}$. First we recall some concepts related to q-calculus, which we need in the development of this article. Let $(a, q) \in \mathbb{C}$ such that $|q| < 1$. The q-analog of a is given by

$$[a]_q = \frac{1 - q^a}{1 - q}, \tag{1}$$

and the q-factorial function is defined by

$$[n]_q! = \prod_{m=0}^{n} [m]_q = \frac{(q;q)_n}{(1-q)^n} \tag{2}$$

with $(q;q)_n = \prod_{m=1}^{n}(1-m^q)$. The corresponding q-binomial coefficient is given by the relation

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}. \tag{3}$$

Finally the q-exponential generating function is defined by

$$e_q(t) = \sum_{n\geq 0}\frac{t^n}{[n]_q!} = \sum_{n\geq 0}\frac{(1-q)^n}{(q;q)_n}t^n. \tag{4}$$

According to these notations, the q-Hermite polynomials $H_{n,q}(x)$ are defined by means of the generating function (see [4, 5])

$$F_q(x,t) = \sum_{n\geq 0}(-1)^n q^{\binom{n}{2}}\frac{e_q(xt)t^{2n}}{[2n]_q!} = \sum_{n\geq 0}H_{n,q}(x)\frac{t^q}{[n]_q!}. \tag{5}$$

Recently, Waseem A. Khan and Divesh Srivastava (see [5, 12–14]) introduced the generalized q-Hermite-based Apostol-type polynomials ${}_H\mathcal{F}^{(\alpha)}_{n,q}(x;a,b;\lambda;\mu,\nu)$ by means of the generating function

$$\left(\frac{2^\mu t^\nu}{\lambda e_q(t)+a^b}\right)^\alpha \sum_{n\geq 0}(-1)^n q^{\binom{n}{2}}\frac{e_q(xt)t^{2n}}{[2n]_q!} = \sum_{n\geq 0}{}_H\mathcal{F}^{(\alpha)}_{n,q}(x;a,b;\lambda;\mu,\nu)\frac{t^q}{[n]_q!} \tag{6}$$

with $\alpha \in \mathbb{N}^\star$, $\lambda, a, b \in \mathbb{C}$ and $|t| < |\log(-\lambda)|$. Letting $x = 0$ in the definition (6):

$${}_H\mathcal{F}^{(\alpha)}_{n,q}(a,b;\lambda;\mu,\nu) = {}_H\mathcal{F}^{(\alpha)}_{n,q}(0;a,b;\lambda;\mu,\nu)$$

are so called q-Hermite-based Apostol-type numbers of order α and generated by the function

$$\left(\frac{2^\mu t^\nu}{\lambda e_q(t)+a^b}\right)^\alpha \sum_{n\geq 0}(-1)^n q^{\binom{n}{2}}\frac{t^{2n}}{[2n]_q!} = \sum_{n\geq 0}{}_H\mathcal{F}^{(\alpha)}_{n,q}(a,b;\lambda;\mu,\nu)\frac{t^q}{[n]_q!}. \tag{7}$$

Other interesting links about q-Hermite-based Apostol-type numbers, $(p;q)$-analogue type of Frobenius Genocchi numbers and polynomials and q-analogue of Hermite poly-Bernoulli numbers and polynomials are illustrated in the works [6–11] of Waseem A. Khan et al.

2 Explicit formula of generalized q-Hermite-based Apostol-type polynomials

The generalized q-Apostol type polynomials $F^{(\alpha)}_{n,q}(x;a,b;\lambda)$ of order $\alpha \in \mathbb{N}^\star$ are defined by means of the generating function

$$\left(\frac{2^\mu t^\nu}{\lambda e_q(t)+a^b}\right)^\alpha e_q(xt) = \sum_{n\geq 0}F^{(\alpha)}_{n,q}(x;a,b;\lambda)\frac{t^n}{[n]_q!} \tag{8}$$

and the generalized q-Apostol type numbers $F^{(\alpha)}_{n,q}(a,b;\lambda) = F^{(\alpha)}_{n,q}(0;a,b;\lambda)$ are given by the generating function

$$\left(\frac{2^\mu t^\nu}{\lambda e_q(t)+a^b}\right)^\alpha = \sum_{n\geq 0}F^{(\alpha)}_{n,q}(a,b;\lambda)\frac{t^n}{[n]_q!}. \tag{9}$$

Based on Cauchy product of series (see [1]); the q-analog Cauchy product of formal q-analog generating functions

$$\sum_{n\geq 0} a_n \frac{t}{[n]_q!} \text{ and } \sum_{n\geq 0} b_n \frac{t}{[n]_q!}$$

is given by the following relation

$$\left(\sum_{n\geq 0} a_n \frac{t}{[n]_q!}\right)\left(\sum_{n\geq 0} bn \frac{t}{[n]_q!}\right) = \sum_{n\geq 0}\sum_{k=0}^{n} \binom{n}{k}_q a_k b_{n-k} \frac{t^n}{[n]_q!}. \tag{10}$$

Regarding the generating function of generalized q-Hermite-based Apostol-type polynomials; ${}_H\mathcal{F}_{n,q}^{(\alpha)}(x;a,b;\lambda;\mu,\nu)$ follows from q-analog Cauchy product of $\left(\frac{2^\mu t^\nu}{\lambda e_q(t)+a^b}\right)^\alpha$ and $F_q(x,t)$. By means of identity (10) we have

$${}_H\mathcal{F}_{n,q}^{(\alpha)}(x;a,b;\lambda;\mu,\nu) = \sum_{k=0}^{n} \binom{n}{k}_q F_{k,q}^{(\alpha)}(a,b;\lambda)\, H_{n-k,q}(x). \tag{11}$$

To get explicit formula of ${}_H\mathcal{F}_{n,q}^{(\alpha)}(x;a,b;\lambda;\mu,\nu)$ we must compute the corresponding explicit formulae of numbers $F_{n,q}^{(\alpha)}(a,b;\lambda)$ and polynomials $H_{n,q}(x)$.

2.1 Explicit formula of q-Hermite polynomials

q-Hermite polynomials follow from q-analog Cauchy product of

$$e_q(xt) \text{ and } F_q(t) = \sum_{n\geq 0}(-1)q^{\binom{n}{2}} \frac{t^n}{[n]_q!}.$$

Explicitly we have the following theorem.

Theorem 2.1.

$$H_{n,q}(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k q^{\binom{k}{2}} \binom{n}{2k}_q x^{n-2k}. \tag{12}$$

Proof. First let the sequence a_n be given by

$$a_n = \frac{1}{2}\left(1+(-1)^n\right)(-1)^{\lfloor \frac{n}{2} \rfloor} q^{\binom{\lfloor \frac{n}{2} \rfloor}{2}}.$$

Then

$$F_q(t) = \sum_{n\geq 0} a_n \frac{t^{2n}}{[2n]_q!}$$

and

$$F_q(x,t) = \left(\sum_{n\geq 0} a_n \frac{t^n}{[n]_q!}\right)\left(\sum_{n\geq 0} \frac{x^n t^n}{[n]_q!}\right).$$

Thus

$$F_q(x,t) = \sum_{n\geq 0}\sum_{k=0}^{n} \binom{n}{k}_q a_k x^{n-k} \frac{t^n}{[n]_q!},$$

but

$$\sum_{k=0}^{n}\binom{n}{k}_q a_k x^{n-k}=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}(-1)^k q^{\binom{k}{2}}\binom{n}{2k}_q x^{n-2k}$$

and the result follows. □

2.2 α-power q-analog generating function

To compute the explicit formula of α-power q-analog generating function; we must revisit some advanced studies in this area. Consider the formal generating function $f(t)=\sum_{n\geq 0} a_n t^n$ with the coefficients a_n are numbers or polynomials and the first term $a_0\neq 0$. Then $f^{\alpha}(t)$ is a generating function too, with hint of umbral calculus we noted in [3] that

$$f^{\alpha}(t)=\sum_{n\geq 0}\sum_{a_{i_1}+\cdots+a_{i_n}=\alpha} a_{i_1}\ldots a_{i_n}t^n. \tag{13}$$

In the general case $\alpha\in\mathbb{C}^{\star}$; an improvement of this result is given in our recent work [2], where

$$f^{\alpha}(t)=a_0^{\alpha}+\sum_{n\geq 1}\sum_{k=1}^{n}\sum_{s_n(k)}\binom{\alpha}{k}\binom{k}{k_1,\ldots,k_n}a_0^{\alpha-k}a_1^{k_1}\ldots a_n^{k_n}t^n, \tag{14}$$

$s_n(k)$ is the set of all $(k_1,\ldots,k_n)\in\mathbb{N}^n$ satisfying conditions $k_1+\cdots+k_n=k$ and $k_1+2k_2+\cdots+nk_n=n$. It is obvious to remark that $k_j=0$ for $j\geq n-k+1$ and $s_n(k)$ reduces to $(n-k+1)$-uplet $(k_1,\ldots,k_{n-k+1})$. We conclude that

$$f^{\alpha}(t)=a_0^{\alpha}+\sum_{n\geq 1}\sum_{k=1}^{n}(\alpha)_k a_0^{\alpha-k}B_{n,k}\left(1!a_1,\ldots,(n-k+1)!a_{n-k+1}\right)\frac{t^n}{n!}. \tag{15}$$

$B_{n,k}$ are exponential partial Bell polynomials given by the expression

$$B_{n,k}\left(x_1,\ldots,x_{n-k+1}\right)=\frac{n!}{k!}\sum_{s_n(k)}\binom{k}{k_1,\ldots,k_{n-k+1}}\prod_{r=1}^{n-k+1}\left(\frac{x_r}{r!}\right)^{k_r} \tag{16}$$

and defined by means of the generating function

$$\frac{1}{k!}\left(\sum_{m\geq 1}x_m\frac{z^m}{m!}\right)=\sum_{n\geq k}B_{n,k}\left(x_1,\ldots,x_{n-k+1}\right)\frac{z^n}{n!}. \tag{17}$$

Stirling numbers $S_2\left(n,k\right)$ obtained by the function

$$\frac{1}{k!}\left(e^t-1\right)^k=\sum_{n\geq 0}S_2\left(n,k\right)\frac{t^n}{n!} \tag{18}$$

are special case of $B_{n,k}$ and we have $B_{n,k}\left(1,\ldots,1\right)=S_2\left(n,k\right)$. Consequently these polynomials admit the following formulation

$$S_2\left(n,k\right)=\frac{1}{k!}\sum_{j=1}^{k}\binom{k}{j}(-1)^{k-j}j^n. \tag{19}$$

According to exponential partial Bell polynomials, the explicit formula of q-analog generating function $f_q(t)=\sum_{n\geq 0} b_n\frac{t^n}{[n]_q!}$ is given by the following theorem.

Theorem 2.2.

$$f_q^\alpha(t) = b_0^\alpha + \sum_{n\geq 1}\sum_{k=1}^{n} \frac{(\alpha)_k b_0^{\alpha-k}[n]_q!}{n!} B_{n,k}\left(\frac{r!b_r}{(q,q)_r}\right)(1-q)^n \frac{t^n}{[n]_q!}, \tag{20}$$

where

$$B_{n,k}\left(\frac{r!b_r}{(q,q)_r}\right) = B_{n,k}\left(\frac{1!b_1}{(q,q)_1}, \ldots, \frac{(n-k+1)!b_{n-k+1}}{(q,q)_{n-k+1}}\right).$$

Proof. Let the sequence $a_n = \frac{b_n}{(n)_q!}$. Then $f_q(t) = \sum_{n\geq 0} a_n t^n$ and by means of the expression (15) we deduce that

$$f_q^\alpha(t) = b_0^\alpha + \sum_{n\geq 1}\sum_{k=1}^{n} \frac{(\alpha)_k a_0^{\alpha-k}[n]_q!}{n!} B_{n,k}\left(\frac{1!b_1}{(1)_q!}, \ldots, \frac{(n-k+1)!b_{n-k+1}}{(n-k+1)_q!}\right)\frac{t^n}{[n]_q!}.$$

But

$$B_{n,k}\left(\frac{1!b_1}{(1)_q!}, \ldots, \frac{(n-k+1)!b_{n-k+1}}{(n-k+1)_q!}\right) = (1-q)^n B_{n,k}\left(\frac{1!b_1}{(q,q)_1}, \ldots, \frac{(n-k+1)!b_{n-k+1}}{(q,q)_{n-k+1}}\right).$$

Then

$$f^\alpha(t) = b_0^\alpha + \sum_{n\geq 1}\sum_{k=1}^{n} \frac{(\alpha)_k a_0^{\alpha-k}[n]_q!}{n!} B_{n,k}\left(\frac{r!b_r}{(q,q)_r}\right)(1-q)^n \frac{t^n}{[n]_q!}. \qquad \square$$

Let auxiliary sequence c_n of numbers be defined by means of the generating function

$$\left(\frac{1}{\lambda e_q(t) + a^b}\right)^\alpha = \sum_{n\geq 0} c_n \frac{t^n}{[n]_q!}.$$

According to Theorem 2.2 it follows that c_n is written in the form given by the following proposition.

Proposition 2.3. *Let* $\lambda + a^b \neq 0$. *Then* $c_0 = \left(\lambda + a^b\right)^{-\alpha}$ *and for* $n \geq 1$ *we have*

$$c_n = \sum_{k=1}^{n} \frac{(-\alpha)_k \left(\lambda + a^b\right)^{-\alpha-k}[n]_q}{n!} \lambda^k (1-q)^n B_{n,k}\left(\frac{r!}{(q,q)_r}\right). \tag{21}$$

Proof. The series expansion of $\lambda e_q(t) + a^b$ is

$$\lambda e_q(t) + a^b = \lambda + a^b + \sum_{n\geq 1} \lambda \frac{t^n}{[n]_q!}.$$

Then

$$\left(\lambda e_q(t) + a^b\right)^{-\alpha} = \left(\lambda + a^b\right)^{-\alpha} + \sum_{n\geq 1}\sum_{k=1}^{n} \frac{(-\alpha)_k \left(\lambda + a^b\right)^{-\alpha-k}[n]_q!}{n!} \lambda^k B_{n,k}\left(\frac{r!}{(q,q)_r}\right)(1-q)^n \frac{t^n}{[n]_q!}.$$

Furthermore $c_0 = \left(\lambda + a^b\right)^{-\alpha}$ and for $n \geq 1$;

$$c_n = \sum_{k=1}^{n} \frac{(-\alpha)_k \left(\lambda + a^b\right)^{-\alpha-k}[n]_q!}{n!} \lambda^k (1-q)^n B_{n,k}\left(\frac{r!}{(q,q)_r}\right). \qquad \square$$

Corollary 2.3.1. *We have* $F_{n,q}^{(\alpha)}(a,b;\lambda)=0$ *for* $n<\nu\alpha$*,* $F_{\nu\alpha,q}^{(\alpha)}(a,b;\lambda)=2^{\mu\alpha}\left(\lambda+a^b\right)^{-\alpha}[\nu\alpha]_q!$ *and for* $n>\nu\alpha$*:*

$$F_{n,q}^{(\alpha)}(a,b;\lambda)=2^{\mu\alpha}[n]_q!\sum_{k=1}^{n-\nu\alpha}\frac{(-\alpha)_k\left(\lambda+a^b\right)^{-\alpha-k}}{(n-\nu\alpha)!}(1-q)^{n-\nu\alpha}\lambda^k B_{n-\nu\alpha,k}\left(\frac{r!}{(q,q)_r}\right). \quad (22)$$

Proof. We have

$$\left(\frac{2^\mu t^\nu}{\lambda e_q(t)+a^b}\right)^\alpha=2^{\mu\alpha}t^{\nu\alpha}\left(\frac{1}{\lambda e_q(t)+a^b}\right)^\alpha.$$

Then,

$$\left(\frac{2^\mu t^\nu}{\lambda e_q(t)+a^b}\right)^\alpha=2^{\mu\alpha}t^{\nu\alpha}\left(\left(\lambda+a^b\right)^{-\alpha}+\sum_{n\geq 1}c_n\frac{t^n}{[n]_q!}\right).$$

Furthermore,

$$\left(\frac{2^\mu t^\nu}{\lambda e_q(t)+a^b}\right)^\alpha=2^{\mu\alpha}\left(\lambda+a^b\right)^{-\alpha}t^{\nu\alpha}+2^{\mu\alpha}\sum_{n\geq\nu\alpha+1}c_{n-\nu\alpha}\frac{t^n}{[n-\nu\alpha]_q!}.$$

Finally,

$$\sum_{n\geq 0}F_{n,q}^{(\alpha)}(a,b;\lambda)\frac{t^n}{[n]_q!}=2^{\mu\alpha}\left(\lambda+a^b\right)^{-\alpha}[\nu\alpha]_q!\frac{t^{\nu\alpha}}{[\nu\alpha]_q!}+2^{\mu\alpha}\sum_{n\geq\nu\alpha}\frac{[n]_q!c_{n-\nu\alpha}}{[n-\nu\alpha]_q!}\frac{t^n}{[n]_q!}.$$

Then $F_{n,q}^{(\alpha)}(a,b;\lambda)=0$ for $n<\nu\alpha$, $F_{\nu\alpha,q}^{(\alpha)}(a,b;\lambda)=2^{\mu\alpha}\left(\lambda+a^b\right)^{-\alpha}[\nu\alpha]_q!$ and for $n\geq\nu\alpha$ we have

$$F_{n,q}^{(\alpha)}(a,b;\lambda)=2^{\mu\alpha}\frac{[n]_q!}{[n-\nu\alpha]_q!}c_{n-\nu\alpha}.$$

Substitute the value of $c_{n-\nu\alpha}$ to get the desired result. □

We have already found the necessary tools for computing the explicit formula of q-Hermite-based Apostol-type polynomial.

Theorem 2.4. *If* $\lambda+a^b\neq 0$ *we have* ${}_H\mathcal{F}_{n,q}^{(\alpha)}(x;a,b;\lambda;\mu,\nu)=0$ *for* $n<\nu\alpha$ *and for* $n\geq\nu\alpha$*:*

$$\begin{aligned}{}_H\mathcal{F}_{n,q}^{(\alpha)}(x;a,b;\lambda;\mu,\nu)&=2^{\mu\alpha}\left(\lambda+a^b\right)^{-\alpha}[\nu\alpha]_q!\binom{n}{\nu\alpha}_q\sum_{l=0}^{\left\lfloor\frac{n-\nu\alpha}{2}\right\rfloor}(-1)^l q^{\binom{l}{2}}\binom{n-\nu\alpha}{2l}_q x^{n-\nu\alpha-2l}\\&+2^{\mu\alpha}\sum\nolimits_1\binom{n}{k}_q\binom{n-k}{2l}_q[k]_q!(1-q)^{k-\nu\alpha}(-1)^l\lambda^j(-\alpha)_j\\&\times q^{\binom{l}{2}}\frac{\left(\lambda+a^b\right)^{-\alpha-j}}{(k-\nu\alpha)!}B_{k-\nu\alpha,j}\left(\frac{r!}{(q,q)_r}\right)x^{n-k-2l},\end{aligned}$$

where $\sum_1$ *is the triple sum* $\sum_{k=\nu\alpha}^{n}\sum_{j=1}^{k-\nu\alpha}\sum_{l=0}^{\left\lfloor\frac{n-k}{2}\right\rfloor}$.

Proof. Since

$${}_H\mathcal{F}_{n,q}^{(\alpha)}(x;a,b;\lambda;\mu,\nu)=\sum_{k=\nu\alpha}^{n}\binom{n}{k}_q F_{k,q}^{(\alpha)}(a,b;\lambda)H_{n-k,q}(x)$$

and

$$H_{n-k,q}(x) = \sum_{l=0}^{\lfloor \frac{n-k}{2} \rfloor} (-1)^l q^{\binom{l}{2}} \binom{n-k}{2l}_q x^{n-k-2l}.$$

Then

$$\begin{aligned} {}_H\mathcal{F}_{n,q}^{(\alpha)}(x;a,b;\lambda;\mu,\nu) &= 2^{\mu\alpha}\left(\lambda + a^b\right)^{-\alpha} [\nu\alpha]_q! \binom{n}{\nu\alpha}_q H_{n-\nu\alpha,q}(x) \\ &\quad + \sum_{k=\nu\alpha+1}^{n} \binom{n}{k}_q F_{k,q}^{(\alpha)}(a,b;\lambda)\, H_{n-k,q}(x) \end{aligned}$$

and the desired result follows. □

Remark 2.5. *In the case $\lambda + a^b = 0$ and $\lambda \neq 0$; the result is totally different. We write*

$$\lambda e_q(t) + a^b = t \sum_{n\geq 0} \lambda \frac{t^n}{[n+1]_q!}.$$

We consider $\nu \geq 1$, then we will have

$$\left(\frac{2^\mu t^\nu}{\lambda e_q(t) + a^b}\right)^\alpha = \left(\frac{2^\mu}{\lambda}\right)^\alpha t^{\nu\alpha-\alpha} \left(\frac{1}{\sum_{n\geq 0} \frac{t^n}{[n+1]_q!}}\right)^\alpha.$$

But

$$\left(\frac{1}{\sum_{n\geq 0} \frac{t^n}{[n+1]_q!}}\right)^\alpha = 1 + \sum_{n\geq 1}\sum_{k=1}^{n} \frac{(-\alpha)_k [n]_q!}{n!} B_{n,k}\left(\frac{r!}{[n+1]_q (q,q)_r}\right) \times (1-q)^n \frac{t^n}{[n]_q!}.$$

Then

$$\begin{aligned} \left(\frac{2^\mu t^\nu}{\lambda e_q(t) + a^b}\right)^\alpha &= \left(\frac{2^\mu}{\lambda}\right)^\alpha t^{\nu\alpha-\alpha} + \left(\frac{2^\mu}{\lambda}\right)^\alpha \sum_{n\geq 1}\sum_{k=1}^{n} \frac{(-\alpha)_k [n]_q!}{n!} \\ &\quad \times B_{n,k}\left(\frac{r!}{[n+1]_q (q,q)_r}\right) (1-q)^n \frac{t^{n+\nu\alpha-\alpha}}{[n]_q!} \end{aligned}$$

and

$$\begin{aligned} \left(\frac{2^\mu t^\nu}{\lambda e_q(t) + a^b}\right)^\alpha &= \left(\frac{2^\mu}{\lambda}\right)^\alpha t^c + \left(\frac{2^\mu}{\lambda}\right)^\alpha \sum_{n\geq c+1}\sum_{k=1}^{n-c} \frac{(-\alpha)_k}{(n-c)!} (1-q)^{n-c} \\ &\quad \times B_{n-c,k}\left(\frac{r!}{[n-c+1]_q (q,q)_r}\right) t^n, \end{aligned}$$

where $c = \nu\alpha - \alpha$. Let us write

$$\left(\frac{2^\mu t^\nu}{\lambda e_q(t) + a^b}\right)^\alpha = \sum_{n\geq 0} d_n \frac{t^n}{[n]_q!}.$$

Then $d_n = 0$ for $n < c$, $d_c = \left(\frac{2^\mu}{\lambda}\right)^\alpha [c]_q!$ and for $n \geq c+1$ we have

$$d_n = \left(\frac{2^\mu}{\lambda}\right)^\alpha \sum_{k=1}^{n-c} \frac{(-\alpha)_k [n]_q!}{(n-c)!} B_{n-c,k}\left(\frac{r!}{[n-c+1]_q (q,q)_r}\right) \times (1-q)^{n-c}.$$

By means of the identity (11) *we will have* ${}_H\mathcal{F}^{(\alpha)}_{n,q}(x;a,b;\lambda;\mu,\nu) = \sum_{k=c}^{n}\binom{n}{k}_q d_k H_{n-k,q}(x)$. *Finally for* $n \geq c$

$$
{}_H\mathcal{F}^{(\alpha)}_{n,q}(x;a,b;\lambda;\mu,\nu) = \left(\frac{2^\mu}{\lambda}\right)^\alpha [c]_q! \sum_{l=0}^{\lfloor\frac{n-c}{2}\rfloor}(-1)^l q^{\binom{l}{2}}\binom{n-c}{2l}_q x^{n-c-2l} +
$$

$$
\left(\frac{2^\mu}{\lambda}\right)^\alpha \sum_{k=0}^{n}\sum_{j=1}^{k-c}\sum_{l=0}^{\lfloor\frac{n-k}{2}\rfloor}\binom{n}{k}_q\binom{n-k}{2l}_q(-1)^l q^{\binom{l}{2}}\frac{(-\alpha)_j[k]_q!}{(k-c)!}B_{k-c,j}\left(\frac{r!}{[k-c+1]_q(q,q)_r}\right)x^{n-k-2l}.
$$

3 Generalized f_q-Hermite-based Apostol-type polynomials

Let $\alpha \neq 0$ be a complex number and β real number. We consider the formal q-analog generating function $f_q(t) = \sum_{n\geq 0} b_n \frac{t^n}{[n]_q!}$ with the condition that b_0 is different from zero. A natural generalization of q-Hermite-based Apostol-type polynomials is given by the following definition

Definition 3.1. *The* f_q*-Hermite-based Apostol-type polynomials* ${}_H\mathcal{F}^{(\alpha)}_{n,f_q}(x;a,b;\lambda;\mu,\nu)$ *are given by the generating function*

$$
\beta t^m f_q^\alpha(t) e_q(xt)\sum_{n\geq 0}(-1)^n q^{\binom{n}{2}}\frac{t^{2n}}{[2n]_q!} = \sum_{n\geq 0} F^{(\alpha)}_{n,f_q}(x;\beta;c)\frac{t^n}{[n]_q!}. \tag{23}
$$

Thereafter the f_q-Hermite-based Apostol-type numbers $F^{(\alpha)}_{n,f_q}(\beta;c) = F^{(\alpha)}_{n,f_q}(0;\beta;c)$ are given by the generating function

$$
\beta t^m f_q^\alpha(t)\sum_{n\geq 0}(-1)^n q^{\binom{n}{2}}\frac{t^{2n}}{[2n]_q!} = \sum_{n\geq 0} F^{(\alpha)}_{n,f_q}(\beta;c)\frac{t^n}{[n]_q!}. \tag{24}
$$

For $-\alpha \in \mathbb{N}^\star, \beta = 2^{\mu\alpha}, m = \nu\alpha$ and $f_q(t) = \lambda e_q(t) + a^b$; $F^{(\alpha)}_{n,f_q}(x;\beta) = {}_H\mathcal{F}^{(\alpha)}_{n,q}(x;a,b;\lambda;\mu,\nu)$. Polynomials $F^{(\alpha)}_{n,f_q}(x;\beta;c)$ follows from q-analog Cauchy product of generating functions

$$
e_q(xt)\sum_{n\geq 0}(-1)^n q^{\binom{n}{2}}\frac{t^{2n}}{[2n]_q!} = \sum_{n\geq 0} H_{n,q}(x)\frac{t^n}{[n]_q!}
$$

and

$$
\beta t^m f_q^\alpha(t) = \sum_{n\geq 0} b_n \frac{t^n}{[n]_q!}.
$$

The closed formula of polynomial $F^{(\alpha)}_{n,f_q}(x;\beta;c)$ is established in the following theorem.

Theorem 3.2.

$$
F^{(\alpha)}_{n,f_q}(x;\beta;c) = \beta b_0^\alpha\binom{n}{m}_q [m]_q! \sum_{l=0}^{\lfloor\frac{n-m}{2}\rfloor}(-1)^l q^{\binom{l}{2}}\binom{n-m}{2l}_q x^{n-m-2l} +
$$

$$
\beta\sum_2\binom{n}{k}_q\binom{n-k}{2l}_q\frac{(\alpha)_j b_0^{\alpha-j}[k]_q!}{(k-m)!}(1-q)^{k-m}(-1)^l q^{\binom{l}{2}}B_{k-m,j}\left(\frac{r!b_r}{(q,q)_r}\right)x^{n-k-2l},
$$

where $\sum_2$ *is the triple sum* $\sum_{r=m+1}^{n}\sum_{j=1}^{k-m}\sum_{l=0}^{\lfloor\frac{n-k}{2}\rfloor}$.

Proof. Since

$$f_q^\alpha(t) = b_0^\alpha + \sum_{n\geq 1}\sum_{k=1}^{n} \frac{(\alpha)_k b_0^{\alpha-k}}{n!} B_{n,k}\left(\frac{r!b_r}{(q,q)_r}\right)(1-q)^n t^n,$$

then

$$t^m f_q^\alpha(t) = b_0^\alpha [m]_q! \frac{t^m}{[m]_q!} + \sum_{n\geq m+1}\sum_{k=1}^{n-m} \frac{(\alpha)_k b_0^{\alpha-k}[n]_q!}{(n-m)!} B_{n-m,k}\left(\frac{r!b_r}{(q,q)_r}\right)(1-q)^{n-m} \frac{t^n}{[n]_q!}.$$

Writing

$$\beta t^m f_q^\alpha(t) = \sum_{n\geq 0} c_n \frac{t^n}{[n]_q!};$$

then $c_n = 0$ for $n < m$, $c_m = \beta b_0^\alpha [m]_q!$ and for $n > m$ we have

$$c_n = \beta \sum_{k=1}^{n-m} \frac{(\alpha)_k b_0^{\alpha-k}[n]_q!}{(n-m)!} B_{n-m,k}\left(\frac{r!b_r}{(q,q)_r}\right)(1-q)^{n-m}$$

Thereafter in means of q-analog Cauchy product of generating functions we have

$$F_{n,f_q}^{(\alpha)}(x;\beta;c) = \sum_{r=m}^{n} \binom{n}{k}_q c_k H_{n-k}(x)$$

and the desired result follows. □

References

[1] Goubi, M. (2018). Successive derivatives of Fibonacci type polynomials of higher order in two variables, *Filomat*, 32, 5149–5159.

[2] Goubi, M. (2020). A new class of generalized polynomials associated with Hermite–Bernoulli polynomials, *J. Appl. Math.* & *Informatics*, 38, 211–220.

[3] Goubi, M. (2019). On the Recursion Formula of Polynomials Generated by Rational Functions, *Inter. Journ. Math. Analysis*, 13, 29–38.

[4] Mahmudov, N. I. (2014). Difference equations of q-Appell polynomials, *Appl. Math. Comput.*, 245, 539–543.

[5] Khan, W. A., & Srivastava, D. (2020). A new class of q-Hermite-based Apostol-type polynomials and its applications, *Notes on Number Theory and Discrete Mathematics*, 26 (1), 75–85.

[6] Khan, W. A., Khan, I. A., & Musharraf A. (2020). Degenerate Hermite poly-Bernoulli numbers and polynomials with q-parameter, *Stud. Univ. Babes-Bolayi, Math.*, 65 (1), 3–15.

[7] Khan, W. A., & Khan, I. A. (2020). A note on (p,q)-analogue type of Frobenius Genocchi numbers and polynomials. *East Asian Mathematical Journal*, 36 (1), 13–24.

[8] Khan, W. A., & Nisar, K. S. (2019). Notes on q-Hermite based unified Apostol type polynomials. *Journal of Interdisciplinary Mathematics*, 22 (7), 1185–1203.

[9] Khan, W. A., Khan, I. A., & Musharraf, A. (2019). A note on q-analogue of Hermite-poly-Bernoulli numbers and polynomials. *Mathematica Morvica*, 23 (2), 1–16.

[10] Kang, J. Y., & Khan, W. A. (2020). A new class of q-Hermite-based Apostol type Frobenius Genocchi polynomials. *Communication of the Korean Mathematical Society*, 35 (3), 759–771.

[11] Khan, W. A., Khan, I. A., Acikgoz, M., & Duran, U. (2020). Multifarious results for q-Hermite based Frobenius type Eulerian polynomials. *Notes on Number Theory and Discrete Mathematics*, 26 (2), 127–141.

[12] Khan, W. A., & Srivastava, D. (2019). A study of poly-Bernoulli polynomials associated with Hermite polynomials with q-parameter. *Honam Mathematical J.*, 41 (4), 781–798.

[13] Khan, W. A., & Srivastava, D. (2019). On the generalized Apostol-type Frobenius–Genocchi polynomials, *Filomat*, 33 (7), 1967–1977.

[14] Khan, W. A., & Srivastava, D. (2021). Certain properties of Apostol-type Hermite-based-Frobenius–Genocchi polynomials, *Kragujevac Journal of Mathematics*, 45 (6), 859–872.